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TRIANGULAR DIOPHANTINE TUPLES FROM $\{1, 2\}$

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Abstract. In this paper, we prove that there does not exist a set of four positive integers $\{1, 2, c, d\}$ such that a product of any two of them increased by 1 is a triangular number

1. Introduction

Long time ago, Diophantus of Alexandria noted that the product of any two distinct elements of the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ increased by 1 is a square of a rational number. Later, especially in the recent years, many generalizations and variations of his original problem were also studied. The following definition describes a more general situation.

Definition 1.1. *Let $m \geq 2$ be an integer, and let R denote a commutative ring with unity. Further let $n \in R$ be a non-zero element, and let $\{a_1, \ldots, a_m\}$ be a set of m distinct non-zero elements from R such that $a_i a_j + n$ is a square in R for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called Diophantine m-tuple with the property $D(n)$ or simply a $D(n)$-m-tuple in R.*

One of the questions of interest is how large these sets can be. The most studied and most well known case is when $R = \mathbb{Z}$ and $n = 1$. In this case, recently, the folklore conjecture of non-existence of quintuple was proved by He, Togbé, and Ziegler in [4]. Beside searching for such sets in different rings, there are also some other generalizations like asking $a_i a_j + n$ to be some other perfect power instead of square. The whole history of the problem, with recent results and up-to-date references can be found on the webpage [3].

In this paper, we introduce one other variation, when instead of squares we want $a_i a_j + n$ to be triangular numbers, i.e. numbers of the form $\triangle_k = \frac{k(k+1)}{2}$, $k \in \mathbb{N}$. To the best of our knowledge this modification has not yet been studied. Thus we introduce the following definition.

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Definition 1.2. *A set* $\{a_1, \ldots, a_m\}$ *of* m *distinct positive integers is called Triangular Diophantine* m*-tuple if* $a_i a_j + 1$ *is a triangular number for* $1 \leq i < j \leq m$.

The natural question asks the cardinality of such Triangular Diophantine tuples. However, in this paper, for the beginning, we start with a Triangular pair $\{1, 2\}$ of the smallest two elements possible, and prove the following theorem.

Theorem 1.3. *There does not exist Triangular Diophantine quadruple of the form* $\{1, 2, c, d\}$.

It states that the pair $\{1, 2\}$ cannot be extended to a quadruple, but we can easily see that there exist infinitely many triples $\{1, 2, c\}$. One of the motivations for proving this result was the paper of Dujella and Pethő [2], where they considered the extension of the $D(1)$-pair $\{1, 3\}$ in the ring $\mathbb{Z}$. This is similar because the pair $\{1, 3\}$ has the smallest integers possible. They showed that $\{1, 3\}$ can be extended to quadruples, but not to a quintuple. More precisely, they proved that the triple of the form $\{1, 3, c\}$ can be extended uniquely to a quadruple.

In the proof of Theorem 1.3, we use the same strategy that appears in their paper, of course with some modifications we need. For example, at some places we use integer points on elliptic curves. Firstly, we find all possible values of c that will extend our Triangular Diophantine pair $\{1, 2\}$ to a triple. Then we transform the problem of extending the triple $\{1, 2, c\}$ to solving a system of simultaneous Pell-like equations. After that we search for the intersection of binary recurrence sequences, which can be solved (for large indices) by combining congruences and Baker's theory on linear forms in logarithms of algebraic numbers. Finally, we need a kind of Baker-Davenport reduction based on continued fractions to prove Theorem 1.3.

We made a computer search for finding Triangular Diophantine quadruples $\{a, b, c, d\}$ in the range $1 \leq a \leq 30$, $a < b \leq 30^2$, $b < c \leq 30^3$, $c < d \leq 30^4$. Triples appear relatively often, the smallest one is $\{1, 2, 27\}$ (this set is under consideration in the present paper), the last one is $\{30, 624, 10101\}$. But we found even a quadruple, namely $\{14, 31, 135, 3510\}$. The quadruples seem to have a sparser occurrence than in case of the classical diophantine quadruples. Because of the similarity we would expect that there is a finite upper bound on the cardinality of Triangular Diophantine sets. Probably this bound is 4.

2. System of Pell-like equations

Let $\{1, 2, c\}$ be a Triangular Diophantine triple with an integer $c \geq 3$. First we find the set of possible values of c. Clearly, the conditions

$$c + 1 = \frac{x(x+1)}{2},$$
$$2c + 1 = \frac{y(y+1)}{2}$$

hold for some integers $x \geq 3$ and $y \geq 5$. The equations above are equivalent to

$$8c + 9 = X_c^2,$$
$$16c + 9 = Y_c^2,$$

where $X_c = 2x + 1 \geq 7$ and $Y_c = 2y + 1 \geq 11$ are odd integers. Combining them leads to the Pell-like equation

$$Y_c^2 - 2X_c^2 = -9. \tag{2.1}$$

The solutions of (2.1) provide the possible values of c as

$$c = c_k = \frac{9}{64}\left((3 + 2\sqrt{2})(17 + 12\sqrt{2})^k + (3 - 2\sqrt{2})(17 - 12\sqrt{2})^k - 6\right), \tag{2.2}$$

where $k \geq 1$ is an integer. The first few terms are $c_k = 0, 27, 945, 32130\ldots.$ It is easy to see that c_k is divisible by 9, which allows us to define $C = C_k = c_k/9$. This sequence begins with $C_k = 0, 3, 105, 10710, \ldots.$

Assume that c is the smallest integer such that $\{1, 2, c, d\}$ is a Triangular Diophantine quadruple. Obviously $d > c$ is a term of the sequence (2.2). Hence there exist positive odd integers X, Y, and Z such that

$$8d + 9 = X^2,$$
$$16d + 9 = Y^2,$$
$$8Cd + 1 = Z^2,$$

where the last equation is the consequence of $8 \cdot 9Cd + 9 = (3Z)^2$. Eliminating d, we obtain the system of simultaneous Pell-like equations

$$Z^2 - CX^2 = 1 - 9C, \tag{2.3}$$

$$2Z^2 - CY^2 = 2 - 9C. \tag{2.4}$$

First we handle the smallest case, when $C = 3$ (which is derived from $c = 27$).

Lemma 2.1. *The Triangular Diophantine triple* $\{1, 2, 27\}$ *cannot be extended to a quadruple.*

Proof. It is sufficient to show that the system (2.3)-(2.4) has no solution for $C = 3$, except the trivial one $X = Y = 3$, $Z = 1$, which leads to the extension with $d = 0$. A user friendly procedure for solving automatically simultaneous Pell-like equations with relatively small coefficients is detailed in [7]. Since the algorithm of [7] was implemented in Magma (thanks to A. Bérczes), we only ran the file, and it resulted no solutions. □

An important consequence of Lemma 2.1 is $c \geq 945$, $d \geq 32130$, and $C \geq 105$. In the sequel, we assume these conditions.

From the theory of Pell and Pell-like equations we know how the (positive) solutions to the individual equation (2.3), and (2.4) are generated, knowing the fundamental solutions of the equations $Z^2 - CX^2 = 1$ and $Z^2 - 2CY^2 = 1$. The fundamental solutions of these associated Pell equations are respectively given by $(T, 4)$ and $(S, 2)$, where S and T are positive integers satisfying $8C + 1 = S^2$ and $16C + 1 = T^2$. To prove that these are really fundamental, i.e. smallest positive solutions, we have to see that $1 + C$, $1 + 2C$, $1 + 4C$, or $1 + 9C$ cannot be square (except in the special cases $C = 0$ and $C = 3$.)

For the proof we use the fact that $8C + 1$ and $16C + 1$ are squares. Then, for instance, if $C + 1$ is also a square, we get that

$$(8C + 1)(16C + 1)(C + 1) = 128C^3 + 152C^2 + 25C + 1$$

should also be a square. Multiplying this by 128^2 and putting $C = x/128$, it yields the elliptic curve

$$y^2 = x^3 + 152x^2 + 3200x + 16384.$$

Using Magma we find the following positive integral solutions in x:

$$12, 40, 112, 208, 384, 952, 992, 3372, 2661328.$$

The only one divisible by 128 is $x = 384$, which gives $C = 3$, a contradiction.

A similar treatment for $(8C+1)(16+1)(2C+1)$, $(8C+1)(16+1)(4C+1)$, and $(8C+1)(16+1)(9C+1)$, respectively shows that the fundamental solutions are really $(T, 4)$, and $(S, 2)$.

Hence, we conclude that the (positive) solutions of (2.3) and (2.4) are given by

$$Z + X\sqrt{C} = (Z_0 + X_0\sqrt{C})(T + 4\sqrt{C})^m, \ m \geq 0,$$
$$Z\sqrt{2} + Y\sqrt{C} = (Z_1\sqrt{2} + Y_1\sqrt{C})(S + 2\sqrt{2C})^n, \ n \geq 0$$

with suitable integers Z_0, X_0, Z_1, Y_1. Thus, we have $Z = V_m$,

$$V_0 = Z_0, \ V_1 = TZ_0 + 4CX_0, \ V_{m+2} = 2TV_{m+1} - V_m,$$

where (Z_0, X_0) is a fundamental solution of (2.3). Similarly, the positive solutions of (2.4) are described by $Z = W_n$,

$$W_0 = Z_1,\, W_1 = SZ_1 + 2CY_1,\, W_{n+2} = 2SW_{n+1} - W_n,$$

where (Z_1, Y_1) is a fundamental solution of (2.4). By [5, Theorem 108a] we have the estimates

$$\begin{aligned} 0 < X_0^2 &\le \frac{8(9C-1)}{T-1}, \\ Z_0^2 &\le \frac{(T-1)(9C-1)}{2}, \\ 0 < Y_1^2 &\le \frac{4(9C-2)}{S-1}, \\ Z_1^2 &\le \frac{(S-1)(9C-2)}{4} \end{aligned}$$

for the fundamental solutions of (2.3) and (2.4). Furthermore, $C \ge 105$ entails

$$\begin{aligned} 0 < X_0 &< 4.29471\sqrt[4]{C}, \\ |Z_0| &< \sqrt{18} \cdot \sqrt{C\sqrt{C}} < 1.3254C, \\ 0 < Y_1 &< 3.63\sqrt[4]{C}, \\ |Z_1| &< 2.52269 \cdot \sqrt{C\sqrt{C}} < 0.7881C. \end{aligned}$$

So we need to solve finitely many Diophantine equations of the form $V_m = W_n$. Of course, we are interested in only the solutions providing extension of a triple to a quadruple. In particular, X, Y and Z should be odd, which implies that Z_0, X_0, Z_1, and Y_1 all are odd. Finally, in this section, we establish the congruences

$$\begin{aligned} V_{2m} &\equiv V_0 \pmod{8C}, \\ V_{2m+1} &\equiv V_1 \pmod{4C}, \\ W_{2n} &\equiv W_0 \pmod{4C}, \\ W_{2n+1} &\equiv W_1 \pmod{4C}, \\ V_m^2 &\equiv Z_0^2 \pmod{8C}, \\ W_{2n}^2 &\equiv W_0^2 \equiv Z_1^2 \pmod{8C}, \\ W_{2n+1}^2 &\equiv W_1^2 \pmod{8C}. \end{aligned}$$

These are easy to prove, here we present only the proof of the first congruence as follows. We will use the fact that if a binary recurrence relation $G_s = AG_{s-1} + BG_{s-2}$ holds for $s \ge 2$, then for every second term we have

$$G_s = (A^2 + 2B)G_{s-2} - B^2 G_{s-4}, \quad s \ge 4.$$

In our case, $V_m = (64C+2)V_{m-2} - V_{m-4}$ follows. Since $V_0 = Z_0$, and

$$V_2 = 2T(TZ_0 + 4CX_0) - Z_0 = (32C+1)Z_0 + 8CTX_0 \equiv Z_0 \pmod{8C},$$

we see

$$V_{2m} \equiv (64C+2)Z_0 - Z_0 \equiv Z_0 = V_0 \pmod{8C}.$$

3. Finding fundamental solutions

In this section, we find all possible values for fundamental solutions (Z_0, X_0), and (Z_1, Y_1) that yield possible extensions of the Triangular Diophantine triple $\{1,2,c\}$ to a quadruple. Recall that $C \geq 105$.

1° Case $V_{2m} = W_{2n}$.

We have $Z_0 \equiv Z_1 \pmod{4C}$. From the estimates for $|Z_0|$ and $|Z_1|$, we can conclude $Z_0 = Z_1$. Moreover,

$$V_{2m}^2 \equiv W_{2n}^2 \equiv Z_0^2 \pmod{8C},$$

together with $8Cd + 1 = Z^2$ implies $Z_0^2 \equiv 1 \pmod{8C}$. Hence there exists an integer d_0 such that

$$d_0 = \frac{Z_0^2 - 1}{8C}.$$

The upper bound on $|Z_0|$ implies $d_0 < c$. Furthermore, we have

$$\begin{aligned} 8d_0 + 9 &= X_0^2, \\ 16d_0 + 9 &= Y_1^2, \\ 8Cd_0 + 1 &= Z_0^2. \end{aligned}$$

It means that $\{a, b, d_0, c\}$ is a Triangular Diophantine quadruple. But by the minimality of c we conclude $d_0 = 0$. Thus $Z_0 = Z_1 = \pm 1$.

2° Case $V_{2m+1} = W_{2n}$.

Now $Z_1 \equiv TZ_0 \pmod{4C}$. If $Z_0 = \pm 1$, then from $4C - T > 3.6C$ and from the bound on $|Z_1|$ we obtain $Z_1 = \pm T$. But then Y_1 is not an integer. Thus $|Z_0| > 1$, which implies $X_0 > 3$. If $X_0 = 5$, then we get $Z_0 = \pm T$, which implies $Z_1 = \pm 1$ and $Y_1 = 3$.

Assume $X_0 \geq 7$. Then we have

$$\begin{aligned} 0 < 4CX_0 - T|Z_0| &= \frac{16C^2X_0^2 - T^2Z_0^2}{4CX_0 + T|Z_0|} = \frac{16C(9C-1) - Z_0^2}{4CX_0 + T|Z_0|} < \frac{144C^2}{53.2982C} \\ &< 2.7018C. \end{aligned}$$

First, if $Z_0 > 0$, then

$$-2.7018C < TZ_0 - 4CX_0 < 0,$$

together with the estimates for $|Z_1|$, gives $Z_1 = TZ_0 - 4CX_0$. Hence we have $Z^2 = W_{2n}^2 \equiv Z_1^2 \pmod{8C}$, which means (via $8Cd + 1 = Z^2$) again $Z_1^2 \equiv 1 \pmod{8C}$, i.e. there exists an integer d_0 with

$$d_0 = \frac{Z_1^2 - 1}{8C}.$$

So $d_0 < c$, and

$$\begin{aligned} 8d_0 + 9 &= \frac{Z_1^2 - 1}{C} + 9 = \frac{(TZ_0 - 4CX_0)^2 - 1 + 9C}{C} = (4Z_0 - TX_0)^2, \\ 16d_0 + 9 &= \frac{2Z_1^2 - 2 + 9C}{C} = Y_1^2, \\ 8Cd_0 + 1 &= Z_1^2. \end{aligned}$$

Consequently, $\{1, 2, d_0, c\}$ is a Triangular Diophantine quadruple. Again $d_0 = 0$ follows, and then $Z_1 = -1$.

Secondly, if $Z_0 < 0$, in the same way, from

$$0 < TZ_0 + 4CX_0 < 2.7018C,$$

we conclude $Z_1 = TZ_0 + 4CX_0$. Defining

$$d_0 = \frac{Z_1^2 - 1}{8C} < c,$$

we observe

$$\begin{aligned} 8d_0 + 9c &= (4Z_0 + TX_0)^2, \\ 16d_0 + 9 &= Y_1^2, \\ 8Cd_0 + 1 &= Z_1^2, \end{aligned}$$

i.e. $\{1, 2, d_0, c\}$ is quadruple, which again means $d_0 = 0$ and $Z_1 = 1$.

In conclusion, $Z_1 = \pm 1$. Multiplying $Z_1 \equiv TZ_0 \pmod{4C}$ by T leads to $Z_0 \equiv \pm T \pmod{4C}$. Consequently, (by the estimate of $|Z_0|$) $Z_0 = \pm T$. Then, $X_0 = 5$, which contradicts $X_0 \geq 7$.

Hence, in the case $V_{2m+1} = W_{2n}$ the only possible fundamental solutions are $Z_0 = \pm T$, $X_0 = 5$, $Z_1 = \pm 1$, $Y_1 = 3$.

3° Case $V_{2m} = W_{2n+1}$.

It is analogous to case 2°. Here we have

$$Z_0 \equiv SZ_1 \pmod{2C} \quad \text{and} \quad Z_0 \equiv SZ_1 \pm 2CY_1 \pmod{4C}.$$

If $Z_1 = \pm 1$, then by $2C - S > 1.7238C$, and by the bound on $|Z_1|$ we get $Z_0 = \pm S$. But then X_0 is not an integer. Hence, $|Z_1| > 1$, which yields $Y_1 > 3$. If $Y_1 = 5$, then $Z_1 = \pm S$, and then $Z_0 = \pm 1$ and $X_0 = 3$.

Suppose now $Y_1 \geq 7$. Clearly,

$$0 < 2CY_1 - S|Z_1| = \frac{4C^2Y_1^2 - S^2Z_1^2}{2CY_1 + S|Z_1|} = \frac{4C(9C-2) - Z_1^2}{2CY_1 + S|Z_1|} < \frac{36C^2}{26.6491C} < 1.3509C.$$

First, if $Z_1 > 0$, then

$$-1.3509C < SZ_1 - 2CY_1 < 0,$$

together with the estimate for $|Z_0|$ (and also from congruence modulo $4C$) shows $Z_0 = SZ_1 - 2CY_1$. A solution implies $Z^2 = V_{2m}^2 \equiv Z_0^2 \pmod{8C}$, and then (again by $8Cd + 1 = Z^2$) $Z_0^2 \equiv 1 \pmod{8C}$. So there exists an integer

$$d_0 = \frac{Z_0^2 - 1}{8C}.$$

We have $d_0 < c$, and then

$$\begin{aligned}
8d_0 + 9 &= \frac{Z_0^2 - 1}{C} + 9 = \frac{Z_0^2 - 1 + 9C}{C} = X_0^2,\\
16d_0 + 9 &= \frac{2(SZ_1 - 2CY_1)^2 - 2 + 9C}{C} = (4Z_1 - SY_1)^2,\\
8Cd_0 + 1 &= Z_0^2.
\end{aligned}$$

Consequently, $\{1, 2, d_0, c\}$ is a Triangular Diophantine quadruple. So $d_0 = 0$ and $Z_0 = -1$ follow from the minimality of c.

If $Z_1 < 0$, then in exactly the same way, by $0 < SZ_1 + 2CY_1 < 1.3509C$, we conclude $Z_0 = SZ_1 + 2CY_1$. Putting

$$d_0 = \frac{Z_0^2 - 1}{8C} < c,$$

it yields

$$\begin{aligned}
8d_0 + 9 &= X_0^2,\\
16d_0 + 9 &= (4Z_1 + SY_1)^2,\\
8Cd_0 + 1 &= Z_0^2.
\end{aligned}$$

Thus $\{1, 2, d_0, c\}$ is quadruple, $d_0 = 0$, and $Z_0 = 1$.

As a summary, we have $Z_0 = \pm 1$. Multiplying $Z_0 \equiv SZ_1 \pmod{2C}$ by S, we get $Z_1 \equiv \pm S \pmod{2C}$, which implies (by the upper bound on $|Z_1|$) $Z_1 = \pm S$. Thus $Y_1 = 5$, a contradiction with $Y_1 \geq 7$.

Hence the only possible fundamental solutions are $Z_1 = \pm S$, $Y_1 = 5$, $Z_0 = \pm 1$, $X_0 = 3$. But it would imply

$$\pm 1 \equiv \pm 1 + 2C \pmod{4C},$$

a contradiction. So there is no solution in the case $V_{2m} = W_{2n+1}$.

4° Case $V_{2m+1} = W_{2n+1}$.

Observe that $TZ_0 \pm 4CX_0 \equiv SZ_1 \pm 2CY_1 \pmod{4C}$. Here we need slightly better estimates than before. First we show that (except in the case $C=3$) we cannot have $X_0=7$ or $Y_1=7$. In order to prove it we have to show that $40C+1$, and $20C+1$ are not square. After the proof of Lemma 2.1 we faced similar problem, so we just refer the changes: the corresponding elliptic equations are $(8C+1)(16C+1)(40C+1)=y^2$ and $(8C+1)(16C+1)(20C+1)=y^2$, respectively.

Now, with the same considerations as before, if $X_0 \geq 9$ and $Y_1 \geq 9$, then we have

$$0 < 4CX_0 - T|Z_0| < 2.0589C,$$
$$0 < 2CY_1 - S|Z_1| < 1.0295C.$$

These estimates, together with

$$TZ_0 \pm 4CX_0 \equiv SZ_1 \pm 2CY_1 \pmod{4C}$$

imply the following four possibilities:

- $TZ_0 - 4CX_0 = SZ_1 - 2CY_1$ if $Z_0,\ Z_1 > 0$,
- $TZ_0 + 4CX_0 = SZ_1 + 2CY_1$ if $Z_0,\ Z_1 < 0$,
- $TZ_0 - 4CX_0 = SZ_1 + 2CY_1$ if $Z_0 > 0,\ Z_1 < 0$,
- $TZ_0 + 4CX_0 = SZ_1 - 2CY_1$ if $Z_0 < 0,\ Z_1 < 0$.

In the first case, we define

$$Z' = TZ_0 - 4CX_0 = SZ_1 - 2CY_1.$$

From $Z = V_{2m+1}$, $8Cd+1 = Z^2$ and $Z'^2 \equiv V_1^2 \equiv V_{2m+1}^2 \pmod{8C}$ we know that there exists an integer d_0 such that

$$d_0 = \frac{Z'^2 - 1}{8C} < c.$$

By

$$\begin{aligned} 8d_0 + 9 &= \frac{(TZ_0 - 4CX_0)^2 - 1 + 9C}{C} = (4Z_0 - TX_0)^2, \\ 16d_0 + 9 &= \frac{2(SZ_1 - 2CY_1)^2 - 2 + 9C}{C} = (4Z_1 - SY_1)^2, \\ 8Cd_0 + 1 &= Z'^2 \end{aligned}$$

the Diophantine quadruple $\{1,2,d_0,c\}$ is Triangular, and the minimality of c implies $d_0 = 0$ and $Z' = \pm 1$. But this is not possible. Indeed $C \geq 105$, and

$$16C(9C-1) - Z_0^2 > 142C^2,$$
$$4CX_0 + T|Z_0| < 35.5C\sqrt[4]{C}$$

imply $4CX_0 - T|Z_0| > 4\sqrt{C\sqrt{C}}$.

The other three cases can be investigated in exactly the same way. Hence, we are left with possibilities $X_0 = 3$, $X_0 = 5$, $Y_1 = 3$ and $Y_1 = 5$. However, as we will show below, these are not possible .

Assume $X_0 = 3$, $Z_0 = \pm 1$, $Y_1 = 3$ and $Z_1 = \pm 1$, with four options for the signs. But

$$\pm T + 12C \equiv \pm S + 6C \pmod{4C},$$
$$\pm T \equiv \pm S + 2C \pmod{4C}$$

leads to a contradiction since $T + S < 0.7C$ is valid for $C \geq 105$.

Suppose now $X_0 = 3$, $Z_0 = \pm 1$, $Y_1 = 5$ and $Z_1 = \pm S$ (again four variations with signs). The congruences

$$\pm T + 12C \equiv \pm 1 + 10C \pmod{4C},$$
$$\pm T \equiv \pm 1 + 2C \pmod{4C},$$

are contradictory because $T + 1 \leq 0.5C$ holds for $C \geq 105$.

Similarly, if $X_0 = 5$, $Z_0 = \pm T$, $Y_1 = 3$ and $Z_1 = \pm 1$, then

$$\pm 1 + 20C \equiv \pm S + 6C \pmod{4C},$$
$$\pm 1 \equiv \pm S + 2C \pmod{4C},$$

a contradiction via $S + 1 < 0.3C$ for $C \geq 105$.

Finally, if $X_0 = 5$, $Z_0 = \pm T$, $Y_1 = 5$ and $Z_1 = \pm S$, we have

$$\pm 1 + 20C \equiv \pm 1 + 10C \pmod{4C},$$
$$\pm 1 \equiv \pm 1 + 2C \pmod{4C},$$

which is obviously not possible. Hence, we conclude that we have no solution if $V_{2m+1} = W_{2n+1}$.

The main result of this section is recorded in

LEMMA 3.1. *The cases providing possible extensions of the Triangular Diophantine triple* $\{1, 2, c\}$ *to a quadruple are*

- $V_{2m} = W_{2n}$, $Z_0 = Z_1 = \pm 1$, $X_0 = Y_1 = 3$;
- $V_{2m+1} = W_{2n}$, $Z_0 = \pm T$, $X_0 = 5$, $Z_1 = \pm 1$, $Y_1 = 3$.

4. LINEAR FORMS IN LOGARITHMS

As far as possible we handle the two cases of Lemma 3.1 together. Assume that $V_m = W_n$ holds with $m, n \neq 0$. From the recurrence relation $V_{m+2} = 2TV_{m+1} - V_m$ and from the initial values V_0 and V_1 we conclude

$$V_m = \frac{1}{2}((Z_0 + X_0\sqrt{C})(T + 4\sqrt{C})^m + (Z_0 - X_0\sqrt{C})(T - 4\sqrt{C})^m),$$

and similarly

$$W_n = \frac{1}{4}((2Z_1 + Y_1\sqrt{2C})(S + 2\sqrt{2C})^n + (2Z_1 - Y_1\sqrt{2C})(S - 2\sqrt{2C})^n).$$

Now the equality $V_m = W_n$ implies

$$(Z_0 + X_0\sqrt{C})(T+4\sqrt{C})^m + (Z_0 - X_0\sqrt{C})(T-4\sqrt{C})^m = \left(Z_1 + \frac{Y_1\sqrt{2C}}{2}\right)(S+2\sqrt{2C})^n + \left(Z_1 - \frac{Y_1\sqrt{2C}}{2}\right)(S-2\sqrt{2C})^n.$$

Defining

$$P = (Z_0 + X_0\sqrt{C})(T+4\sqrt{C})^m \quad \text{and} \quad Q = \left(Z_1 + \frac{Y_1\sqrt{2C}}{2}\right)(S+2\sqrt{2C})^n$$

the equation $V_m = W_n$ implies

$$P - (9C-1)P^{-1} = Q - \left(\frac{9C}{2} - 1\right)Q^{-1}.$$

Clearly, we have $P, Q > 1$, and

$$P - Q = (9C-1)P^{-1} - \left(\frac{9C}{2} - 1\right)Q^{-1} > (9C-1)(P^{-1} - Q^{-1}) = (9C-1)(Q-P)P^{-1}Q^{-1},$$

which proves $P > Q$. Furthermore,

$$P - Q < (9C-1)P^{-1},$$

and then

$$\frac{P-Q}{P} < (9C-1)P^{-2}$$

follows. From $m \geq 1$ and $C \geq 105$ we get

$$P \geq (5\sqrt{C} - T)(T + 4\sqrt{C}) > 0.9988\sqrt{C} \cdot 8\sqrt{C} > 7.99C,$$

consequently

$$P^{-1} < \frac{1}{7.99C}.$$

Hence,

$$(9C-1)P^{-2} < \frac{9C-1}{(7.99C)^2} < 0.0014.$$

Now

$$\begin{aligned}
0 < \log\frac{P}{Q} &= -\log\left(1 - \frac{P-Q}{P}\right) < (9C-1)P^{-2} + (9C-1)^2P^{-4} \\
&= (1 + (9C-1)P^{-2})(9C-1)P^{-2} < 1.0014(9C-1)P^{-2} \\
&= 1.0014(9C-1)\cdot\frac{(T+4\sqrt{C})^{-2m}}{(Z_0 + X_0\sqrt{C})^2} < \frac{1.0014(9C-1)}{(0.9988\sqrt{C})^2}(T+4\sqrt{C})^{-2m} \\
&< 9.035(T+4\sqrt{C})^{-2m}.
\end{aligned}$$

Thus we have proved the following lemma.

Lemma 4.1. *Let $V_m = W_n$, with $m, n \neq 0$. Then*

$$0 < m\log(T+4\sqrt{C}) - n\log(S+2\sqrt{2C}) + \log\frac{Z_0 + X_0\sqrt{C}}{Z_1 + 0.5Y_1\sqrt{2C}}$$
$$< 9.035(T+4\sqrt{C})^{-2m}.$$

5. Connection between indices and their lower bound

The next step is to find connection between indices m and n if $V_m = W_n$, $m, n \neq 0$. We will see that the indices are not far from each other. We have

$$V_m \geq (2T-1)^{m-1}V_1 \quad \text{and} \quad W_n < (2S)^{n-1}W_1.$$

Considering the possible cases, if $Z_0 = Z_1 = \pm 1$, $X_0 = Y_1 = 3$, then obviously $m < n$ follows from

$$(2T-1)^{m-1}(12C \pm T) < (2S)^{n-1}(6C \pm S).$$

In case of $Z_0 = \pm T$, $X_0 = 5$, $Z_1 = \pm 1$, $Y_1 = 3$ we see

$$(2T-1)^{m-1}(20C - 16C - 1) < (2S)^{n-1}(6C + S),$$

i.e. for $C \geq 105$ we conclude

$$3.99(2T-1)^{m-1} < 6.28(2S)^{n-1}.$$

The last inequality appears in the case $V_{2m+1} = W_{2n}$, i.e. we have

$$3.99(2T-1)^{2m} < 6.28(2S)^{2n-1},$$

which is obviously impossible if $2m > 2n - 1$. Thus, $V_{2m+1} = W_{2n}$ yields $2m + 1 < 2n$.

Hence $V_m = W_n$ always implies $n > m$. To get upper bound on n in terms of m we use the inequality of linear form in logarithms. Consider the "worst" case providing

$$n\log(S+2\sqrt{2C}) < m\log(T+4\sqrt{C}) + \log\frac{Z_0 + X_0\sqrt{C}}{Z_1 + 0.5Y_1\sqrt{2C}}$$
$$< m\log(T+4\sqrt{C}) + \log\frac{5\sqrt{C}+T}{1+1.5\sqrt{2C}} < m\log(T+4\sqrt{C}) + 1.44532.$$

Thus

$$n < m\cdot\frac{\log(T+4\sqrt{C})}{\log(S+2\sqrt{2C})} + \frac{1.44532}{\log(S+2\sqrt{2C})} < m\cdot\frac{\log(8.0012\sqrt{C})}{\log(4\sqrt{2}\cdot\sqrt{C})} + 0.356004$$
$$< 1.085404m + 0.356004 < 1.44108m.$$

Summarizing the estimates, we have proved the following statement.

Lemma 5.1. *Let $V_m = W_n$ with $m, n \neq 0$. Then $m < n < 1.44108m$.*

The useful congruence statements

$$\begin{aligned} V_{2m} &\equiv (32m^2C+1)Z_0 + 8mTCX_0 \pmod{64C^2},\\ V_{2m+1} &\equiv (32m(m+1)C+1)TZ_0 + 4(2m+1)CX_0 \pmod{64C^2},\\ W_{2n} &\equiv (16n^2C+1)Z_1 + 4nSCY_1 \pmod{64C^2},\\ W_{2n+1} &\equiv (16n(n+1)C+1)SZ_1 + 2(2n+1)CY_1 \pmod{64C^2} \end{aligned}$$

can be proved easily by induction.

If $V_{2m} = W_{2n}$, then we get

$$\pm(32m^2C+1) + 24TCm \equiv \pm(16n^2C+1) + 12SCn \pmod{64C^2}.$$

Divide it by $4C$ to find

$$\pm 8m^2 + 6mT \equiv \pm 4n^2 + 3nS \pmod{16C}. \tag{5.1}$$

If we assume $m \le 0.5\sqrt{C}$, then we see that this congruence actually becomes equation. Indeed, we have

$$\begin{aligned} &8m^2 < 8\cdot 0.25C = 2C,\\ &6mT < 6\cdot 0.5\sqrt{C}\cdot\sqrt{16C+1} < 3\sqrt{C}\cdot 4.0012\sqrt{C} = 12.0036C,\\ &4n^2 < 4\cdot(1.44108m)^2 <^8 .307m^2 < 2.08C,\\ &3nS < 6mT < 12.0036C. \end{aligned}$$

We can also see that both sides of (5.1) are positive. Indeed,

$$\begin{aligned} 8m^2 &< 8m\cdot 0.5\sqrt{C} = 4m\sqrt{C} < mT < 6mT,\\ 4n^2 &< 4n\cdot 1.44108m < 5.77n\cdot 0.5\sqrt{C} < 3nS. \end{aligned}$$

Hence, we arrived at

$$\pm 8m^2 + 6mT = \pm 4n^2 + 3nS.$$

The left hand side is obviously larger in case of the '+' sign. If we consider '−' case, then

$$6mT - 8m^2 = (6T-8m)m > 20m\sqrt{C}.$$

On the other hand

$$3nS - 4n^2 < 3nS < 8.4904n\sqrt{C}$$

holds. The last two inequalities imply $n > 2.355m$, a contradiction. Consequently, we may assume $m > 0.5\sqrt{C}$, and in our case ($V_{2m} = W_{2n}$) it becomes $2m > \sqrt{C}$.

Let us now consider the second case $V_{2m+1} = W_{2n}$. Then the corresponding congruences modulo $64C^2$ give us

$$\pm(32m(m+1)+1)T^2 + 20(2m+1)C \equiv \pm(16n^2C+1) + 12nSC \pmod{64C^2}.$$

Using $T^2 = 16C + 1$ we straightforwardly have

$$\pm(32m(m+1)C + 16C) + 20(2m+1)C \equiv \pm 16n^2C + 12nSC \pmod{64C^2},$$

and finally dividing it by $4C$ leads to

$$\pm 4(2m(m+1)+1) + 5(2m+1) \equiv \pm 4n^2 + 3nS \pmod{16C}.$$

Now, if we assume $n \le 0.79\sqrt{C}$, the last congruence becomes

$$\pm 4(2m(m+1)+1) + 5(2m+1) = \pm 4n^2 + 3nS.$$

Then the assumption $n \le 0.79\sqrt{C}$ would imply that the right hand side is always larger, so we derive $n > 0.79\sqrt{C}$, and then

$$1.44108(2m+1) > 2n > 2 \cdot 0.79\sqrt{C} = 1.58\sqrt{C},$$

which shows $2m + 1 > \sqrt{C}$. Thus we obtain the following conclusion.

LEMMA 5.2. *Let $V_m = W_n$, $m, n \neq 0$. Then $m > \sqrt{C}$.*

6. Proof of Theorem 1.3

For the linear form in logarithms of Lemma 4.1 we will use the following theorem of Baker-Wűstholz [1]:

THEOREM 6.1. *Let $\Lambda \neq 0$ be linear form in logarithms of ℓ algebraic numbers $\alpha_1, \dots, \alpha_\ell$ with rational integer coefficients $b_1, \dots, b_\ell$. Then*

$$\log \Lambda \ge -18(\ell+1)!\ell^{\ell+1}(32d)^{\ell+2}h'(\alpha_1)\cdots h'(\alpha_\ell)\log(2\ell d)\log B,$$

where $B = \max\{|b_1|, \dots, |b_\ell|\}$, d is the degree of the number field generated by $\alpha_1, \dots, \alpha_\ell$. Here

$$h'(\alpha) = \max\left\{h(\alpha), \frac{|\log \alpha|}{d}, \frac{1}{d}\right\}$$

and $h(\alpha)$ denotes the standard Weil logarithmic height of α.

We apply this theorem to the linear form from Lemma 4.1 with $\ell = 3$, $d = 4$, and $B = n$, moreover with

$$\alpha_1 = T + 4\sqrt{C}, \quad \alpha_2 = S + 2\sqrt{2C}, \quad \alpha_3 = \frac{Z_0 + X_0\sqrt{C}}{Z_1 + 0.5Y_1\sqrt{2C}}.$$

By the condition $C \ge 105$ we have

$$h'(\alpha_1) = \frac{1}{2}\log(T + 4\sqrt{C}) < 0.4735 \log C,$$

$$h'(\alpha_2) = \frac{1}{2}\log(S + 2\sqrt{2C}) < 0.4363 \log C.$$

Obviously, all conjugates of α_3 are $\frac{Z_0 \pm X_0\sqrt{C}}{Z_1 \pm 0.5Y_1\sqrt{2C}}$.

Let $\varepsilon, \delta \in \{1, -1\}$. Thus the minimal polynomial of α_3 over $\mathbb{Q}$ is

$$\prod_{\varepsilon,\delta} \left(X - \frac{Z_0 + \varepsilon \cdot X_0\sqrt{C}}{Z_1 + \delta \cdot 0.5Y_1\sqrt{2C}} \right).$$

So when computing $h'(\alpha_3)$ we have

$$h'(\alpha_3) = \frac{1}{4} \log \left(a_d \prod \max\{\alpha', 1\} \right),$$

where a_d is the leading coefficient of the minimal polynomial in $\mathbb{Z}[X]$ and the product goes over all conjugates of α_3. It is enough to get an upper bound on $h'(\alpha_3)$ without computing it exactly. Notice that, among the conjugates of α_3, $\frac{5\sqrt{C}-T}{1.5\sqrt{2C}\pm 1}$ are less than 1. It means that we have

$$h'(\alpha_3) \leq \frac{1}{4} \log \left((9C-2)^2 \cdot \frac{2(T+5\sqrt{C})^2}{(9C-2)} \right) < 0.89133 \log C.$$

From Lemma 4.1 we also conclude

$$\log(9.035(T+4\sqrt{C})^{-2m}) < \log(T+4\sqrt{C})^{-2m+1} < 0.9469(-2m+1)\log C,$$

which together with Baker-Wűstholz theorem implies

$$0.9469(2m-1)\log C < 3.822 \cdot 10^{15} \cdot 0.4735 \cdot 0.4363 \cdot 0.89133 \cdot (\log^3 C) \cdot \log n.$$

Using $n < 1.44108m$ we conclude

$$2m - 1 < 7.4325 \cdot 10^{14} \log(1.44108m) \log^2(m^2),$$

where we have used the bound $C < m^2$. It finally yields

$$m < 1.5024 \cdot 10^{20},$$

and then $C < 2.2573 \cdot 10^{40}$ and $k \leq 27$. Hence, we have just proved the following.

Proposition 6.2. *Let $k > 27$ be an integer. Then the Triangular Diophantine triple of the form $\{1, 2, c_k\}$ cannot be extended to a quadruple with a larger element.*

To finish the proof of Theorem 1.3, we applied a version of Baker-Davenport reduction (see [2, Lemma 5]) to the linear form from Lemma 4.1, which is already well known method in proving similar results. We have implemented it in GP-Pari, and in all cases, after one step of reduction, we got $m \leq 5$ which is small enough to get a contradiction (and then no extension to a quadruple except the trivial one with $d = 0$), because we have already proved that $m \geq \sqrt{C} \geq \sqrt{105} > 10$ if $m, n \neq 0$.

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Trokutasti Diofantovi skupovi koji sadrže par $\{1, 2\}$

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Sažetak. U ovom řadu smo dokazali kako ne postoji četveročlani skup prirodnih brojeva $\{1, 2, c, d\}$ tako da je produkt bilo koja dva njegova elementa uvećan za 1 jednak nekom trokutastom broju.

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