RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 27 = 555 (2023): 203-218
DOI: https://doi.org/10.21857/moxpjh1l0m

ON THE VARCHENKO DETERMINANT FORMULA FOR ORIENTED BRAID ARRANGEMENTS

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Abstract. In this paper, we first consider the arrangement of hyperplanes and then the corresponding oriented arrangement of hyperplanes in n-dimensional real space. Following the work of Varchenko, who studied the determinant of the quantum bilinear form of a real configuration and the determinant formula for a matroid bilinear form, we discuss here first some of the main properties of the braid arrangement and then of the oriented braid arrangements in n-dimensional real space. The main result of this study is a theorem that provides an explicit formula for determining the determinant of the matrix associated with the oriented braid arrangement. The proof of this theorem is based on the results of two different approaches. One is to determine the space of all constants in the multiparametric quon algebra equipped with a multiparametric q-differential structure, and the other is to study the feasibility of multiparametric quon algebras in Hilbert space.

1. Introduction

Here we show the connection between the determinant of Varchenko's bilinear form [15] (i.e., the Varchenko matrix) and the more explicit determinant of the quantum bilinear form of the oriented braid arrangement in $\mathbb{R}^n$, which has been studied by many researchers. We will also discuss the results of the author [12] and Meljanac-Svrtan [7], who had different approaches to prove the same formula.

In Section 2 we will first discuss some of the main properties related to the arrangement $\mathcal{A}$ of hyperplanes in the real affine space $\mathbb{R}^n$, $n \geq 1$ studied by Varchenko. He assigned to each weighted arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$ a symmetric matrix whose rows and columns are indexed by the regions of the arrangement $\mathcal{A}$, i.e., by components of the complement of the union of all hyperplanes of the arrangement $\mathcal{A}$. The entries of the Varchenko matrix are products of the weights a_i of the hyperplanes separating the respective regions,

2020 *Mathematics Subject Classification.* 52C35, 05E16.
Key words and phrases. Arrangements of hyperplanes, braid arrangement, oriented braid arrangements, quantum bilinear form, multiparametric quon algebra.

where to each hyperplane $H_i \in \mathcal{A}$ is associated a weight $a_i = a(H_i)$ of the hyperplane H_i. These weights are variables in a polynomial ring. Varchenko gave an explicit combinatorial formula for the determinant of this matrix, as a product of terms of the form $(1-a_L^2)^{l(L)}$, where L passes through the edges in intersection poset (except $\mathbb{R}^n$) and $l(L)$ denotes a computable multiplicity of the edge L, see Theorem 2.4.

Several other works have dealt with this subject, including those of Varchenko with Brylawski [2] or Schechtman [11] and those of Denham and Hanlon [3], Hochstättler and Welker [6]. It should be noted that the study of arrangements of hyperplanes is a much more complicated area of research than is described in this article. We note that the study of hyperplane arrangements was a continuation of the geometric study of Knizhnik-Zamolodchikov differential equations, which first appeared in conformal field theory. Briefly, in the papers [11,15] studies are given on the cohomology of certain quantum groups with special interest in the cohomology of one-dimensional local systems over complements of hyperplanes in complex affine spaces with specialization to arbitrary real affine arrangements.

Following the paper [2], in which the authors introduced a symmetric bilinear form of a weighted matroid, and the paper [6], in which the authors generalized the Varchenko matrix to oriented matroids, we are concerned here with oriented real arrangements and, in a special case, with the oriented braid arrangement in the space $\mathbb{R}^n$, $n \geq 2$. Consequently, we introduce the orientation of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ by choosing a unit normal vector to each hyperplane $H_i \in \mathcal{A}$. Then each hyperplane H_i partitions the n-dimensional space $\mathbb{R}^n$ into an open half-space containing a unit normal vector, an open half-space not containing it, and the hyperplane itself. In particular, each hyperplane H_i can be obtained as the intersection of the closures of the half-spaces containing H_i. Then each region is a nonempty intersection of the corresponding open half-spaces. Therefore, we consider here two open half-spaces $H_i^+ \subset \mathbb{R}^n \backslash H_i$ and $H_i^- \subset \mathbb{R}^n \backslash H_i$ associated with the oriented hyperplane $H_i \in \mathcal{A}^*$ with the weight a_i^+ assigned to H_i^+ and the weight a_i^- to H_i^-. These weights are variables in a polynomial ring. We denote by $\mathcal{A}^*$ an arrangement $\mathcal{A}$ provided with an orientation and $\mathcal{A}^*$ we call an oriented arrangement. To an oriented arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbb{R}^n$ one associates a matrix whose rows and columns are indexed by the regions D_i^*, $1 \leq i \leq m$ of the arrangement $\mathcal{A}^*$. The (D_i^*, D_j^*)-entry of this matrix is the product of weights of all open half-spaces containing the region D_i^* and not containing the region D_j^*, see identity (2.10). By examining in detail the edges L in the arrangement $\mathcal{A}$ and the edges L^* in the oriented arrangement $\mathcal{A}^*$ and their associated weights a_L and a_{L^*}, we found that the relation between them can be expressed in the form $a_L^2 = a_{L^*}$, where we obtained $a_L^* = a_{L^*}^+ \cdot a_{L^*}^-$, see Remark 2.7 Thus, the determinant of the matrix associated with the oriented arrangement $\mathcal{A}^*$ can be obtained as a generalization of the determinant of the matrix associated with the arrangement $\mathcal{A}$, see Theorem 2.9 and Theorem 2.4.

In Section 3 we first describe the basic concepts of the braid arrangement in $\mathbb{R}^n$, denoted by B_n, and then introduce the orientation of the braid arrangement. Then we show the relationship between the matrix associated with the braid arrangement and the matrix associated with the oriented braid arrangement, and their determinants. The main result of this study is Theorem 3.8, which provides an explicit formula for determining the determinant of the matrix associated with the oriented braid arrangement. The proof of Theorem 3.8 is based on the results of two different approaches [7, 12], which are detailed below.

2. Preliminaries on arrangements

A finite set of hyperplanes in the n-dimensional real space $\mathbb{R}^n$, $n \geq 1$ is called a *hyperplane arrangement* or simply *an arrangement*, where a hyperplane is a $(n-1)$-dimensional subspace of $\mathbb{R}^n$, $n \geq 1$ (i.e., a subspace of codimension one).

Let $\mathcal{A} = \{H_1, H_2, \dots, H_k\}$ be an arrangement consisting of k hyperplanes H_i in $\mathbb{R}^n$, where $k = \#\mathcal{A}$ denotes the number of hyperplanes in $\mathcal{A}$. A connected component

$$D \subseteq \mathbb{R}^n \setminus \bigcup_{1 \leq i \leq k} H_i \tag{2.1}$$

in the complement of the union of all hyperplanes in $\mathcal{A}$ is called *a region* or a chamber. The regions are open and convex. Let $\mathcal{D}(\mathcal{A}) = \{D_1, D_2, \dots, D_m\}$ be the set of all regions of $\mathcal{A}$, where $m = \#\mathcal{D}(\mathcal{A})$ denotes the number of regions of $\mathcal{A}$. A fundamental object associated with the arrangement $\mathcal{A}$ is its intersection poset $L_{\mathcal{A}}$, which is partially ordered by reverse inclusion and is defined as follows. The elements of $L_{\mathcal{A}}$, called *edges* of $\mathcal{A}$, are the nonempty intersections of subsets of hyperplanes in $\mathcal{A}$, including the empty intersection, where the space $\mathbb{R}^n$ can be viewed as the intersection of the empty set of hyperplanes. Consequently, $\mathbb{R}^n$ is a single minimal element of $L_{\mathcal{A}}$. The poset $L_{\mathcal{A}}$ has a single maximal element, the centre of $\mathcal{A}$, if and only if the intersection of all hyperplanes in $\mathcal{A}$ is nonempty. An arrangement $\mathcal{A}$ is called *centred* if $L_{\mathcal{A}}$ has a single maximal element, otherwise it is called $\mathcal{A}$ *centerless*. In the general case, $L_{\mathcal{A}}$ has no single maximal element. In particular, a single maximal element is a zero-dimensional edge called a *vertex*. If the center of an arrangement $\mathcal{A}$ is a vertex, then $\mathcal{A}$ is called *central*, see [8, 9] for more details.

We mention a very important result discovered by Zaslavsky [17]. He proved that the number of regions of $\mathcal{A}$ is equal to $\|p_{\mathcal{A}}(-1)\|$, where $p_{\mathcal{A}}(\lambda)$

denotes the characteristic polynomial of $\mathcal{A}$ given by

$$p_{\mathcal{A}}(\lambda) = \sum_{t \in L_{\mathcal{A}}} = \mu(\hat{0}, t)\lambda^{\dim t}. \tag{2.2}$$

Here μ denotes the Möbius function of $L_{\mathcal{A}}$, see also [14].

Let us briefly recall the result of Varchenko [15], which is designated as follows. Let $R_{\mathcal{A}} = \mathbb{Z}[a_i \mid 1 \le i \le k]$ be the polynomial ring in variables a_i.

Definition 2.1. *To each hyperplane $H_i \in \mathcal{A}$ is associated a commutative variable $a_i = a(H_i) \in R_{\mathcal{A}}$, which is called the* weight of the hyperplane H_i.

Definition 2.2. *The* weight of an edge $L \in L_{\mathcal{A}}$ *is defined as the product of the weights of all hyperplanes containing the edge L.*

In particular, the weight of the space $\mathbb{R}^n$ is equal to one. Since $\mathbb{R}^n$ can be viewed as the intersection of the empty set of hyperplanes, in what follows we consider intersection poset $L'_{\mathcal{A}} = L_{\mathcal{A}} \backslash \mathbb{R}^n$ (except $\mathbb{R}^n$). In other words, all edges without $\mathbb{R}^n$ are treated. The arrangement of hyperplanes, where to each hyperplane H_i is associated a particular monomial of weight a_i (i.e., a variable in a polynomial ring) is sometimes called a *weighted arrangement*.

Remark 2.3. Suppose that r hyperplanes $H_{j_1}, \dots, H_{j_r} \in \mathcal{A}$ are all hyperplanes that contain an edge $L \in L'_{\mathcal{A}}$. Then

$$L = \bigcap_{1 \le p \le r} H_{j_p}.$$

Thus, according to Definition 2.2, the weight of the edge L is given by

$$a_L = \prod_{1 \le p \le r} a_{j_p}. \tag{2.3}$$

Let $M_{\mathcal{A}}$ be the module of $R_{\mathcal{A}}$-linear combinations of regions of arrangement $\mathcal{A}$. Then a $R_{\mathcal{A}}$-bilinear symmetric form B in the module $M_{\mathcal{A}}$ is defined as follows

$$B(D_p, D_q) = \prod_{1 \le i \le l} a_{j_i} \tag{2.4}$$

where the product runs over all hyperplanes $H_{j_i} \in \mathcal{A}$ separating regions D_p and D_q. Here we have used $a_{j_i} = a(H_{j_i})$ for $1 \le i \le l$ and $l \le k$, where $k = \#\mathcal{A}$. It is easy to see that for any two regions $D_p, D_q \in \mathcal{D}(\mathcal{A})$, $1 \le p, q \le m$, $m = \#\mathcal{D}(\mathcal{A})$ it follows $B(D_p, D_q) = B(D_q, D_p)$ and $B(D_p, D_p) = 1$. Consequently, the matrix B with entries (2.4) is a symmetric square matrix. The matrix B is called the *quantum bilinear form of the arrangement $\mathcal{A}$*.

Theorem 2.4. *The determinant of the matrix B is given by*

$$\det B = \prod_{L \in L'_{\mathcal{A}}} (1 - a_L^2)^{l(L)} \tag{2.5}$$

where a_L is the weight of the edge $L \in L'_{\mathcal{A}}$ and $l(L)$ is a certain natural number called the multiplicity of the edge L.

Briefly, the natural number $l(L)$ is defined as the product of the numbers $n(L)$ and $p(L)$, where $n(L)$ is the number of regions of the arrangement $\mathcal{A}_L = \{H_i \cap L \mid H_i \in \mathcal{A}, L \not\subset H_i\}$ induced in L, and $p(L)$ is the number of regions of the projective localization $P\mathcal{A}^L$ such that the closures of these regions do not intersect the hyperplane, see [11, 15].

In the following we introduce the orientation of an arrangement $\mathcal{A}$ consisting of k hyperplanes H_i in $\mathbb{R}^n$. Let $\mathbf{n}_i$ denote a unit normal vector to a hyperplane $H_i \in \mathcal{A}$. Then each hyperplane H_i, $1 \leq i \leq k$ divides $\mathbb{R}^n$ into three parts:

1. H_i the hyperplane itself,
2. H_i^+ the open half-space in $\mathbb{R}^n$ containing $\mathbf{n}_i$,
3. H_i^- the open half-space in $\mathbb{R}^n$ which does not contain $\mathbf{n}_i$.

The arrangement $\mathcal{A}$ which is provided with an orientation is denoted by $\mathcal{A}^*$ and is called an oriented arrangement. We introduce the abbreviations $H_i^{\varepsilon_i}$, $\varepsilon_i \in \{+,-\}$ for the corresponding open half-spaces H_i^+ and H_i^- in $\mathbb{R}^n$. It is straightforward to check the following two properties.

(OA1) Any hyperplane $H_i \in \mathcal{A}^*$ can be obtained as the intersection of the closures of the half-spaces containing H_i:

$$H_i = \mathrm{Cl}(H_i^+) \cap \mathrm{Cl}(H_i^-) \quad \text{for each } 1 \leq i \leq k,$$

where k denotes the number of hyperplanes in $\mathcal{A}^*$.

(OA2) A region of an oriented arrangement $\mathcal{A}^*$ (consisting of k hyperplanes in $\mathbb{R}^n$) is a nonempty intersection

$$\bigcap_{1\leq i\leq k} H_i^{\varepsilon_i} \neq \emptyset$$

of the corresponding open half-spaces $H_i^{\varepsilon_i} \subset \mathbb{R}^n \backslash H_i$, $\varepsilon_i \in \{+,-\}$.

Note that each region can be associated with the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ consisting of plus or minus signs. It is obvious that the length of this sequence is equal to the number of hyperplanes in $\mathcal{A}^*$. In agreement with (OA1) as well as the predefined weight of the hyperplane, it is sufficient to define the weight of the open half-space in $\mathbb{R}^n$. Let $R_{\mathcal{A}^*} = \mathbb{Z}[a_i^+, a_i^- \mid 1 \leq i \leq k]$ be the polynomial ring in variables $a_i^{\varepsilon_i}$ for each $\varepsilon_i \in \{+,-\}$, $i \in \{1, 2, \ldots, k\}$.

Definition 2.5. *To each open half-space $H_i^{\varepsilon_i} \subset \mathbb{R}^n \backslash H_i$ with $\varepsilon_i \in \{+,-\}$ for all $1 \leq i \leq k$ is associated a commutative variable $a_i^{\varepsilon_i} \in R_{\mathcal{A}^*}$ called the* weight of the open half-space *$H_i^{\varepsilon_i}$, $\varepsilon_i \in \{+,-\}$.*

Thus, the ring $R_{\mathcal{A}^*}$ is generated by two sets of variables consisting of the weights of the open half-spaces distinguishing between the intersection of a hyperplane in the positive and negative directions. According to Definition 2.2

and Property (OA1), we define the weight of an edge $L^* \in L'_{\mathcal{A}^*}$ of an oriented arrangement $\mathcal{A}^*$ as follows.

Definition 2.6. *The weight of an edge $L^* \in L'_{\mathcal{A}^*}$ is defined as the product of the weights of all open half-spaces in $\mathbb{R}^n$ whose closures contain the edge L^*.*

Remark 2.7. The edge $L \in L'_{\mathcal{A}}$ from Remark 2.3 leads to the edge $L^* \in L'_{\mathcal{A}^*}$, where the oriented arrangement $\mathcal{A}^*$ descends to the arrangement $\mathcal{A}$, after forgetting orientations. Consequently, we can write

$$L^* = \bigcap_{1 \le p \le r} \Big(\mathrm{Cl}(H^+_{j_p}) \cap \mathrm{Cl}(H^-_{j_p}) \Big). \tag{2.6}$$

Let a_{L^*} be the weight of the edge L^*. Then

$$a_{L^*} = \prod_{1 \le p \le r} a^+_{j_p} \cdot a^-_{j_p} \tag{2.7}$$

can be rewritten as follows

$$a_{L^*} = a^+_{L^*} \cdot a^-_{L^*} \tag{2.8}$$

with

$$a^+_{L^*} = \prod_{1 \le p \le r} a^+_{j_p}, \qquad a^-_{L^*} = \prod_{1 \le p \le r} a^-_{j_p},$$

where the products run over all corresponding open half-spaces whose closures contain the edge L^*. Comparing (2.3) with (2.7) and applying Property (OA1), we find that the relation between the weights of edges L and L^* can be expressed in the form: $a_L^2 = a_{L^*}$. Thus we obtain

$$a_L^2 = a^+_{L^*} \cdot a^-_{L^*}\,. \tag{2.9}$$

In particular, it follows from (2.9) $a_i^2 = a_i^+ a_i^-$, where $a_i = a(H_i)$ denotes the weight of the hyperplane $H_i \in L'_{\mathcal{A}}$ and $a_i^{\varepsilon_i} = a(H_i^{\varepsilon_i})$ with $\varepsilon_i = \{+,-\}$ are the weights of the corresponding open half-spaces $H_i^{\varepsilon_i}$ whose closures contain the hyperplane H_i.

Let $\mathcal{D}(\mathcal{A}^*) = \{D_1^*, D_2^*, \ldots, D_m^*\}$ be the set of all regions of $\mathcal{A}^*$ and let B^* be the quantum bilinear form of the oriented arrangement $\mathcal{A}^*$ (associated with the arrangement $\mathcal{A}$).

Proposition 2.8. *The entries of B^* are monomials of the form*

$$B^*(D_p^*, D_q^*) = \prod_{D_p^* \subset H_{j_i}^{\varepsilon_{j_i}},\, D_q^* \not\subset H_{j_i}^{\varepsilon_{j_i}}} a_{j_i}^{\varepsilon_{j_i}}, \qquad \varepsilon_{j_i} \in \{+,-\} \tag{2.10}$$

where the product is taken over all open half-spaces containing the region D_p^* and not containing the region D_q^*.

It is now obvious that $a_{j_i}^+ \neq a_{j_i}^-$ implies $B^*(D_q^*, D_p^*) \neq B^*(D_p^*, D_q^*)$ for $p \neq q$. Clearly, $B^*(D_p^*, D_p^*) = 1$. Consequently, the square matrix B^* is not symmetric. Note that it follows directly from (2.10) that the product of the quantum forms $B^*(D_p^*, D_q^*)$ and $B^*(D_q^*, D_p^*)$ is equal to the square of the corresponding associated bilinear symmetric form $B(D_p, D_q)$ given by (2.4). See also two examples below. From this it follows the following theorem.

Theorem 2.9. *The determinant of the matrix B^* is given by the formula*

$$\det B^* = \prod_{L^* \in L'_{\mathcal{A}^*}} (1 - a_{L^*}^+ \cdot a_{L^*}^-)^{l(L^*)}, \tag{2.11}$$

where $a_{L^*}^+ \cdot a_{L^*}^-$ is the weight of the edge $L^* \in L'_{\mathcal{A}^*}$ and $l(L^*)$ is a certain natural number called the multiplicity of the edge L^*.

The following two examples are intended to illustrate the relationship between the formulas (2.5) and (2.11).

Example 2.10. Figure 1 shows an arrangement $\mathcal{A}$ of two points on a line and Figure 2 illustrates the corresponding oriented arrangement $\mathcal{A}^*$ of the arrangement $\mathcal{A}$. The orientations should be designated on a real line. Let us denote by $D_j = D_j^*$, $j = 1, 2, 3$ the regions of $\mathcal{A}$ and $\mathcal{A}^*$, respectively.

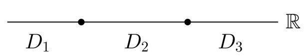

Figure 1. An arrangement of two points on a line

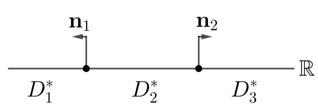

Figure 2. Oriented arrangement of two points on a line

Then by applying the formulas (2.5) and (2.11) we obtain that the corresponding matrices B_1 and B_1^* and also their determinants are given as follows:

$$B_1 = \begin{pmatrix} 1 & a_1 & a_1a_2 \\ a_1 & 1 & a_2 \\ a_1a_2 & a_2 & 1 \end{pmatrix}, \quad B_1^* = \begin{pmatrix} 1 & a_1^- & a_1^-a_2^+ \\ a_1^+ & 1 & a_2^+ \\ a_1^+a_2^- & a_2^- & 1 \end{pmatrix},$$

$$\det B_1 = \left(1 - a_1^2\right) \cdot \left(1 - a_2^2\right), \qquad \det B_1^* = \left(1 - a_1^+a_1^-\right) \cdot \left(1 - a_2^+a_2^-\right).$$

Example 2.11. Let $\mathcal{A} = \{H_1, H_2\}$ be an arrangement of two lines in a plane as shown in Figure 3, and let $\mathcal{A}^*$ be the associated oriented arrangement as shown in Figure 4.

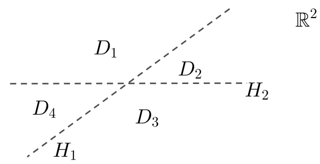

Figure 3. An arrangement of two lines in a plane

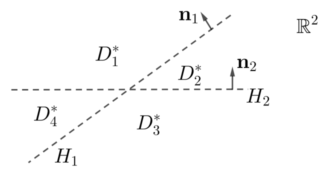

Figure 4. Oriented arrangement of two lines in a plane

Then the corresponding matrices B_2 and B_2^* and their determinants are given as follows:

$$B_2 = \begin{pmatrix} 1 & a_1 & a_1a_2 & a_2 \\ a_1 & 1 & a_2 & a_1a_2 \\ a_1a_2 & a_2 & 1 & a_1 \\ a_2 & a_1a_2 & a_1 & 1 \end{pmatrix}, \quad \det B_2 = \left(1 - a_1^2\right)^2 \cdot \left(1 - a_2^2\right)^2,$$

$$B_2^* = \begin{pmatrix} 1 & a_1^- & a_1^-a_2^- & a_2^- \\ a_1^+ & 1 & a_2^- & a_1^+a_2^- \\ a_1^+a_2^+ & a_2^+ & 1 & a_1^+ \\ a_2^+ & a_1^-a_2^+ & a_1^- & 1 \end{pmatrix},$$

$$\det B_2^* = \left(1 - a_1^+a_1^-\right)^2 \cdot \left(1 - a_2^+a_2^-\right)^2.$$

3. The oriented braid arrangement

In this section we study a particularly important arrangement in a real affine space $\mathbb{R}^n$, $n \geq 2$, the *braid arrangement*, denoted by B_n, consisting of $n(n-1)/2$ hyperplanes

$$H_{ij} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}, \quad 1 \leq i < j \leq n. \tag{3.1}$$

Note that although this arrangement is centered, it is not a central arrangement because its center (=the unique maximal element of $L_{\mathcal{A}}$) is the line given by $T(\mathcal{A}) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_n\}$. From the definition of the rank of an arrangement $\mathcal{A}$, denoted by $r(\mathcal{A})$, see [8, Definition 2.10], it follows that the rank of the braid arrangement is equal to $n-1$. We recall briefly that the rank of an arrangement $\mathcal{A}$ is given by $r(\mathcal{A}) = r(T(\mathcal{A}))$, where $T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H \neq \emptyset$ is the single maximal element of the poset $L_{\mathcal{A}}$, called the center of $\mathcal{A}$. Thus $\mathcal{A}$ is an essential arrangement if and only if the center is a vertex. In other words, an arrangement $\mathcal{A}$ in space $\mathbb{R}^n$ is essential if and only if $r(\mathcal{A}) = r(T(\mathcal{A})) = n$. Consequently, the braid arrangement B_n is not an essential arrangement. In each space $\mathbb{R}^n$, $n \geq 2$ there is a unique braid arrangement, denoted by B_n.

Example 3.1. In the plane $\mathbb{R}^2$ there is the braid arrangement $\mathsf{B}_2 = \{H_{12}\}$ consisting of a line $H_{12} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$, which is also the center of B_2.

From the definition of the reflection arrangement, see [9], it follows that the braid arrangement is the reflection arrangement of the symmetric group S_n. We can thus say that B_n consists of the reflecting hyperplanes, see [3]. Moreover, the regions of the braid arrangement B_n are directly connected to S_n, i.e., to the set of all permutations of the first n natural numbers. Each region of B_n is indexed by exactly one permutation in S_n, and conversely, each permutation $\sigma \in S_n$ is connected to exactly one region $P_\sigma \in \mathcal{D}(\mathsf{B}_n)$ via the correspondence

$$\sigma \leftrightarrow P_\sigma = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n}\}.$$

Let $\sigma \in S_n$ be any permutation in S_n. Two points $x, y \in \mathbb{R}^n$ lie in the same region $P_\sigma \in \mathcal{D}(\mathsf{B}_n)$ if and only if the relative orders of their coordinates are equal. Then we can write

$$x, y \in P_\sigma \;\Leftrightarrow\; x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n},\; y_{\sigma_1} < y_{\sigma_2} < \cdots < y_{\sigma_n}.$$

Consequently, $\#\mathcal{D}(\mathsf{B}_n) = \text{Card } S_n = n!$. We recall that $\#\mathcal{D}(\mathsf{B}_n)$ denotes the number of all regions of the braid arrangement B_n. Given the notation introduced above, a commutative variable $a_{ij} = a(H_{ij})$ denotes the weight of the hyperplane $H_{ij} \in \mathsf{B}_n$. The identity $a_{ji} = a_{ij}$ follows directly from the identity $H_{ji} = H_{ij}$ for each $1 \leq i < j \leq n$, $n \geq 2$. Let us denote by B_n the quantum bilinear form of the braid arrangement B_n. Then the (P_σ, P_τ)-entry of the matrix B_n is equal to the product of all weights a_{ij}, so that the index i appears to the left of the index j in the one-line form of σ and to the right in the one-line form of τ.

Let $I(\sigma) = \{(p, q) \mid p < q,\ \sigma(p) > \sigma(q)\}$ be the set of inversions of the permutation $\sigma \in S_n$. Then the following proposition is obtained.

Proposition 3.2. *The entries of the matrix B_n are the monomials of the form*

$$B_n(P_\sigma, P_\tau) = \prod_{(p,q)\in I(\tau^{-1}\sigma)} a_{\sigma(p)\sigma(q)}, \tag{3.2}$$

where $a_{ij} = a_{ji}$, for i, j distinct and $a_{ii} = 1$, $i, j = 1, \dots, n$.

Comparing (3.2) with (2.4), it is easy to see that the entries of the matrix B_n are a special case of the entries of the Varchenko matrix.

Example 3.3. The braid arrangement B_3 consists of three diagonal planes H_{12}, H_{13}, H_{23}, the line $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$ is the center of B_3. Then $\#\mathcal{D}(\mathsf{B}_3) = 6$. From the identity (3.2) it follows

$$B_3 = \begin{pmatrix} 1 & a_{23} & a_{13}a_{23} & a_{12}a_{13}a_{23} & a_{12}a_{13} & a_{12} \\ a_{23} & 1 & a_{13} & a_{12}a_{13} & a_{12}a_{13}a_{23} & a_{12}a_{23} \\ a_{13}a_{23} & a_{13} & 1 & a_{12} & a_{12}a_{23} & a_{12}a_{13}a_{23} \\ a_{12}a_{13}a_{23} & a_{12}a_{13} & a_{12} & 1 & a_{23} & a_{13}a_{23} \\ a_{12}a_{13} & a_{12}a_{13}a_{23} & a_{12}a_{23} & a_{23} & 1 & a_{13} \\ a_{12} & a_{12}a_{23} & a_{12}a_{13}a_{23} & a_{13}a_{23} & a_{13} & 1 \end{pmatrix}.$$

We have used here the Johnson-Trotter ordering of permutations in S_3 given by 123, 132, 312, 321, 231, 213. It is easy to check

$$\det B_3 = (1 - a_{12}^2)^2 \cdot (1 - a_{13}^2)^2 \cdot (1 - a_{23}^2)^2 \cdot (1 - a_{12}^2 a_{13}^2 a_{23}^2). \tag{3.3}$$

Remark 3.4. The quantum bilinear form and its determinant of the braid arrangement B_2 are given by

$$B_2 = \begin{pmatrix} 1 & a_{12} \\ a_{12} & 1 \end{pmatrix}, \qquad \det B_2 = 1 - a_{12}^2.$$

In particular, in the one-parameter case where all weights a_{ij} are equal to q, the (P_σ, P_τ)-entry of the matrix B_n is expressed by $q^{i(\tau^{-1}\sigma)}$, where $i(\sigma)$ denotes the number of inversions of the permutation $\sigma \in S_n$, see [16] for more details.

Let us introduce the orientation of the braid arrangement B_n in $\mathbb{R}^n$, $n \geq 2$ similar to Section 2. Then, for each $1 \leq i < j \leq n$, we denote by

$$H_{ij}^+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i > x_j\},$$

$$H_{ij}^- = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i < x_j\}.$$

To each open half-space H_{ij}^+ is associated a weight $q_{ij} = a(H_{ij}^+)$ and to each open half-space H_{ij}^- is associated a weight $q_{ji} = a(H_{ij}^-)$ such that $q_{ji} \neq q_{ij}$ for all $1 \leq i < j \leq n$. Thus, the matrix B_n^* is not symmetric. In what follows we use the same notation as above. Then Proposition 3.2 can be reformulated as follows.

PROPOSITION 3.5. *The entries of the matrix B_n^* are the monomials of the form*

$$B_n^*(P_\sigma, P_\tau) = \prod_{(a,b)\in I(\tau^{-1}\sigma)} q_{\sigma(a)\sigma(b)} \tag{3.4}$$

where $I(\tau^{-1}\sigma) = \{(a,b) \mid a < b,\ \tau^{-1}\sigma(a) > \tau^{-1}\sigma(b)\}$ *denotes the set of inversions of* $\tau^{-1}\sigma$.

We consider here that $q_{\sigma(b)\sigma(a)} \neq q_{\sigma(a)\sigma(b)}$. In the special case $q_{ji} = q_{ij}$ for each $1 \leq i < j \leq n$ it follows that the matrix B_n arises from the matrix B_n^*. Then the symmetric matrix B_n is derived from the corresponding non-symmetric matrix B_n^*; compare Example 3.3 with Example 3.6.

EXAMPLE 3.6. Let B_3^* be the oriented braid arrangement associated to the braid arrangement B_3. Then we obtain that the quantum bilinear form of the oriented braid arrangement B_3^* is given in the form

$$B_3^* = \begin{pmatrix} 1 & q_{23} & q_{13}q_{23} & q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} \\ q_{32} & 1 & q_{13} & q_{12}q_{13} & q_{12}q_{13}q_{32} & q_{12}q_{32} \\ q_{31}q_{32} & q_{31} & 1 & q_{12} & q_{12}q_{32} & q_{12}q_{31}q_{32} \\ q_{21}q_{31}q_{32} & q_{21}q_{31} & q_{21} & 1 & q_{32} & q_{31}q_{32} \\ q_{21}q_{31} & q_{21}q_{31}q_{23} & q_{21}q_{23} & q_{23} & 1 & q_{31} \\ q_{21} & q_{21}q_{23} & q_{21}q_{13}q_{23} & q_{13}q_{23} & q_{13} & 1 \end{pmatrix},$$

where

$$\det B_3^* = (1-q_{12}q_{21})^2\cdot(1-q_{13}q_{31})^2\cdot(1-q_{23}q_{32})^2\cdot(1-q_{12}q_{21}q_{13}q_{31}q_{23}q_{32}). \tag{3.5}$$

To write more efficiently, we will first introduce the following abbreviations. Let $Q = \{l_1, l_2, \ldots, l_n\}$ be the set of cardinality n. Then for each $2 \leq m \leq n$ we define

$$(Q;m) := \{T \subseteq Q \mid \text{Card } T = m\}, \tag{3.6}$$

where Card T denotes the cardinality of the set T. Moreover, for each set $T \in (Q;m)$ we define

$$\sigma_T := \prod_{\{i,j\}\subset T} \sigma_{ij} \tag{3.7}$$

with $\sigma_{ij} := q_{ij}q_{ji}$ for i,j distinct. (We may consider $\sigma_{ii} := 1$, it will be consistent with $q_{ii} = 1$.) It is obvious that the formula (3.7) coincides with

$$\sigma_T = \prod_{a\neq b\in T} q_{ab} \tag{3.8}$$

see eq. (4.1) in [5].

REMARK 3.7. The formula (3.7) implies that the identity (3.5) can be rewritten in the form $\det B_3^* = (1-\sigma_{12})^2\cdot(1-\sigma_{13})^2\cdot(1-\sigma_{23})^2\cdot(1-\sigma_{123})$. In particular, $\det B_2^* = 1-\sigma_{12}$ for $n = 2$ coincides with $\det B_2$ in Remark 3.4.

THEOREM 3.8. *The determinant of the quantum bilinear form B_n^* of the oriented braid arrangement B_n^* is given by the following formula*

$$\det B_n^* = \prod_{\substack{T\in(Q;m)\\ 2\le m\le n}} (1-\sigma_T)^{(m-2)!\,(n-m+1)!}. \tag{3.9}$$

PROOF. The formula (3.9) has already been discussed in several articles. We will prove Theorem 3.8 here on the basis of the author study [12], in which the problem of determining the space of all constants in the multiparametric quon algebra $\mathcal{B}$ equipped with a multiparametric $\boldsymbol{q}$-differential structure was studied in detail. Of particular interest in [12] is the matrix $\mathbf{A}_Q$, which in the generic case (when Q is a set) plays a crucial role in computing the determinant of the matrix B_n^*. We note that the more general matrices $\mathbf{A}_Q$ for any multiset $Q=\{l_1\le\cdots\le l_n\}$ of cardinality n have been studied in [12], where $S_nQ=\{\sigma(l_1\ldots l_n)\mid\sigma\in S_n\}$ the set of all permutations of the multiset Q, was considered.

We denote by $\underline{j}=j_1,\ldots,j_n$, $\underline{k}=k_1,\ldots,k_n$ any two permutations in S_n and let $g\in S_n$ satisfies the following condition

$$\underline{k}=g.\underline{j}$$

i.e., $j_{g^{-1}(p)}=k_p$ for all $1\le p\le n$. Then the $(\underline{k},\underline{j})$-entry of the matrix $\mathbf{A}_Q$ (in the generic case) is given by

$$(\mathbf{A}_Q)_{\underline{k},\,\underline{j}} = \prod_{(a,b)\in I(g)} q_{j_b}q_{j_a} \tag{3.10}$$

where $I(g)=\{(a,b)\mid a<b,\ g(a)>g(b)\}$ denotes the set of inversions of $g\in S_n$. Note that (3.10) can be rewritten (as an operator on a free non-commutative algebra) as follows

$$\mathbf{A}_Q\,e_{\underline{j}} = \sum_{g\in S_n}\left(\prod_{(a,b)\in I(g)} q_{j_b}q_{j_a}\,e_{\underline{k}}\right) \tag{3.11}$$

with $\underline{k}=g.\underline{j}$, where $\underline{j},\underline{k}\in S_n$ (see [12] and also Proposition 4.8 in [13], and also [7], Proposition 1.6.1. (*ii*) in slightly different notation). Moreover, comparision of the matrix $\mathbf{A}_Q$ given by (3.11) with the quantum bilinear form B_n^* of the oriented braid arrangement B_n^*, $n\ge 2$ leads to the conclusion that the matrix B_n^* is equal to the matrix $\mathbf{A}_Q$. Consequently, $\det\mathsf{B}_n^*=\det\mathbf{A}_Q$. By applying the study in [12], it is then shown that $\det\mathbf{A}_Q$ can be represented by the identity (3.9) from Theorem 3.8. ∎

We also emphasize that the matrix $\mathbf{A}_Q$ (i.e., B_n^*) can be written as a product of simpler matrices $\mathbf{B}_{Q,k}$, $2\le k\le n$ given in [12] by $\mathbf{B}_{Q,n-k+1}$, $1\le k\le n-1$, where $\mathbf{B}_{Q,1}=\mathbf{I}$ is equal to the unit matrix $\mathbf{I}$. Moreover, each matrix $\mathbf{B}_{Q,n-k+1}$ can be expressed as a product of even simpler matrices, which are not considered here because they are not relevant to the following considerations. Thus it turns out

$$\det \mathbf{B}_{Q,n-k+1} = \prod_{\substack{T\subseteq Q\\ 2\leq \mathrm{Card}\ T\leq n-k+1}} (1-\sigma_T)^{(\mathrm{Card}\ T-2)!\,(n-\mathrm{Card}\ T)!} \tag{3.12}$$

for each $1 \leq k \leq n-1$, where Card T denotes the cardinality of the set T and σ_T is defined by (3.8) and

$$\det \mathbf{A}_Q = \prod_{1\leq k\leq n-1} \det(\mathbf{B}_{Q,n-k+1}) \tag{3.13}$$

where the formula (3.9) is obtained directly.

Note that the matrix B_n^*, $n \geq 2$ is a part of the study [7] by Meljanac and Svrtan, who worked out matrix-level factorizations in the interest of the feasibility of multiparametric quon-algebras in Hilbert space. The authors studied matrices $A^{(\nu)}$, $\|\nu\| = n$, their factorizations as well as their determinants first in the generic case and then for general ν. They proved the determinant formula of the matrix $A^{(\nu)}$, $\|\nu\| = n$ for generic ν. Comparing the matrix $A^{(\nu)}$ with the matrix $\mathbf{A}_Q$ discussed in [12], we find that they coincide. Moreover, in the generic case, the determinant of the matrix $A^{(\nu)}$ given in [7, Theorem 1.9.2.] can be expressed by the formula (3.9) in Theorem 3.8. So we can assume that $\mathsf{B}_n^* = \mathbf{A}_Q = A^{(\nu)}$. We note that the same matrices $A^{(\nu)}$ and $\mathbf{A}_Q$ were obtained in different contexts. In particular, the matrix $A^{(\nu)}$ in [7] was obtained in the study of the explicit Fock representation of a multiparametric quon algebra on the free associative algebra of noncommutative polynomials with multiparametric partial derivatives. There, the basic problem was to compute the inverse of the matrix $A^{(\nu)}$ both in its original form and in the generic case, see [7, Theorem 2.2.6. and Theorem 2.2.17.]. Consequently, the factorizations are given from the right. On the other hand, another approach was made based on [12], where the author was motivated by the problem of computing the constants in the multiparametric quon algebra $\mathcal{B}$. Therefore, the factorizations from the left are shown because they were more appropriate for determining the constants in the algebra $\mathcal{B}$. See also [4] for more general factorizations in braid group algebra.

We note that there are in [7] applications of the above results to arrangements of hyperplanes also to contravariant forms of certain quantum groups. There the authors proved that Theorem 1.9.2 can be reinterpreted by Theorem 3.1.3, which we reproduce here because of its importance.

Theorem 3.9 (Meljanac-Svrtan). *The determinant of the quantum bilinear form B_n of the arrangement $\mathcal{A}_n$ is given by the formula*

$$\det B_n = \prod_{L\in\varepsilon'(\mathcal{A}_n)} (1-a(L)^2)^{l(L)} \tag{3.14}$$

where for $L = \{x_{i_1} = x_{i_2} = \cdots = x_{i_k}\} \in \mathcal{A}_{n,k} \subset \varepsilon'(\mathcal{A}_n)$ *we have*

$$a(L) = \prod_{1 \le a < b \le k} q_{i_a i_b}, \qquad l(L) = (k-2)!\,(n-k+1)!$$

It should be noted that the authors argue that the formula (3.14) is more explicit than Varchenko's formula, see [7, Applications], which refers to non-oriented arrangements, expressed here as (2.5) in Theorem 2.4. It is obvious that the identity (3.14) corresponds to the identity (2.5) where $a(L)^2$ coincides with a_L^2. Similarly, the two sets $\varepsilon'(\mathcal{A}_n)$ and $L'_{\mathcal{A}}$ denote a set of all edges of the corresponding arrangement without $\mathbb{R}^n$; we recall, $l(L)$ is a certain natural number called the multiplicity of edge L.

However, we would like to emphasize that the quantum bilinear form B_n of the arrangement $\mathcal{A}_n$ given in Theorem 3.9 is compatible with the quantum bilinear form B_n of the non-oriented braid arrangement B_n discussed above. According to their compatibility there is a correlation between the determinants of the quantum bilinear form of the arrangement and the quantum bilinear form of the corresponding oriented braid arrangement B_n^*. Furthermore, by applying the identity (2.7) with

$$a_{L^*}^+ = \prod_{1 \le a < b \le k} q_{i_a i_b}, \qquad a_{L^*}^- = \prod_{1 \le a < b \le k} q_{i_b i_a},$$

we obtain

$$a(L)^2 = a_{L^*}^+ \cdot a_{L^*}^- = \prod_{1 \le a < b \le k} \sigma_{i_a i_b} \tag{3.15}$$

where we used $\sigma_{i_a i_b} = q_{i_a i_b} q_{i_b i_a}$.

Let $\xi'_* := L'_{\mathsf{B}_n^*}$ be a set of all edges of the oriented braid arrangement B_n^* without $\mathbb{R}^n$. Then the expression $L = \{x_{i_1} = x_{i_2} = \cdots = x_{i_k}\} \in \mathcal{A}_{n,k}$ in Theorem 3.9 can be treated in our notation as $L^* \in \xi'_*$ with $L^* = \{x_{i_1} = \cdots = x_{i_m}\}$, $2 \le m \le n$. From this we have concluded that Theorem 3.9 can be reinterpreted as follows.

Theorem 3.10. *The determinant of the quantum bilinear form* B^* *of the oriented braid arrangement* B_n^* *is given by the following formula*

$$\det B_n^* = \prod_{L^* \in \xi'_*} (1 - a_{L^*}^+ \cdot a_{L^*}^-)^{l(L^*)} \tag{3.16}$$

where for $L^* = \{x_{i_1} = \cdots = x_{i_m}\}$ *we get*

$$a_{L^*}^+ \cdot a_{L^*}^- = \prod_{1 \le a < b \le m} \sigma_{i_a i_b}, \qquad l(L^*) = (m-2)!\,(n-m+1)!$$

Theorem 3.10 is directly related to Theorem 2.9, where the product in formula (3.16) runs over all subsets $T \in (Q; m)$, $2 \le m \le n$ with Card $Q = n$;

see also the identities (3.6) and (3.9). Theorem 3.10 then corresponds to Theorem 3.8 and shows the connection between Theorem 3.8 and Theorem 2.9.

Acknowledgements.

This work was fully supported by the University of Rijeka under project number uniri-prirod-18-9.

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O Varchenkovoj determinanti za orijentirane pletenične aranžmane

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Sažetak. U ovom radu prvo razmatramo aranžmane hiperravnina, a zatim odgovarajuće orijentirane aranžmane hiperravnina u n-dimenzionalnom realnom prostoru. Slijedeći rad od Varchenka, koji je proučavao determinantu kvantne bilinearne forme realne konfiguracije i formulu za izračunavanje determinante bilinearne forme matroida, ovdje prvo raspravljamo o nekim od glavnih svojstava pleteničnih aranžmana (diskriminantnih aranžmana), a zatim orijentiranih pleteničnih aranžmana (orijentiranih diskriminantnih aranžmana) u n-dimenzionalnom realnom prostoru. Glavni rezultat ove studije je teorem kojim se iskazuje eksplicitna formula za određivanje determinante matrice pridružene orijentiranom pleteničnom aranžmanu. Dokaz ovog teorema temelji se na rezultatima dva različita pristupa. Jednog utemeljenog na određivanju prostora svih konstanti u višeparametarskoj kuonskoj algebri snabdjevenoj višeparametarskom q-diferencijalnom strukturom, a drugog na proučavanju reprezentabilnosti višeparametarskih kuonskih algebri u Hilbertovom prostoru.

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Received: 15.10.2022.
Revised: 31.1.2023.
Accepted: 14.2.2023.