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BOUNDS FOR CONFLUENT HORN FUNCTION Φ_2 DEDUCED BY MCKAY I_ν BESSEL LAW

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Abstract. The main aim of this article is to derive by probabilistic method new functional and uniform bounds for Horn confluent hypergeometric Φ_2 of two variables and the incomplete Lipschitz–Hankel integral, among others. The main mathematical tools are the representation theorems for the McKay I_ν Bessel probability distribution's cumulative distribution function (CDF) and certain known and less known properties of CDF.

1. Introduction and motivation

The first results about probability distributions involving Bessel functions can be traced back to the early work of McKay [14] in 1932 who considered two classes of continuous distributions called Bessel function distributions. In contemporary fashion speaking the random variable (rv) ξ defined on a standard probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is distributed according to McNolty's version of the McKay I_ν Bessel law means that the related probability density function (PDF) reads [16, p. 496, Eq. (13)]

$$f_I(x; a, b; \nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}\, \mathrm{e}^{-bx} x^\nu I_\nu(ax), \qquad x \geq 0,$$

where $\nu > -1/2$ and $b > a > 0$. Here the modified Bessel function of the first kind of order ν [18, p. 249, Eq. 10.25.2]

$$I_\nu(x) = \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)\, n!} \left(\frac{x}{2}\right)^{2n+\nu}, \qquad \Re(\nu) > -1,\, x \in \mathbb{C}.$$

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The related cumulative distribution function (CDF) is

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^x \mathrm{e}^{-bt} t^\nu I_\nu(at)\,\mathrm{d}t, \qquad x \geq 0. \tag{1.1}$$

The properties of this distribution including its characteristic functions, estimation issues and extensions have been subjects of several investigations [15–17, 20]. The applications of such PDF concerns also the strong activity in the electrical and electronic engineering literature that requires the study of further generalizations of such PDFs, see [4, 10, 17] and the related references therein.

In this note we select another point of view to the rv having McKay I_ν Bessel distribution, considering the special functions which consist its PDF and CDF and using properties of probability distributions we establish several functional and uniform bounds upon the involved special functions, such as Horn's confluent hypergeometric Φ_2 of two variables, consult [13]. Also the CDF can be expressed in terms the incomplete Lipschitz–Hankel integral (ILHI), [10]. These CDF formulae results occur in (1.1) and (2.2). Finally, in the discussion section we give critical observations of some results coming from the cited references which are erroneous, e.g. [19]. We correct them and obtain further bounding inequalities for the confluent Horn Φ_3 function and the Marcum Q function, in terms of exponential and/or the familiar modified Bessel I_0.

2. Bounding inequalities involving Φ_2

Recently, Jankov Maširević and Pogány [13] presented a computational series formula for the CDF of McKay I_ν Bessel distribution in terms of the Horn confluent hypergeometric function [21, p. 25, Eq. (17)]

$$\Phi_2(b, b'; c; x, y) = \sum_{m,n\geq 0} \frac{(b)_m (b')_n}{(c)_{m+n}} \frac{x^m\, y^n}{m!\, n!}, \qquad \max\big(|x|, |y|\big) < \infty\,. \tag{2.1}$$

In [13, p. 149, Theorem 3] we prove that for all $x \geq 0$, when $\nu > -1/2$ and $b > a > 0$ there holds

$$\begin{aligned} F_I(x;a,b;\nu) &= \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+2)}\, x^{2\nu+1} \\ &\quad \cdot \Phi_2\big(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; 2\nu+2; (a-b)x, -(a+b)x\big). \end{aligned} \tag{2.2}$$

The relation (2.2) implies the functional upper bound for the Horn confluent hypergeometric function of two variables Φ_2.

Proposition 2.1. *For all $b > a > 0$; $\nu > -1/2$ and for all $x \geq 0$ there holds*

$$x^{2\nu+1}\, \Phi_2\big(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; 2\nu+2; x(a-b), -x(b+a)\big) \leq \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+1/2}}\,.$$

Different approach in treating CDF is enabled by the following result.

LEMMA 2.2. [3, p. 45, 2.1.7] *Let* $F(x)$ *be a* CDF *and* $h > 0$. *Then*

$$H_1(x) = \frac{1}{h} \int_x^{x+h} F(t)\,\mathrm{d}t\,, \tag{2.3}$$

and

$$H_2(x) = \frac{1}{2h} \int_{x-h}^{x+h} F(t)\,\mathrm{d}t\,,$$

are also CDF*s.*

THEOREM 2.3. *For all* $x \ge 0, h > 0$ *and* $\nu > -\frac{1}{2}$, $b > a > 0$ *there holds*

$$\frac{\Phi_2\big(\nu+\frac12, \nu+\frac12; 2\nu+2; (a-b)x, -(a+b)x\big)}{\Phi_2\big(\nu+\frac12, \nu+\frac12; 2\nu+2; (a-b)(x+h), -(a+b)(x+h)\big)} \le \Big(1 + \frac{h}{x}\Big)^{2\nu+2}.$$

Moreover, for the same values of parameters and $x \ge 0$ *we have*

$$\frac{\Phi_2\big(\nu+\frac12, \nu+\frac12; 2\nu+2; (a-b)(x-h), -(a+b)(x-h)\big)}{\Phi_2\big(\nu+\frac12, \nu+\frac12; 2\nu+2; (a-b)(x+h), -(a+b)(x+h)\big)} \le \Big(\frac{x+h}{x-h}\Big)^{2\nu+2}.$$

PROOF. Let $\chi_S(x)$ denote the characteristic function of the set S. Accordingly, the Mellin transform of $\Phi_2(b, b'; c; px, qx)\, \chi_{[\alpha,\beta]}(x)$ on the finite positive interval $[\alpha, \beta]$, $\alpha \ge 0$ turns out to be

$$\begin{aligned}
\mathcal{I}_c(\alpha,\beta) &:= \int_0^\infty t^{c-1}\, \Phi_2(b, b'; c; pt, qt)\, \chi_{[\alpha,\beta]}(t)\,\mathrm{d}t \\
&= \int_\alpha^\beta t^{c-1}\, \Phi_2(b, b'; c; pt, qt)\,\mathrm{d}t \\
&= \frac{1}{c}\Big\{\beta^c\, \Phi_2(b, b'; c+1; p\beta, q\beta) - \alpha^c\, \Phi_2(b, b'; c+1; p\alpha, q\alpha)\Big\}\,,
\end{aligned} \tag{2.4}$$

where both Horn functions converge by virtue of (2.1). Indeed,

$$\begin{aligned}
\mathcal{I}_c(\alpha,\beta) &= \sum_{m,n\ge 0} \frac{(b)_m\,(b')_n\, p^m\, q^n}{(c)_{m+n}\, m!\, n!} \int_\alpha^\beta t^{m+n+c-1}\,\mathrm{d}t \\
&= \sum_{m,n\ge 0} \frac{(b)_m\,(b')_n\, p^m\, q^n}{(c)_{m+n}\, m!\, n!}\, \frac{\beta^{m+n+c} - \alpha^{m+n+c}}{m+n+c} \\
&= \sum_{m,n\ge 0} \frac{(b)_m\,(b')_n\, \Gamma(m+n+c)\, p^m\, q^n}{(c)_{m+n}\, \Gamma(m+n+c+1)\, m!\, n!}\big(\beta^{m+n+c} - \alpha^{m+n+c}\big) \\
&= \frac{1}{c} \sum_{m,n\ge 0} \frac{(b)_m\,(b')_n\, p^m\, q^n}{(c+1)_{m+n}\, m!\, n!}\big(\beta^{m+n+c} - \alpha^{m+n+c}\big) \\
&= \frac{1}{c} \sum_{m,n\ge 0} \Big\{\beta^c \frac{(b)_m\,(b')_n\,(p\beta)^m\,(q\beta)^n}{(c+1)_{m+n}\, m!\, n!} - \alpha^c \frac{(b)_m\,(b')_n\,(p\alpha)^m\,(q\alpha)^n}{(c+1)_{m+n}\, m!\, n!}\Big\},
\end{aligned}$$

which is equivalent to the asserted formula (2.4).

Consider (2.3) with the baseline CDF F_I, which implies, since the (2.2) and (2.4) (in which we specify $\alpha = x$, $\beta = x + h$, $h > 0$; $b = b' = \nu + \frac{1}{2}$; $c = 2\nu + 2$ and $p = a - b$, $q = -(a + b)$), that

$$\begin{aligned} H_1(x) &= \frac{1}{h} \int_x^{x+h} F_I(t)\,\mathrm{d}t = \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+3)\,h}\, \mathfrak{I}_{2\nu+2}(x, x+h) \\ &= \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+3)\,h} \Big\{ (x+h)^{2\nu+2}\, \Phi_2^{[2\nu+3]}(x+h) - x^{2\nu+2} \Phi_2^{[2\nu+3]}(x) \Big\}, \end{aligned}$$

where $\Phi_2^{[\eta]}(x) = \Phi_2\big(\nu+\frac{1}{2}, \nu+\frac{1}{2}; \eta; (a-b)x, -(a+b)x\big)$. As for all $x \geq 0$ the CDF $H_1(x) \in [0,1]$, we have the first statement.

Applying the same property $H_2(x) \geq 0$ as above for $H_1(x)$, the second claim follows. ∎

Another result concerning CDFs for positive rv reads as follows.

Lemma 2.4. [3, p. 45, 2.1.8] *Let $F(x)$ be a* CDF *of a continuous rv with $F(0) = 0$. Then*

$$G(x) = \begin{cases} F(x) - F(x^{-1}), & x \geq 1 \\ 0, & x < 1 \end{cases}$$

is also a CDF*.*

Proof. Consider the probability $\mathsf{P}\big\{ \max\big(\xi, \xi^{-1}\big) < x \big\}$. When $x \geq 1$

$$\mathsf{P}\big\{ \max\big(\xi, \xi^{-1}\big) < x \big\} = \mathsf{P}\big\{ x^{-1} < \xi < x \big\} = F(x) - F(x^{-1}),$$

which completes the proof. ∎

Bearing in mind this result valid for the baseline CDF $F_I(x)$, we conclude the following bounds.

Theorem 2.5. *For all $\nu > -\frac{1}{2}$, $b > a > 0$ and for all $x \geq 1$ we have*

$$\frac{\Phi_2\big(\nu+\frac{1}{2}, \nu+\frac{1}{2}; 2\nu+2; (a-b)/x, -(a+b)/x\big)}{\Phi_2\big(\nu+\frac{1}{2}, \nu+\frac{1}{2}; 2\nu+2; (a-b)x, -(a+b)x\big)} \leq x^{4\nu+2}.$$

Moreover, for the same domain and parameter space there holds true the uniform bound

$$\begin{aligned} & x^{2\nu+1}\, \Phi_2\big(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; 2\nu+2; (a-b)x, -(a+b)x\big) \\ & \qquad - x^{-(2\nu+1)} \Phi_2\Big(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; 2\nu+2; \frac{a-b}{x}, -\frac{a+b}{x}\Big) \leq \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+\frac{1}{2}}}. \end{aligned}$$

The next derivation method depends on the incomplete Lipschitz–Hankel integral built by the modified Bessel functions of the first kind

$$I_{e_{\mu,\nu}}(z; a, b) = \int_0^z \mathrm{e}^{-bt} t^{\mu} I_\nu(at)\,\mathrm{d}t,$$

where $a, b > 0$, the argument and another two parameters $z, \nu, \mu \in \mathbb{C}$ and since convergence reasons should $\Re(\mu+\nu) > -1$ (for more details on ILHI consult for instance [10] and the there listed references). Obviously, the special case $I_{e_{\nu,\nu}}(x; a, b)$ occurs in (1.1) and [10, Corollary 1]

$$F_I(x; a, b; \nu) = \frac{[(b/a)^2-1]^{\nu+\frac{1}{2}}}{2^\nu(\frac{1}{2})_\nu} \, I_{e_{\nu,\nu}}\left(ax; 1, \frac{b}{a}\right), \tag{2.5}$$

where $x \geq 0$, and $b > a > 0$ and $\nu > -1/2$, having in mind that in the Pochhammer notation writing we have

$$\sqrt{\pi}\,(\tfrac{1}{2})_\nu = \Gamma(\nu + \tfrac{1}{2})\,.$$

From (2.5) readily follows the following uniform bound result. For all $b > a > 0$; $\nu > -1/2$ and for all $x \geq 0$ we have

$$I_{e_{\nu,\nu}}\left(ax; 1, \frac{b}{a}\right) \leq \frac{2^\nu(\frac{1}{2})_\nu}{[(b/a)^2-1]^{\nu+\frac{1}{2}}}\,.$$

Remark 2.6. Functional and uniform bounds for ILHI integrals are presented by Gaunt in a series of articles, for different values of parameters a, b, ν. However, we are interested in the case $b > a > 0$, which is opposite to his assumption in the coefficient of the exponential term [7–9].

3. Discussion about related results

In this section we discuss some already known results concerning CDF F_I, in order to point out the existing erroneous places in published works in the elementary case when $\nu = 0$. Namely, for $\nu = 0$, the CDF (1.1) is the special case of (2.5):

$$F_I(x; a, b; 0) = \frac{1}{a}\sqrt{b^2-a^2} \cdot I_{e_{0,0}}\left(ax; \frac{b}{a}\right),$$

where $x \geq 0$, whilst $b > a > 0$ ensures the convergence of the ILHI.

Also, to present our results we need the generalized Marcum Q_ν–function of the order ν, defined by [12]

$$Q_\nu(a, b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu \mathrm{e}^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\,\mathrm{d}t,$$

where $a, \nu > 0$ and $b \geq 0$. In fact, we are interested in the case $\nu = 1$, which is usually denoted as $Q_1 \equiv Q$.

Further, let us mention that Paris *et al.* [19, p. 2816, Lemma 3] derive the identity

$$I_{e_{0,0}}(x; \alpha) = -2\overline{\alpha}\, Q\left(\frac{\sqrt{x}}{\sqrt{\alpha}+\sqrt{\alpha^2-1}}, \sqrt{x(\alpha+\sqrt{\alpha^2-1})}\right) + \overline{\alpha}\big[1 + \mathrm{e}^{-\alpha x} I_0(x)\big], \tag{3.1}$$

where $\overline{\alpha} = 1/\sqrt{\alpha^2-1}$. Subsequently, using (3.1), in [13] the authors proved that for all $x \ge 0$ and $b > a > 0$ there holds [13, p. 11, Corollary 4]

$$F_I(x;a,b;0) = -2Q\left(\frac{a\sqrt{ax}}{\sqrt{ab}+\sqrt{b^2-a^2}}, \sqrt{x(b+\sqrt{b^2-a^2})}\right) + \mathrm{e}^{-bx}I_0(ax)+1, \tag{3.2}$$

and, bearing in mind the equality [5, p. 179, Eq. (12)]

$$Q(a,b) = \exp\left(-\frac{a^2+b^2}{2}\right)\Phi_3\left(1,1;\frac{a^2}{2},\frac{a^2b^2}{4}\right),$$

they also concluded that [13, p. 11, Corollary 5]

$$\begin{aligned} F_I(x;a,b;0) &= 1-2\exp\left(\frac{-x(b+\sqrt{b^2-a^2})}{2} - \frac{1}{2}\left(\frac{a\sqrt{ax}}{\sqrt{ab}+\sqrt{b^2-a^2}}\right)^2\right) \\ &\cdot \Phi_3\left(1,1;\frac{a^3x}{2(\sqrt{ab}+\sqrt{b^2-a^2})^2};\frac{a^3x^2(b+\sqrt{b^2-a^2})}{4(\sqrt{ab}+\sqrt{b^2-a^2})^2}\right) + \mathrm{e}^{-bx}I_0(ax), \end{aligned} \tag{3.3}$$

where the confluent Horn function Φ_3 (which is also known as the Humbert function denoted by the same symbol) is defined by the series [6, p. 3]

$$\Phi_3\big(\alpha;\beta;z,w\big) = \sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(\alpha)_k}{(\beta)_{k+m}}\frac{z^k}{k!}\frac{w^m}{m!}, \qquad \max(|z|,|w|)<\infty.$$

In the meanwhile, we recognized that the formula (3.1) is unfortunately erroneously written in its source publication [19, p. 2816, Lemma 3]. The corrected version of the formula (3.1), as well as its proof, is given below in Theorem 3.1. Also, we need the auxiliary relation [1, p. 139, Eq. (6.15)]

$$\int_0^z \mathrm{e}^{\mp\alpha t}I_0(t)\,\mathrm{d}t = \frac{1}{\sqrt{\alpha^2-1}}\left\{1-\mathrm{e}^{\mp\alpha z}\left[I_0(z)+2Y_2\left(\frac{z}{c},z\right)\pm 2Y_1\left(\frac{z}{c},z\right)\right]\right\}. \tag{3.4}$$

Here $2\alpha = c+c^{-1}$ and $Y_\nu(w,z)$ stands for the von Lommel function of two variables of the order ν, defined by the Neumann series of the first type [2], built by modified Bessel functions of the first kind [1, p. 138, Eq. (6.5)]:

$$Y_\nu(w,z) = \sum_{k\ge0}\left(\frac{w}{z}\right)^{\nu+2k} I_{\nu+2k}(z).$$

THEOREM 3.1. *For all $\alpha, x \in \mathbb{C}$ there holds*

$$\begin{aligned} I_{e_{0,0}}(x;\alpha) = \frac{1}{\sqrt{\alpha^2-1}}\Bigg\{&1+\mathrm{e}^{-\alpha x}I_0(x) \\ &- 2\,Q\left(\frac{\sqrt{x}}{\sqrt{\alpha+\sqrt{\alpha^2-1}}}, \sqrt{x}\sqrt{\alpha+\sqrt{\alpha^2-1}}\right)\Bigg\}. \end{aligned}$$

PROOF. With the help of (3.4) we obtain

$$
\begin{aligned}
I_{e_{0,0}}(x;\alpha) &= \frac{1}{\sqrt{\alpha^2-1}}\Big\{1-\mathrm{e}^{-\alpha x}\Big[I_0(x)+2Y_1\left(\frac{x}{c},x\right)+2Y_2\left(\frac{x}{c},x\right)\Big]\Big\} \\
&= \frac{1}{\sqrt{\alpha^2-1}}\Big\{1-\mathrm{e}^{-\alpha x}\Big[-I_0(x)+2\sum_{m\geq 0}\left(\frac{1}{c}\right)^{2m} I_{2m}(x) \\
&\qquad + 2\sum_{k\geq 0}\left(\frac{1}{c}\right)^{1+2k} I_{1+2k}(x)\Big]\Big\} \\
&= \frac{1}{\sqrt{\alpha^2-1}}\Big\{1-\mathrm{e}^{-\alpha x}\Big[-I_0(x)+2\sum_{m\geq 0}\left(\frac{1}{c}\right)^{m} I_{m}(x)\Big]\Big\}, \qquad (3.5)
\end{aligned}
$$

where at the end we used the elementary transformation

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} a_{2n} + \sum_{n\geq 0} a_{2n+1};$$

also, considering the relation $2\alpha = c + c^{-1}$ we see that $c = \alpha + \sqrt{\alpha^2-1}$.

Further, applying the identity [11, p. 169]

$$Q(a,b) = \exp\left(-\frac{a^2+b^2}{2}\right)\sum_{n\geq 0}\left(\frac{a}{b}\right)^n I_n(ab),$$

with $a = \sqrt{x/(\alpha+\sqrt{\alpha^2-1})}$ and $b = \sqrt{x(\alpha+\sqrt{\alpha^2-1})}$, after simplification we get

$$\sum_{m\geq 0}\left(\frac{1}{\alpha+\sqrt{\alpha^2-1}}\right)^m I_m(x) = \mathrm{e}^{\alpha x} Q\left(\frac{\sqrt{x}}{\sqrt{\alpha+\sqrt{\alpha^2-1}}}, \sqrt{x}\sqrt{\alpha+\sqrt{\alpha^2-1}}\right).$$

Finally, substituting this display into (3.5) we conclude the assertion. $\square$

At the end of this chapter, in the following proposition, we state the corrected version of the formulae (3.2) and (3.3). It is also worth to mention that further inter–connection formulae between Marcum Q function and the Horn confluent hypergeometric functions of two variables Φ_2, Φ_3 are published in the recent article [6].

PROPOSITION 3.2. *For all $x \geq 0$ and $b > a > 0$ there holds*

$$F_I(x;a,b;0) = 1 + \mathrm{e}^{-bx} I_0(ax) - 2Q\left(\frac{a\sqrt{x}}{\sqrt{b+\sqrt{b^2-a^2}}}, \sqrt{x}\sqrt{b+\sqrt{b^2-a^2}}\right),$$

and

$$F_I(x;a,b;0) = 1 + \mathrm{e}^{-bx}\left[I_0(ax) - 2\Phi_3\left(1,1;\frac{x}{2}\big(b-\sqrt{b^2-a^2}\big);\frac{a^2x^2}{4}\right)\right].$$

Now we obtain the following elegant, simple bilateral bounds upon Marcum Q and Horn's confluent Φ_3.

Corollary 3.3. *For all $x \ge 0$ and $b > a > 0$ we have*

$$0 \le 2\,Q\left(\frac{a\sqrt{x}}{\sqrt{b+\sqrt{b^2-a^2}}}, \sqrt{x}\sqrt{b+\sqrt{b^2-a^2}}\right) - \mathrm{e}^{-bx}\, I_0(ax) \le 1\,,$$

and

$$\frac{1}{2}\, I_0(ax) \le \Phi_3\left(1, 1; \frac{x}{2}\big(b-\sqrt{b^2-a^2}\,\big); \frac{a^2x^2}{4}\right) \le \frac{1}{2}\left(\mathrm{e}^{bx} + I_0(ax)\right).$$

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Granice za konfluentnu Hornovu funkciju Φ_2 izvedene McKay I_ν Besselovom razdiobom

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Sažetak. U ovom radu izvedene su nove funkcionalne i uniformne ocjene za Hornov konfluentni hipergeometrijski red dviju varijabli Φ_2 kao i za nekompletni Lipschitz–Hankelov integral. Dobiveni rezultati temelje se na vjerojatnosnim metodama. Naime, osnovni matematički alati u radu su McKayeva I_ν Bessel vjerojatnosna razdioba te neke poznate, kao i manje poznate osobine kumulativnih funkcija razdiobe.

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