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ON SOME PROPERTIES OF KIEPERT PARABOLA IN THE ISOTROPIC PLANE

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Abstract. In this paper we consider the curve which is an envelope of the axes of homology of a given triangle and the corresponding Kiepert triangles in the isotropic plane – the Kiepert parabola of the given triangle. We derive the equation of this parabola by using appropriate coordinate system. We give some new significant characterizations of this curve which are not valid in the Euclidean plane. We have also studied the relationships between Kiepert parabola and the Steiner point, the tangential triangle as well as the Jeřabek hyperbola of the given triangle.

1. Introduction and Motivation

The isotropic (or Galilean) plane is a projective-metric plane, where the absolute consists of one line, absolute line $\omega_{\mathcal{A}}$, and one point on that line, absolute point $\Omega_{\mathcal{A}}$. The lines through the point $\Omega_{\mathcal{A}}$ are isotropic lines, and the points on the line $\omega_{\mathcal{A}}$ are isotropic points. Points with the same abscissa, i.e., which lie on the same isotropic line, are called *parallel points*.

For two non-parallel points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the *isotropic distance* is defined by $d(P_1, P_2) := x_2 - x_1$. The isotropic distance is directed. For two parallel points $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$, the isotropic span is defined by $s(P_1, P_2) := y_2 - y_1$ ([4], [5]).

A triangle is called allowable if none of its sides is isotropic ([4]). If we choose the coordinate system in a way that the circumscribed circle of an allowable triangle ABC has the equation $y = x^2$, and therefore its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$, then we say that the triangle ABC is in the standard position, or that ABC is a standard triangle, for short. Its sides BC, CA, and AB have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. Denoting $p := abc$ and $q := bc + ca + ab$, the authors proved a number of useful equalities in [2], e.g. $a^2 = bc - q$, $q + 3bc = -(b - c)^2$, $2q - 3bc = (c - a)(a - b)$. To prove geometric

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facts for all allowable triangles it is sufficient to provide a proof for a standard triangle [2].

Conics which touch the absolute line $\omega_{\mathcal{A}}$ at the absolute point $\Omega_{\mathcal{A}}$ are circles. However, singular circles (circles of the second kind) are sets of points which are equidistant from a given point and they consist of pairs of isotropic lines.

In the isotropic plane we have the following formula for Brocard angle of standard triangle $\omega = -\frac{1}{3q}(b-c)(c-a)(a-b)$ ([6]).

In [6] we have proved that if A_m, B_m, and C_m are the midpoints of the sides BC, CA, and AB of the allowable triangle ABC, and A', B', and C' are points on the perpendicular bisectors of these sides such that the spans $s(A_m, A')$, $s(B_m, B')$, and $s(C_m, C')$ are proportional to the lengths of the sides BC, CA, and AB, then the points $BC \cap B'C'$, $CA \cap C'A'$, and $AB \cap A'B'$ lie on a line $\mathcal{T}$ (see Figure 1). In the case of a standard triangle ABC, the line $\mathcal{T}$ has the equation

$$\mathcal{T} \ldots\ y = \frac{6pt}{q(2t+3\omega)}x + \frac{q}{6t}(2t+3\omega), \tag{1.1}$$

where t is a parameter, and ω is the Brocard angle of a standard triangle ABC. The triangles $A'B'C'$ are the so-called *Kiepert triangles* of the triangle ABC, and the line $\mathcal{T}$ is the axis of homology of the triangle ABC and the corresponding Kiepert triangle $A'B'C'$. In this paper we are going to determine what curve envelopes the lines $\mathcal{T}$ for variable t.

2. Kiepert parabola of a triangle in the isotropic plane

Our first result concerns the curve which is an envelope of the axes of homology of a given triangle and the corresponding Kiepert triangles in the isotropic plane.

Theorem 2.1. *Axes of homology of an allowable triangle ABC and its Kiepert triangles, envelope a parabola $\mathcal{P}$ (see Figure 1), which for the standard triangle ABC has the equation*

$$y^2 = 4px. \tag{2.1}$$

The line $\mathcal{T}$ given by (1.1) *has a point of tangency T with this parabola, where*

$$T = (x_t, y_t) = \left(\frac{q^2}{36pt^2}(2t+3\omega)^2, \frac{q}{3t}(2t+3\omega)\right). \tag{2.2}$$

Proof. The point T from (2.2) lies on the parabola with equation (2.1), and has the polar $y_t y = 2p(x + x_t)$ with respect to this parabola. Substituting the coordinates x_t and y_t from (2.2) into equation of this polar, and dividing by $\dfrac{q}{3t}(2t+3\omega)$, equation of the polar assumes the form (1.1). ∎

By analogy with the Euclidean case, the parabola $\mathcal{P}$ from Theorem 2.1 will be called the *Kiepert parabola* of the triangle ABC.

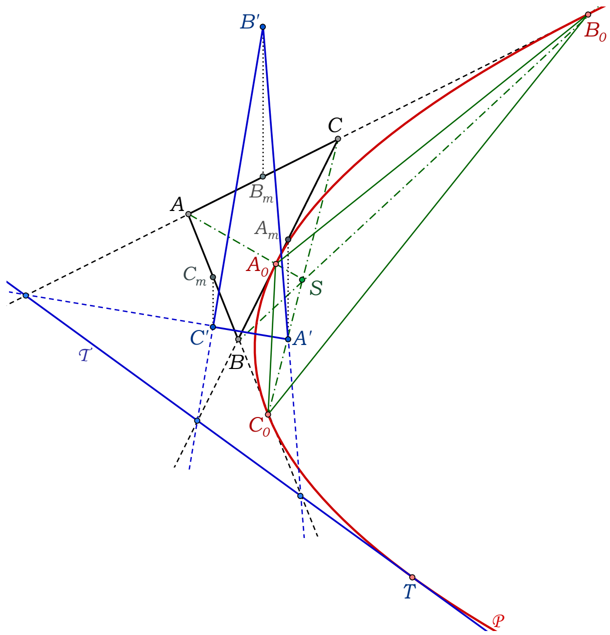

FIGURE 1. The axis of homology $\mathcal{T}$ of the triangle ABC and its Kiepert triangle $A'B'C'$, Kiepert parabola $\mathcal{P}$ of the triangle ABC, Ceva's triangle $A_0B_0C_0$ of the Steiner point S of the triangle ABC.

COROLLARY 2.2. *The Kiepert parabola of a standard triangle ABC has the equation* (2.1)*, and it has the tangent $\mathcal{T}$ given by* (1.1) *at the point T given by* (2.2) *(see Figure 1).*

Equation (1.1) is of the form $y = kx + l$ with

$$k = \frac{6pt}{q(2t+3\omega)}, \quad l = \frac{q}{6t}(2t+3\omega),$$

from which it follows that

$$kl = p. \tag{2.3}$$

COROLLARY 2.3. *The line given by the equation $y = kx + l$ touches the Kiepert parabola of a standard triangle ABC if and only if the equality* (2.3) *holds.*

3. Kiepert parabola and some significant elements of a triangle in the isotropic plane

In this section we are going to study the relationships between Kiepert parabola and some elements of the given triangle.

Theorem 3.1. *The Kiepert parabola $\mathcal{P}$ of an allowable triangle is inscribed in that triangle, i.e., it touches its sides (see Figure 1). In the case of a standard triangle ABC, the points of tangency of parabola $\mathcal{P}$ with lines BC, CA, AB are*

$$A_0 = \left(\frac{bc}{a}, -2bc\right), \quad B_0 = \left(\frac{ca}{b}, -2ca\right), \quad C_0 = \left(\frac{ab}{c}, -2ab\right).$$

Proof. By (2.1) and the equation $y = -ax - bc$ of the line BC, it follows that $4px = a^2x^2 + 2px + b^2c^2$, i.e., $a^2x^2 - 2px + b^2c^2 = 0$ or $(ax - bc)^2 = 0$, with double solution $x = \frac{bc}{a}$, and on the other hand, it follows that $y^2 = -4p \cdot \frac{y + bc}{a}$, i.e., $y^2 + 4bcy + 4b^2c^2 = 0$ or $(y + 2bc)^2 = 0$, with double solution $y = -2bc$. ∎

Points A_0, B_0, and C_0 in Theorem 3.1 coincide with vertices of the Ceva's triangle of the Steiner point of triangle ABC from [3, Corollary 2.7], so we have

Corollary 3.2. *Using the notations from Theorem* 3.1*, lines AA_0, BB_0, and CC_0 pass through the Steiner point S of the triangle ABC (see Figure 1).*

The isotropic line $x = 0$, which is according to [2] the Euler line of triangle ABC, touches parabola (2.1) at the point $\Phi_t = (0, 0)$, which is by [8] the Feuerbach point of the tangential triangle $A_tB_tC_t$ of triangle ABC. We may consider Φ_t as the vertex of parabola $\mathcal{P}$. According to terminology in [4], this point is the *focus* of parabola $\mathcal{P}$.

Letting $t \to \infty$, from (2.2) it follows that $T = \left(\frac{q^2}{9p}, \frac{2}{3}q\right)$, and equation (1.1) assumes the form $y = \frac{3p}{q}x + \frac{q}{3}$, and it is by [8] the equation of the Lemoine line of triangle ABC. With respect to the circumscribed circle of triangle ABC given by $y = x^2$, the point (x_0, y_0) has the polar line given by $y + y_0 = 2x_0x$. By [2], for the centroid $G = (x_0, y_0)$ of triangle ABC, we get $x_0 = 0$, $y_0 = -\frac{2}{3}q$, and its polar line has the equation $y = \frac{2q}{3}$. Obviously, it passes through the point of tangency T of parabola $\mathcal{P}$ with the Lemoine line of triangle ABC (see Figure 2).

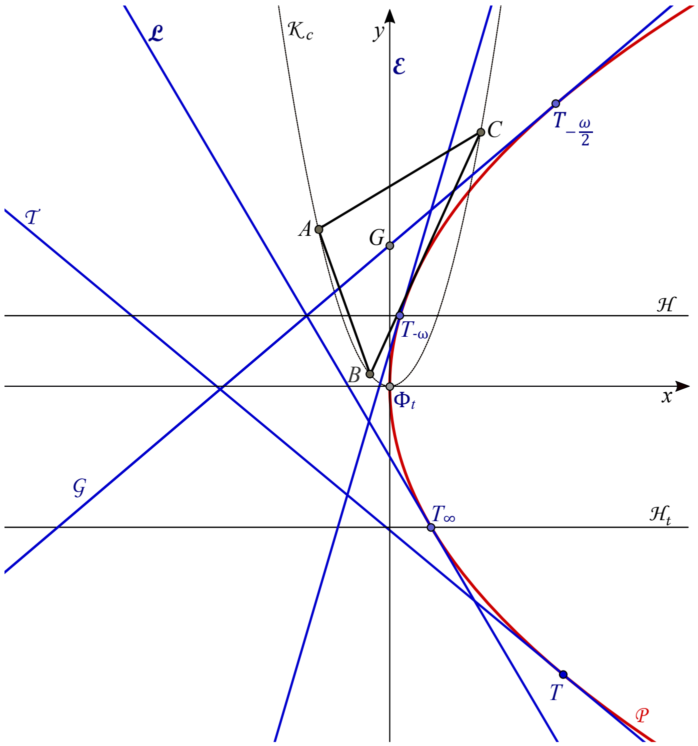

FIGURE 2. Euler line $\mathcal{E}$, Lemoine line $\mathcal{L}$, Steiner axis $\mathcal{G}$, orthic axis $\mathcal{H}$ of the triangle ABC, and orthic axis $\mathcal{H}_t$ and the Feuerbach point Φ_t of the tangential triangle $A_tB_tC_t$.

For $t = -\dfrac{\omega}{2}$, from (2.2) we obtain $T = \left(\dfrac{4q^2}{9p}, -\dfrac{4}{3}q\right)$, and equation (1.1) becomes $y = -\dfrac{3p}{2q}x - \dfrac{2}{3}q$. By [8], it is the equation of the Steiner axis of triangle ABC.

For $t = -\omega$, equation (1.1) of the line $\mathcal{T}$ is $y = -\dfrac{6p}{q}x - \dfrac{q}{6}$, the point T from (2.2) becomes $T = \left(\dfrac{q^2}{36p}, -\dfrac{q}{3}\right)$, and it lies on the orthic line of triangle ABC, which by [2] has the equation $y = -\dfrac{q}{3}$. In [6], it is shown that e.g. the vertex A' of the Kiepert triangle $A'B'C'$ is of the form

$$A' = \left(-\frac{a}{2}, -\frac{1}{2}(q + bc) + \frac{1}{2}(b - c)t\right).$$

For $t = -\omega$, the ordinate of A' is

$$
\begin{aligned}
y &= -\frac{1}{2}(q+bc) - \frac{1}{2}(b-c)\omega = -\frac{1}{2}(q+bc) + \frac{1}{6q}(b-c)^2(c-a)(a-b) \\
&= -\frac{1}{6q}[3q(q+bc) + (q+3bc)(2q-3bc)] = -\frac{1}{6q}(5q^2 + 6bcq - 9b^2c^2) \\
&= -\frac{1}{6q}[5q^2 + 6(q+a^2)q - 9bc(q+a^2)] = -\frac{1}{6q}[11q^2 + 6a^2q - 9(q+a^2)q - 9ap] \\
&= \frac{1}{6q}(9ap + 3a^2q - 2q^2),
\end{aligned}
$$

and then $A' = A_1$, where

$$A_1 = \left(-\frac{a}{2}, \frac{1}{6q}(9ap + 3a^2q - 2q^2)\right).$$

Tangent lines of the Kiepert parabola of a triangle can be characterized in several other geometric ways.

THEOREM 3.3. *Let $A_tB_tC_t$ be the tangential triangle of an allowable triangle ABC, i.e., let the lines B_tC_t, C_tA_t, and A_tB_t be tangents to the circumscribed circle of the triangle ABC at the points A, B, and C. Let $A''B''C''$ be a triangle obtained from the triangle $A_tB_tC_t$ by any translation in the isotropic direction. The points $BC \cap B''C''$, $CA \cap C''A''$, and $AB \cap A''B''$ lie on a line $\mathcal{T}'$, which envelopes the Kiepert parabola $\mathcal{P}$ of triangle ABC.*

PROOF. In [1], it is proved that in the case of standard triangle ABC the line $\mathcal{T}'$ has the equation

$$\mathcal{T}' \ldots\ y = \frac{3p}{t+q}x + \frac{t+q}{3},$$

where t is a parameter. By Corollary 2.3, the line $\mathcal{T}'$ touches parabola $\mathcal{P}$. $\square$

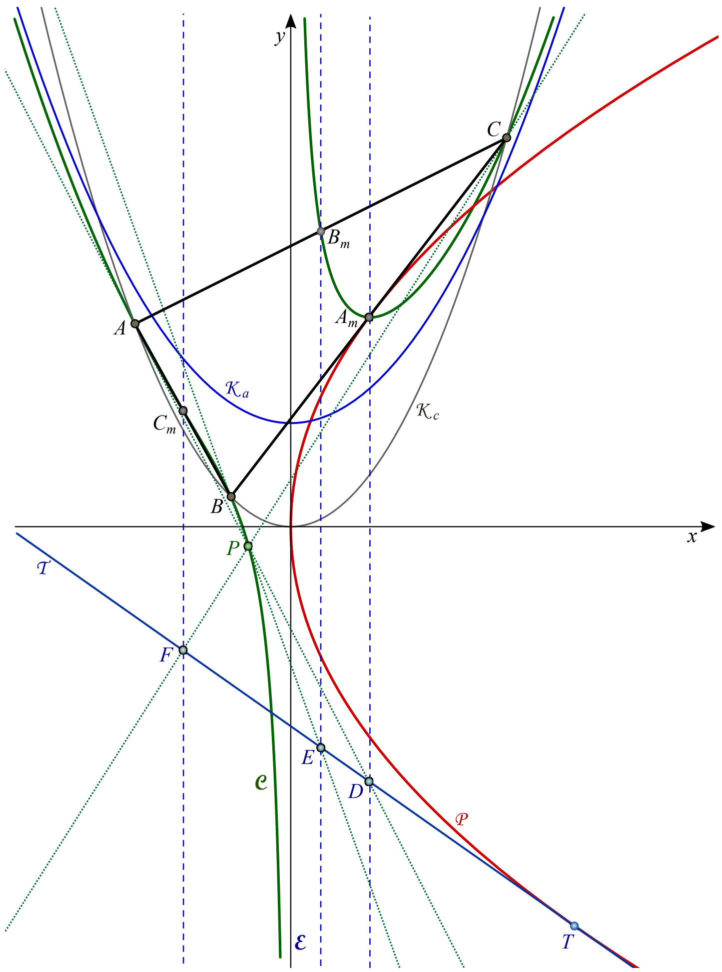

FIGURE 3. Visualization of the statements of Theorems 3.4 and 3.5.

THEOREM 3.4. *Let D, E, and F be intersections of the line $\mathcal{T}$ with bisectors of the sides BC, CA, and AB of an allowable triangle ABC. The lines AD, BE, and CF pass through one point P if and only if the line $\mathcal{T}$ touches the Kiepert parabola of the triangle ABC (see Figure 3).*

PROOF. Let the line $\mathcal{T}$ have the equation $y = kx + l$. As the midpoint of BC has the abscissa $\frac{b+c}{2} = -\frac{a}{2}$ (because $a + b + c = 0$), the bisector of the side BC has the equation $x = -\frac{a}{2}$ and we get $y = l - \frac{ak}{2}$, and then

$$D = \left(-\frac{a}{2}, l - \frac{ak}{2}\right).$$

The line given by the equation

$$3y = \left(k + 2a - \frac{2l}{a}\right) x + a^2 - ak + 2l \tag{3.1}$$

passes through the point $A = (a, a^2)$ and through the point D because we have

$$\left(k + 2a - \frac{2l}{a}\right) a + a^2 - ak + 2l = 3a^2,$$

$$\left(k + 2a - \frac{2l}{a}\right) \left(-\frac{a}{2}\right) + a^2 - ak + 2l = 3l - \frac{3}{2}ak,$$

and this is the line AD. Equations of lines BE and CF look analogously. These three lines pass through a common point P under the condition

$$0 = \begin{vmatrix} k + 2a - \frac{2l}{a} & a^2 - ak + 2l & 1 \\ k + 2b - \frac{2l}{b} & b^2 - bk + 2l & 1 \\ k + 2c - \frac{2l}{c} & c^2 - ck + 2l & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} a - \frac{l}{a} & a^2 - ak & 1 \\ b - \frac{l}{b} & b^2 - bk & 1 \\ c - \frac{l}{c} & c^2 - ck & 1 \end{vmatrix}.$$

We have

$$\begin{aligned} \left(b - \frac{l}{b}\right)(c^2 - ck) - \left(c - \frac{l}{c}\right)(b^2 - bk) &= bc(c-b) + \frac{l}{bc}(b^3 - c^3) + \frac{kl}{bc}(c^2 - b^2) \\ &= \frac{c-b}{bc}[b^2c^2 - (b^2 + bc + c^2)l + (b+c)kl] \\ &= -\frac{b-c}{p}(bcp + aql - a^2kl), \end{aligned}$$

and similarly for the other two terms, and therefore this condition becomes

$$(bcp + aql - a^2kl)(b-c) + (cap + bql - b^2kl)(c-a) + (abp + cql - c^2kl)(a-b) = 0,$$

i.e.,

$$[bc(b-c) + ca(c-a) + ab(a-b)]p - [a^2(b-c) + b^2(c-a) + c^2(a-b)]kl = 0,$$

and since the terms in each of the square brackets are equal to $-(b-c)(c-a)(a-b)$, the condition $kl - p = 0$ follows again. □

What curve does the point P in Theorem 3.4 describe? Since $l = \frac{p}{k}$, we obtain

$$k + 2a - \frac{2l}{a} = k + 2a - \frac{2bc}{k},$$

$$a^2 - ak + 2l = a^2 - ak + \frac{2p}{k},$$

and hence the line AD from (3.1) has the equation

$$3y = \left(k + 2a - \frac{2bc}{k}\right) x + a^2 - ak + \frac{2p}{k}.$$

It passes through the point

$$P = \left(\frac{k}{2}, \frac{1}{3}\left(\frac{k^2}{2} - q + \frac{2p}{k}\right)\right)$$

because we get

$$\left(k + 2a - \frac{2bc}{k}\right) \cdot \frac{k}{2} + a^2 - ak + \frac{2p}{k} = \frac{k^2}{2} - bc + a^2 + \frac{2p}{k} = \frac{k^2}{2} - q + \frac{2p}{k},$$

and the lines BE and CF also pass through this point. The point P describes the curve with the parametric equations

$$x = \frac{k}{2}, \quad 3y = \frac{k^2}{2} - q + \frac{2p}{k},$$

where k is a parameter. Substituting $k = 2x$ into the second equation, we obtain

$$3y = 2x^2 - q + \frac{p}{x}, \tag{3.2}$$

and the derived curve $\mathcal{C}$ is a curve of third order with the equation

$$2x^3 - 3xy - qx + p = 0. \tag{3.3}$$

By (3.2), this curve has an isotropic asymptote with the equation $x = 0$ and an asymptotic circle with the equation

$$y = \frac{2}{3}x^2 - \frac{q}{3}.$$

Abscissae of intersections of the cubic $\mathcal{C}$ with the circumscribed circle $\mathcal{K}_c$ of the triangle ABC are found by substituting $y = x^2$ into (3.3), giving $-x^3 - qx + p = 0$, i.e. $(x-a)(x-b)(x-c) = 0$. The solutions $x = a$, $x = b$, and $x = c$ show that these intersections are precisely the points A, B, and C. Therefore we obtain

THEOREM 3.5. *The point P from Theorem 3.4 describes a circular cubic circumscribed to the triangle ABC whose asymptote is the line $x = 0$, the Euler line of this triangle (see Figure 3).*

THEOREM 3.6. *Let $\mathcal{T}$ be a non-isotropic line that is not parallel to lines BC, CA and AB, and let D, E, and F be its intersections with these lines. Let $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ be singular circles with centers D, E, F, which pass through the points A, B and C. The circles $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ belong to a pencil of circles if and only if the line $\mathcal{T}$ is tangent to the Kiepert parabola of the triangle ABC, and in that case the common radical center S of circles $\mathcal{D}, \mathcal{E}$, $\mathcal{F}$ and the circumscribed circle $\mathcal{K}_c$ of the triangle ABC describes its Jeřabek hyperbola.*

PROOF. Let the line $\mathcal{T}$ have the equation $y = kx + l$, $k \notin \{-a, -b, -c\}$. From this equation and the equation $y = -ax - bc$ of the line BC, for abscissa of the point D, we get the equation $(k+a)x = -(l+bc)$ with the solution $x = d$, where

$$d = -\frac{l+bc}{k+a}.$$

In addition, denoting

$$d' = 2d - a = -a - 2\frac{l+bc}{k+a},$$

the circle $\mathcal{D}$ has the equation $(x-a)(x-d') = 0$, i.e.,

$$(x-a)\left(x + a + 2\frac{l+bc}{k+a}\right) = 0$$

or

$$\mathcal{D} \ldots\ x^2 + 2\frac{l+bc}{k+a}x - a^2 - 2\frac{al+p}{k+a} = 0. \tag{3.4}$$

Equations of the circles $\mathcal{E}$ and $\mathcal{F}$ look analogously. The radical axis of any two of these three circles is an isotropic line. Radical axis $\mathcal{F}'$ of the circles $\mathcal{D}$ and $\mathcal{E}$ has the equation

$$2\left(\frac{l+bc}{k+a} - \frac{l+ca}{k+b}\right)x = a^2 - b^2 + 2\left(\frac{al+p}{k+a} - \frac{bl+p}{k+b}\right),$$

which after multiplication by $(k+a)(k+b)$ and rearrangement becomes

$$2[ck(b-a)+l(b-a)+c(b^2-a^2)]x = (k^2-ck+ab)(a^2-b^2)+2[kl(a-b)+p(b-a)],$$

and after cancelling the factor $b-a$ we obtain

$$2(ck+l-c^2)x = c(k^2-ck+ab) + 2p - 2kl,$$

i.e.,

$$2(ck+l-c^2)x = k(ck+l-c^2) + 3(p-kl),$$

or finally,

$$\mathcal{F}' \ldots\ x = \frac{k}{2} + \frac{3}{2}\cdot\frac{p-kl}{ck+l-c^2}.$$

Analogously, the circles $\mathcal{D}$ and $\mathcal{F}$ have the radical axis $\mathcal{E}'$ given by

$$\mathcal{E}' \ldots\ x = \frac{k}{2} + \frac{3}{2}\cdot\frac{p-kl}{bk+l-b^2}.$$

These two radical axes coincide either under the condition $kl = p$ or the condition $bk - b^2 = ck - c^2$, i.e., $(b-c)(k-b-c) = 0$, or finally, $k+a = 0$, which is not true. Therefore, the circles $\mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ belong to the same pencil subject to (2.3), which means that the line $\mathcal{T}$ is a tangent of the parabola $\mathcal{P}$, and then the common radical axis of these circles has the equation $x = \dfrac{k}{2}$. When $kl = p$, we obtain

$$l + bc = l + \frac{p}{a} = l + \frac{kl}{a} = \frac{l}{a}(k+a),$$

$$al + p = al + kl = l(k + a),$$

so equation (3.4) becomes

$$\mathcal{D} \ldots \; x^2 + \frac{2l}{a}x - a^2 - 2l = 0.$$

Therefore, the circle $\mathcal{K}_c$ with the equation $y = x^2$ and the circle $\mathcal{D}$ have the radical axis with the equation

$$y = -\frac{2l}{a}x + a^2 + 2l.$$

For $x = \frac{k}{2}$, we get

$$y = -\frac{kl}{a} + a^2 + 2l = -\frac{p}{a} + bc - q + 2l = 2l - q,$$

and the common radical center of circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{K}_c$ is the point

$$P = \left(\frac{k}{2}, 2l - q\right).$$

It describes a curve with parametric equations $x = \frac{k}{2}$, $y = 2l - q$, from where $x(y + q) = kl$, i.e. $x(y + q) = p$, and it is by [1] the equation of the Jeřabek hyperbola of the triangle ABC. □

The Kiepert parabola of a triangle has interesting relations with some other curves of that triangle.

THEOREM 3.7. *The Jeřabek hyperbola of the tangential triangle is the polar to the Kiepert parabola of the given triangle with respect to its circumscribed circle.*

PROOF. With respect to the circumscribed circle, the point (x_0, y_0) has the polar line with equation $y + y_0 = 2x_0x$. If (x_0, y_0) lies on parabola (2.1), then $x_0 = \frac{y_0^2}{4p}$, and this polar has the following equation

$$y = \frac{y_0^2}{2p}x - y_0.$$

Therefore, it is necessary to find the equation of the envelope of lines with equation

$$y = \frac{t^2}{2p}x - t, \tag{3.5}$$

where t is a parameter. The point

$$\left(\frac{2p}{t + t'}, -\frac{tt'}{t + t'}\right)$$

lies on line (3.5) because $t^2 - t(t+t') = -tt'$, and it also lies on an analogous line with the parameter t' instead of t. For $t' = t$ we get that points with coordinates

$$x = \frac{p}{t}, \quad y = -\frac{1}{2}t \tag{3.6}$$

are points of tangency of lines with equation (3.5) with their envelope, i.e., (3.6) are parametric equations of this envelope. From (3.6) follows $xy = -\frac{p}{2}$, and by [1, Theorem 15] this is the equation of the Jeřabek hyperbola of the tangential triangle of triangle ABC. ◻

THEOREM 3.8. *The intersections (except vertices A, B, C) of the Kiepert hyperbola of an allowable triangle ABC with sides of its anticomplementary triangle $A_nB_nC_n$ are the points of tangency of these sides with the Kiepert parabola of triangle $A_nB_nC_n$.*

PROOF. By [7], the Kiepert hyperbola of a standard triangle ABC has the equation

$$3px^2 + 2qxy + 2q^2x - 3py - 2pq = 0, \tag{3.7}$$

and by [2], the line B_nC_n is given by $y = -ax + 2a^2$, which, substituted into (3.7), for the abscissa x of the intersection yields the equation

$$(3p - 2aq)x^2 + (4a^2q + 2q^2 + 3ap)x - (6a^2p + 2pq) = 0.$$

Because

$$\begin{aligned} 4a^2q + 2q^2 + 3ap &= 4(bc - q)q + 2q^2 + 3bc(bc - q) \\ &= 3b^2c^2 + bcq - 2q^2 = (3bc - 2q)(bc + q) \\ &= (3bc - 2q)(2bc - a^2), \\ 6a^2p + 2pq &= 2p(q + 3a^2) = 2p(3bc - 2q), \end{aligned}$$

and

$$3p - 2aq = a(3bc - 2q),$$

after dividing by $3bc - 2q = -(c - a)(a - b) \neq 0$, this equation becomes $ax^2 + (2bc - a^2)x - 2p = 0$, i.e., $(x - a)(ax + 2bc) = 0$. That is why the abscissa of the second intersection is $-\frac{2bc}{a}$ (with exception of the vertex A). This point on the line B_nC_n has for its complementary point on the line BC, the point with abscissa $\frac{bc}{a}$, and this is the point A_0 from Theorem 3.1. ◻

THEOREM 3.9. *With respect to its circumscribed Steiner ellipse, the Kiepert hyperbola of a triangle has the Kiepert parabola of its anticomplementary triangle for a polar.*

PROOF. With respect to the circumscribed Steiner ellipse which is by [8] given by

$$q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 = 0,$$

the point $T_0 = (x_0, y_0)$ has the polar $\mathcal{T}_0$ with the equation

$$2q^2x_0x - 9p(x_0y + y_0x) - 6qy_0y - 6pq(x + x_0) - 4q^2(y + y_0) + 18p^2 = 0,$$

i.e.,

$$(2q^2x_0 - 9py_0 - 6pq)x - (9px_0 + 6qy_0 + 4q^2)y - (6pqx_0 + 4q^2y_0 - 18p^2) = 0,$$

or $y = kx + l$, where

$$k = \frac{2q^2x_0 - 9py_0 - 6pq}{9px_0 + 6qy_0 + 4q^2}, \quad l = -\frac{6pqx_0 + 4q^2y_0 - 18p^2}{9px_0 + 6qy_0 + 4q^2}.$$

Hence, we get

$$l + 2q = \frac{12pqx_0 + 8q^2y_0 + 18p^2 + 8q^3}{9px_0 + 6qy_0 + 4q^2}$$

and then

$$\begin{aligned}&(9px_0+6qy_0 + 4q^2)^2[k(l + 2q) + 2p]\\ &=(2q^2x_0-9py_0-6pq)(12pqx_0+8q^2y_0+18p^2+8q^3) + 2p(9px_0+6qy_0+4q^2)^2\\ &=(54p^2 + 8q^3)(3px_0^2 + 2qx_0y_0 + 2q^2x_0 - 3py_0 - 2pq) = 0\end{aligned}$$

if the point T_0 lies on the Kiepert hyperbola (3.7). Therefore, the line $\mathcal{T}_0$ satisfies the following condition

$$k(l + 2q) + 2p = 0. \tag{3.8}$$

Line complementary to the line $\mathcal{T}_0$ is the line $\mathcal{T}_0'$ with equation $y = kx+l'$, for which $l + 2l' = -2q$ since $L + 2L' = 3G$ holds for $L = (0, l)$, $L' = (0, l')$ and $G = \left(0, -\frac{2}{3}q\right)$ where L and L' are corresponding points on the lines $\mathcal{T}_0$ and $\mathcal{T}'_0$ lying on the isotropic line $x = 0$.

That is why $l + 2q = -2l'$ and then condition (3.8) becomes $kl' = p$. By Corollary 2.3, the line $\mathcal{T}_0'$ touches the Kiepert parabola of the triangle ABC and then the line $\mathcal{T}_0$ touches the Kiepert parabola of the anticomplementary triangle $A_nB_nC_n$ of the triangle ABC. □

COROLLARY 3.10. *The line with equation $y = kx + l$ touches the Kiepert parabola of the anticomplementary triangle of the standard triangle ABC if and only if* (3.8) *holds.*

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Neka svojstva Kiepertove parabole u izotropnoj ravnini

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Sažetak. U ovom radu razmatramo krivulju koja je omotaljka osiju homologije dopustivog trokuta i njegovog varijabilnog Kiepertovog trokuta u izotropnoj ravnini, koju ćemo po analogiji s euklidskim slučajem zvati Kiepertovom parabolom danog trokuta. Izvodimo jednadžbu ove parabole i navodimo neke nove značajne karakteristike ove krivulje koje ne vrijede u Euklidskoj ravnini. Proučavamo veze Kiepertove parabole i Steinerove točke, tangencijalnog trokuta, kao i Jeřabekove hiperbole danog trokuta.

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