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REFINED EULER'S INEQUALITIES IN PLANE GEOMETRIES AND SPACES

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Abstract. Refined famous Euler's inequalities $R \geq nr$ of an n-dimensional simplex for $n = 2, 3$ and 4 as well as of non-Euclidean triangles in terms of symmetric functions of edge lengths of a triangle or a simplex in question are shown. Here R is the circumradius and r the inradius of the simplex. We also provide an application to geometric probabilities of our results and an example from astrophysics to the position of a planet within the space of four stars. We briefly discuss a recursive algorithm to get similar inequalities in higher dimensions.

1. Introduction

In papers [6] and [5] are presented improvements of the well known Euler's inequality $R \geq 2r$ from 1765, where R and r are the circumradius and inradius of a triangle, respectively, in terms of symmetric functions of triangle's side lengths and similarly for tetrahedra. In [4] we established non-Euclidean versions of Euler's inequality: $\tan(R) \geq 2\tan(r)$ in spherical and $\tanh(R) \geq 2\tanh(r)$ in hyperbolic geometry (when the triangle has the circumcircle). In [1] the authors reproved our Theorem 1 and tried to adapt it to hyperbolic (and spherical) triangles, but it did not work directly. Instead, we found in [6] the right analogue of our improvement of Euler's inequality in non-Euclidean geometries of the previous results. This was, in fact, an initial motivation for [6]. In this paper we give one more intrinsic improvement of $R \geq 2r$, and hence one more non-Euclidean analogue. The second motivation was to refine the 3D-analogue of $R \geq 3r$ of Euler's inequality and possibly in higher dimensions, as well as applications in astrophysics of positions of a planet within the space between four stars. In this work we shall briefly recall main results from previous works and make advances as well as some new interpretations of the results.

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2. Triangle

We stick with standard triangle notations. Let $T = \triangle ABC$ be a triangle with side lengths a, b, c, S its area, $2s$ the perimeter, R the circumradius and r the inradius of T. With these notations, recall, we have

Theorem 2.1.

$$R/r \geq (abc + a^3 + b^3 + c^3)/2abc \geq 2. \tag{2.1}$$

Equalities hold in both inequalities if and only if T is an equilateral triangle.

Note that the right inequality (2.1) is just the AM-GM inequality (i.e., the arithmetic-geometric means inequality) for numbers a^3, b^3, c^3. We presented three proofs of (2.1), two in [6] and another one in [5], modifying and simplifying proof given in [1]. The algebraic proof from [6] basically reduces to the inequality $d_3(a, b, c)^2 - d_3(a^2, b^2, c^2) \geq 0$ (inequality (2.4) in [6]). Here in general, $d_3(u, v, w) := (u+v-w)(u-v+w)(-u+v+w)$. The second proof of (2.1) in [6] is shown to be equivalent to Gerretsen's inequality (from 1953) which in turn follows from Blundon's inequality (from 1965): $s \leq 2R + (3\sqrt{3} - 4)r$ as explained in [4] (note, $s = 2R + r$ holds only for right triangles).

Proofs in [1] and [5] have more geometric flavor by considering tangent segments from vertices to the incircle. Gerretsen's inequalities can also be proved geometrically by computing distances between the incenter I, centroid G and orthocenter H. Namely, $9IG^2 = s^2 - 16Rr + 5r^2 \geq 0$ and $IH^2 = 4R^2 + 4Rr - s^2 \geq 0$. These formulas can also be proved geometrically by using vectors and Euler's line.

Here we provide yet another (a bit weaker) refinement of Euler's basic inequality $R \geq 2r$ in terms of symmetric functions of a, b, and c.

Theorem 2.2.

$$(R/r)^2 \geq 12(a^2 + b^2 + c^2)/(a + b + c)^2 \geq 4. \tag{2.2}$$

Equality holds only for an equilateral triangle.

Proof. Let us make the following splitting of the fraction $(R/r)^2$. Recall first $rs = S$:

$$(R/r)^2 = R^2 \cdot s^2/S \cdot 1/S. \tag{2.3}$$

Now use the well known inequality $9R^2 \geq a^2 + b^2 + c^2$ (see also later in the text the general inequality $(n+1)^2R^2 \geq \sum a^2$ for any n-simplex, here we use the case $n = 2$). Next, use the standard triangle inequality $s^2/S \geq 3\sqrt{3}$, and once again in the form $1/S \geq 3\sqrt{3}/s^2$ (see also later in the text the standard isoperimetric for simplices, but we use here the case $n = 2$). By inserting all three estimates in (2.3) yields the left hand side inequality (2.2). The right hand side of (2.2) follows from the quadratic-algebraic mean inequality for numbers a, b and c. Again, (2.2) is tight because both inequalities become equalities only in the case of an equilateral triangle. □

As it was explained in [6], if we express (2.1) in the form $R/r = 2abc/d_3(a,b,c)$ and express everything in terms of elementary symmetric functions e_1, e_2 and e_3 in variables a, b and c, the left hand side inequality (2.1) is equivalent to the inequality

$$e_1^6 + 12e_1^3e_3 + 12e_1^2e_2^2 + 36e_3^2 \geq 7e_1^4e_2 + 40e_1e_2e_3. \tag{2.4}$$

By using Lemma 1.4 from [6], we have the following theorem which improves non-Euclidean versions of Euler: $\tan(R) \geq 2\tan(r)$ and $\tanh(R) \geq 2\tanh(r)$ from [4].

Theorem 2.3. *In the spherical geometry (2.4) holds but for variables $s(a)$, $s(b)$, $s(c)$, where $s(x) = \sin(x/2)$ and in the hyperbolic geometry (2.4) holds but for variables $h(a)$, $h(b)$, $h(c)$, where $h(x) = \sinh(x/2)$. (In (2.4) the variables are $a/2$, $b/2$, $c/2$.). Theorem 2.2 would also give an inequality in elementary symmetric functions of degree 8, and hence the corresponding inequality in non-Euclidean planes which also improves the standard Euler's inequality.*

3. Tetrahedron

3D-Euler's inequality for a tetrahedron T is $R \geq 3r$. A short elegant argument is worth repeating. Consider the tetrahedron T' with centroids of faces of T as vertices. Since T and T' are similar, the circumradius R' of T' is $R' = R/3$, since the similarity coefficient is $1/3$. But $R' \geq r$, because the smallest ball that touches all faces of T is just the inscribed ball of T. So, $R \geq 3r$. The same proof works in all dimensions (including $n = 2$), but of course then $R \geq nr$.

This is the inequality which reflects only the ambient dimension and not the simplex itself. So, the motivation in [6] (and [5]) was to find an intrinsic refinement of $R \geq nr$.

Let $T = ABCD$ be any tetrahedron (3-simplex) with edge lengths a, b, c, a', b', c' with a, b, c forming the base triangle ABC, a opposite to a' etc. Let V, S, R and r be the volume, surface area, circumradius and inradius of T, respectively. In [6] we proved the following refinement of 3D Euler's inequality.

Theorem 3.1.

$$(R/r)^2 \geq 3(aa' + bb' + cc')/d_3(aa', bb', cc')^{1/3} \geq 9. \tag{3.1}$$

The left hand side of (3.1) becomes equality only when T is equifacial, i.e., all faces of T are congruent, that is $a = a'$, $b = b'$, $c = c'$. The right *hand side inequality in (3.1) becomes equality only when $aa' = bb' = cc'$, that is when Crelle's triangle of T with these side lengths is equilateral.*

Note that the right hand side of (3.1) is the AM-GM inequality combined with Schur's inequality $uvw \geq d_3(u, v, w)$, which in turn follows from ordinary Euler's $R \geq 2r$, since $R/r = 2abc/d_3(a, b, c)$ and by continuity in general.

Recall that the main ingredients of the proof of Theorem 3.1 are Crelle's formula $C = 6RV$ for the area C of Crelle's triangle (proved by inversion e.g. in [3]), the standard isoperimetric inequality for tetrahedra $2^3 3^4 V^2 \leq \sqrt{3} S^3$ (proved in [6], and also in [3]), and finally Mazur's inequality for volume proved in [2]:

$$2^3 3^2 V^2 \leq d_3(aa', bb', cc'). \tag{M}$$

As Gerretsen's and Blundon's inequalities encompass the triangle's perimeter in terms of R and r, so Theorem 3.1 implies lower and upper bounds for the surface area S of T as follows:

$$2^3 \cdot 3\sqrt{3} \cdot r^2 \leq S \leq \sqrt{2 \cdot d_3(aa', bb', cc')}/4r \text{ and } 3kV/R \leq S \leq 8\sqrt{3} \cdot R^3/9r.$$

Here k^2 is the middle term in (3.1). Equalities hold in all these inequalities only when T is a regular tetrahedron. Note that we can use our results to many other situations. For instance, looking at exradii of T, r_A the exradius of T against A is given by $r_A = 3V/(S - S_A)$, where S_A is the area of the face against A. Since $1/r = 3S/V$, it follows that $\sum 1/r_A + 1/r = 4S/3V$. So, we can estimate this sum from above and below by using the above inequalities and (M).

An analogue of Theorem 3.1 for non-Euclidean tetrahedra (inscribed in a sphere) would be a big advance, but at the time it is not clear how to do it.

4. Probability and Euler's inequalities

Let us give an interesting probabilistic interpretation of our results. Let P be a randomly and uniformly chosen point from the circumball of the tetrahedron T. What is the probability that P lies in the inscribed ball of T? One can imagine four stars with known mutual distances and a small planet orbiting around them. What is the utmost chance that the planet is deeply within the stars (in the sense that it belongs to the inball of the stars)?

Theorem 4.1. *The probability that a randomly and uniformly chosen point within the circumsphere of T is within the inscribed sphere of T is at most equal to*

$$\sqrt{d_3/(3e_1)^3}. \tag{4.1}$$

Here $d_3 = d_3(aa', bb', cc')$ and $e_1 = aa' + bb' + cc'$.

PROOF. The probability in question is equal to the quotient of volumes of the inball and the circumball of T. But the ratio of these volumes is equal to the ratio $(r/R)^3$. The upper bound $(r/R)^3 \leq \sqrt{d_3/(3e_1)^3}$ follows directly from the left hand side of inequality (3.1). $\square$

Note that bound (4.1) is $\leq 1/3^3 = 0.037037\ldots$ (by $R \geq 3r$). For an n-simplex the corresponding probability is $\leq 1/n^n$, so very fast goes to zero in higher dimensional simplices; of course, this is due to Euler's $R \geq nr$.

For a triangle we can compute the exact probability. It is equal to $(r/R)^2 = (S/s : abc/4S)^2$ and by using Heron's formula $16S^2 = 2sd_3(a,b,c)$ we get the exact probability $(d_3(a,b,c)/2abc)^2 \leq 0.25$. Now we can use Theorem 2.1 to get good upper bound for this probability. Note that Theorem 2.2 gives us also an upper bound $(a+b+c)^2/12(a^2+b^2+c^2)$.

For one more application in astrophysics see [7].

5. 4-simplex and n-simplex

Let $T = A_0A_1\ldots A_n$ be an n-dimensional simplex (or simply an n-simplex) with edge lengths $a_{ij} = A_iA_j$ and volume V, the surface area S as the sum of all face (n-1)-dimensional volumes of S_i, $i = 0, 1, \ldots, n$, R the circumradius and r the inradius of T. The general inequalities (see [6] and literature cited there) are as follows:

$$(n!V)^2 n^n \leq (n+1)^{n+1} R^{2n}, \quad (n!S^n/V^{n-1})^2 \geq n^{3n}(n+1)^{n+1},$$

and this is the standard isoperimetric inequality for simplices, and

$$n^{3n} V^{2(n-1)} \leq (n+1)^{n-1}(n!)^2 (S_0 \times S_1 \times \cdots \times S_n)^{2n/(n+1)},$$

referred to as the "volume-faces inequality". And one more is $(n+1)^2R^2 \geq \sum a^2$, the sum of squared edge lengths of T.

The next theorem can be considered as the 4D analogue of Mazur's inequality (M) in 3D.

THEOREM 5.1. *Let $T = A_0A_1A_2A_3A_4$ be a 4-simplex, $a_{ij} = A_iA_j, i < j$, the edge lengths of T and V its volume. Denote by $(ijkl) := d_3(a_{ij}a_{kl}, a_{ik}a_{jl}, a_{il}a_{jk})$ for all $0 \leq i < j < k < l \leq 4$. Then we have the following upper bound of the volume V in terms of edges:*

$$(2^5 \cdot 3 \cdot V)^3 \leq 5\sqrt{5} \cdot \left[\prod (ijkl)\right]^{2/5}. \tag{5.1}$$

Equality holds for a regular 4-simplex.

PROOF. The volume-faces inequality for $n = 4$ yields $4^{12} \cdot V^6 \leq 5^3 \cdot 24^2(S_0 \times \cdots \times S_4)^{8/5}$. Now apply to each of the five factors S_i Mazur's inequality (M). Then a bit of checking confirms (5.1). $\square$

Now we get a refinement of Euler's $R \geq 4r$ for a 4-simplex in terms of symmetric functions of edge lengths of T, a 4D analogue of Theorem 3.1 (formula (3.1)).

THEOREM 5.2.

$$(R/r)^2 \geq (8 \cdot \sum a^2)/5 \prod (ijkl)^{1/15} \geq 2^4. \tag{5.2}$$

Equality is achieved for a regular 4-simplex.

PROOF. From $rS = 4V$ we have the following splitting

$$(R/r)^2 = R^2 \cdot S^2/(4^2V^2) = (R/4)^2 \cdot [(24S^4/V^3)^2]^{1/4} \cdot 24^{-1/2} \cdot V^{-1/2}. \tag{5.3}$$

Now use the inequality $25R^2 \geq \sum a^2$ to get $(R/4)^2 \geq 2^{-2} \cdot 5^{-2} \sum a^2$. Next use the standard isoperimetric inequality for $n = 4$ to get the estimate $\left[(24S^4/V^3)^2\right]^{1/4} \geq 4^3 \cdot 5^{5/4} \cdot 24^{-1/2}$. Finally, use Theorem 5.1, that is the upper bound (5.1) for V to obtain $V^{-1/2} \geq (2^{10} \cdot 3/5)^{1/4} \cdot \prod (ijkl)^{-1/15}$. By inserting all these estimates to factors in (5.3) yields the left hand side inequality (5.2) in terms of edge lengths of T. The right inequality again follows by using Schur's inequality and the AM-GM inequality. Note if all $a_{ij} = a = 1$, then clearly all d_3 are equal to 1, so all $(ijkl) = 1$ and $\sum a^2 = 10$. Hence, for a regular simplex (5.2) implies $R = 4r$ as we know it should be by the standard Euler's inequality. $\square$

Let us only remark that Theorem 5.2 also has the probabilistic interpretation in the Euclidean dimension 4. But the hyperbolic 3D and 4D ("spacetime") analogues are still missing.

Now it is clear how to proceed with refining of Euler's inequality to the next dimension: a splitting like (5.3) and volume estimates like (5.1) by iterating inequalities as used above. This recursive algorithm works for any dimension. As we mentioned in [6] the concept of Crelle's $(n-1)$-simplex $C_{n-1}^{(i)}$ of an n-simplex can be useful in this respect. The volume $vol_{n-1}(C_{n-1}^{(i)}) = 2nRV\rho_i^{2(n-3)}$, where ρ_i^2 is the product of all edge lengths of T ending at $A_i (i = 0, 1, \ldots, n)$, is the generalisation of Crelle's formula for $n = 3$ as we used it in the proof of Theorem 3.1.

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Profinjene Eulerove nejednakosti u ravninskim geometrijama te u prostorima

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Sažetak. Profinjene su poznate Eulerove nejednakosti $R \geq nr$ za n-dimenzionalni simpleks kad je $n = 2$, 3 ili 4, te za neeuklidske trokute u terminima simetričnih funkcija duljina bridova trokuta odnosno simpleksa. Pritom su R i r redom polumjeri opisane i upisane sfere simpleksa. Nadalje, primijenjeni su dobiveni rezultati u vidu geometrijskih vjerojatnosti, a kao primjer iz astrofizike je položaj planeta unutar prostora između četiriju zvijezda. Na kraju je razmatran rekurzivni algoritam za dobivanje analognih nejednakosti u višim dimenzijama.

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