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FUZZY AGGREGATORS – AN OVERVIEW

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ABSTRACT

The article deals with mathematical formalism of the process of combining several inputs into a single output in fuzzy inteligent systems, the process known as aggregation. We are interested in logic aggregation operators. Such aggregators are present in most decision problems and in fuzzy expert systems. Fuzzy inteligent systems are equipped with aggregation operators (aggregators) with which reasoning models adapt well to human reasoning. A brief overview of the field of fuzzy aggregators is given. Attention is devoted to so called graded logic aggregators.. The role of fuzzy agregators in modelling reasoning and the way they are chosen in modelling are pointed out. The conclusions are given and research in the field is pointed out.

KEY WORDS

aggregation, fuzzy intelligent systems, conjunction, disjunction, compensatory operators

CLASSIFICATION

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JEL: O31
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INTRODUCTION

In order to achieve an intelligent system, we need intelligence and a device – a computer. In order to implement intelligence with a computer, we need to model intelligence (knowledge representation), we need the automation of the process of (intelligent) reasoning to get new ideas about the world, and we need to implement the process of intelligent action based on new ideas [1].

Logic is one of tools for modelling the observable properties of human reasoning. We use logic to implement decision-making process or knowledge representation and automatic reasoning. In the last century, it has been noticed that the classical two-value logic is a limited framework for modelling the representation of knowledge and human reasoning. The ways to expand the possibilities of representation by logic have been proposed. One of the most fruitful of these attempts was initiated by Lotfi Zadeh [2].

Zadeh has expanded the idea of the degree to which an element belongs to a set from two values, 0 (for non-belonging), and 1 (for belonging), to a range between 0 and 1, which allows the development of models in which key elements are not precise numbers but vague sets, i.e. a class of objects in which the transition from non-belonging to belonging is gradual, not abrupt. Zadeh described the mathematical theory of fuzzy sets and the corresponding fuzzy logic (a kind of a continuous logic with truth value from [0, 1], instead as in standard logic where each sentences have truth value from {0, 1}, there is no "in between"). Zadeh, also, proposed appropriate set and logical operations, which improved the expressiveness of the model, i.e. enabled dealing with uncertain and vague information common in human reasoning. Operations on fuzzy sets of unions, intersection and complement are defined using *max*, *min* and $1 - \mu(x)$ operations, (where μ is degree of membership of element x in a fuzzy set), which correspond to fuzzy logic functions disjunction, conjunction, and negation. In fuzzy intelligent systems [3], one of the key issues is the problem of aggregation of fuzzy information represented by membership functions (whose values are in [0, 1]). Fuzzy membership can be interpreted as a degree of truth, so we have fuzzy logic aggregation. Aggregation operators combine multiple input values into one output value, which represents all input values.

In this article, the aggregation operator (aggregator), present in fuzzy intelligent systems, is considered. In Section 2, the considered problem is formulated. In Section 3 a formal definition of aggregator is given, as well as main classes of that operator. Section 4 deals with compensatory aggregators. Special attention is devoted to the aggregator called graded conjunction/disjunction. The selection of an aggregator is discussed in Section 5. Section 6 contains the conclusions. A list of references is given.

AGGREGATION

In fuzzy intelligent systems, one of the key problems is the problem of agrregating fuzzy information represented by membership functions (whose values are in [0, 1]). Aggregators combine multiple input values into one output value, which represents all input values.

For example, the general form of a fuzzy multicriteria decision-making system is shown in the Figure 1.

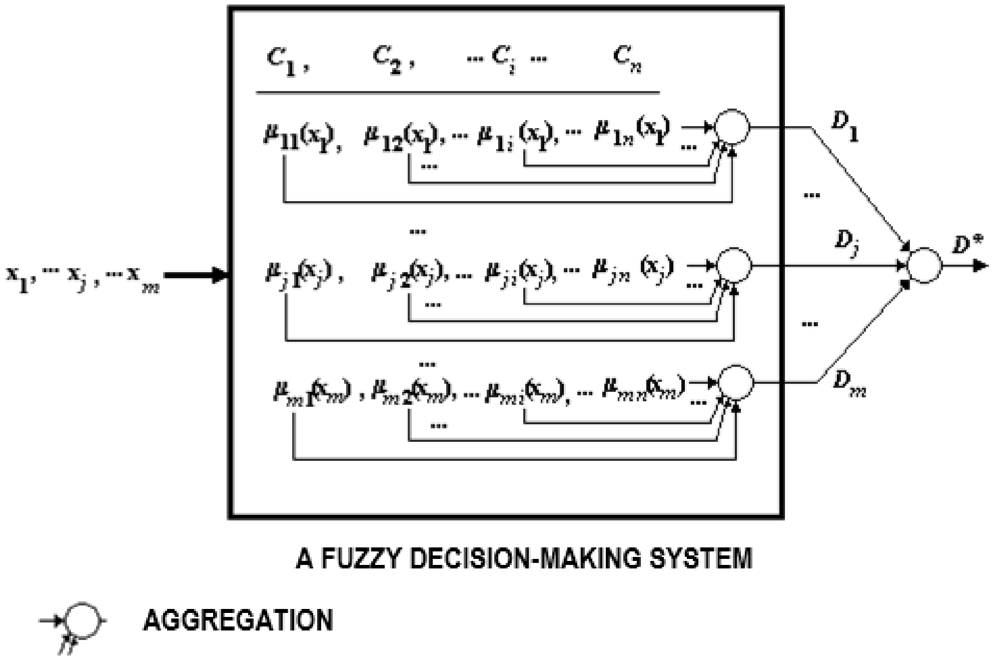

Figure 1.* *Aggregation in a type of fuzzy multicriteria decision-making system.

In Figure 1 meanings of symbols are as follows:

- $\mathbf{x}_i$, i = 1, 2, ..., m, are vectors of object properties, which are considered in decision-making process;
- C_j, j = 1, 2, ..., n, are decision-making criteria;
- $\mu_{ji}(\mathbf{x}_j)$, j = 1, 2, ..., m, i = 1,2, ...,n, $\mu_{ji} \in [0, 1]$, are scores – degrees in which an object $\mathbf{x}_i$ (or its property) satisfies the criteria C_j, μ_{ji} is the degree of fuzzy membership in a fuzzy set of object property that completely satisfies criterion C_j;
- D_i, i = 1, 2, ..., m, $D_i \in [0, 1]$, are decisions (performance indices) of an object $\mathbf{x}_i$ with respect to all the criteria C_j; decisions D_i are obtained by aggregation of information $\mu_{ji}(\mathbf{x}_j)$, using appropriate aggregation operation.
- The decision D^*, on object $\mathbf{x}_i$ that best satisfies all the criteria C_j, j = 1, 2, ..., m, is obtained by aggregation of decisions D_i – using suitable aggregation operation, appropriate for the considered problem.

The procedure used to combine the scores by which the object $\mathbf{x}_i$, or one of its characteristics, satisfies the criteria C_i into one decision D_j, i.e. D^*, is:

$$D_j = A_1(\mu_{j1}(\mathbf{x}_j), ..., \mu_{jn}(\mathbf{x}_j)), D^* = A_2(D_1, ..., D_m). \qquad (1)$$

The symbol A in the above expressions indicates aggregators. In the more general case, expressions (1) can be given in the form

$$a = A(a_1, ..., a_r), \qquad (2)$$

where a_j, j = 1, ... , r, $r \in \{n, m\}$, and a are values from interval of degrees of membership [0, 1].

Fuzzy operators, min for conjunction and max for disjunction, for A_1 or A_2 in (1), are to restrictive in practice and do not coincide with how people perform this operations. This lead to studies of other aggregators. In the huge majority of applications, primarily in decision-support systems, aggregators are developed as models of observable human reasoning.

So, we are interested in graded logic aggregators, i.e., aggregators that aggregate degrees of truth. Such aggregators are present in most decision problems. We assume that decision-making commonly includes evaluation of alternatives and selection of the most suitable alternative, Figure 1.

Some other examples of applications of fuzzy set theory, for modelling complex and perhaps incompletely defined systems, use knowledge bases in which knowledge is represented by a base of fuzzy rules. These applications include fuzzy rule-based systems (and fuzzy logic control). What is typical for these situations is the set of rules, which emphasizes the aggregation components, also.

DEFINITION AND CLASSES OF AGGREGATORS

Let us aggregate n degrees of truth $\mathbf{x} = (x_1,...,x_n)$, $n > 1$, $x_i \in I = [0, 1]$, i=1, ..., n. A *general logic aggregator* A: $I^n \rightarrow I$ is defined as a continuous function that is nondecreasing in all components of $\mathbf{x}$:

$\mathbf{x} \leq \mathbf{y}$ implies $A(\mathbf{x}) \leq A(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, (nondecreasing monotonicity);

and satisfies the boundary conditions (idempotency in extreme points):

$$A(\underbrace{0, 0, \ldots, 0}_{n \text{ times}}) = 0 \text{ and } A(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}) = 1.$$

It is assumed that the vector inequality is componentwise.

Typical examples of aggregators are: weighted means, medians, *OWA* operators and *t*-norms / *t*-conorms. But there are many other aggregators and an infinite number of aggregator members in most families. Not all aggregators have the same properties, so they are grouped into separate classes according to the properties they satisfy.

ClassES of aggregators

Some classes of aggregators are, as follows:

conjunctive aggregators A have the following property:
$\min(\mathbf{x}) \geq A(\mathbf{x})$;
disjunctive aggregators A have the following property:
$A(\mathbf{x}) \geq \max(\mathbf{x})$;
averaging aggregators A if they are bound by:
$min(\mathbf{x}) = \min_{i=1,\ldots,n} x_i \leq A(\mathbf{x}) \leq \max_{i=1,\ldots,n} x_i = max(\mathbf{x})$;
mixed, if they are neither conjunctive, disjunctive or averaging;
idempotent, if $A(t, \ldots, t) = t$ for any $t \in [0, 1]$;
symmetric (commutative) if $A(\mathbf{x}) = A(\mathbf{x}_P)$ for any $\mathbf{x} \in [0, 1]^n$ and any permutation P of $\{1, \ldots, n\}$.

Monotonicity and idempotency implies averaging behavior.

Main classes

Conjunctive/Disjunctive Aggregators

For this class of aggregators holds duality: for strong negation N,

$A_N(\mathbf{x}) = N(A(N(\mathbf{x}))$,

is N-dual of operator A.

In a special case of standard negation:

$A_d(x_1, x_2, \ldots, x_n) = 1 - A(1 - x_1, 1 - x_2, \ldots, 1 - x_n)$.

Duals of conjunctive operators are disjunctive operators, and vice versa, duals of disjunctive operators are conjunctive operators

Among conjunctive/disjunctive aggregators are t (triangular) norms – conorms, copulas and their duals, and others, [3].

Averaging Operators

Averaging operators model trade-offs between goals. These include:

- weighted arithmetic means: $M_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$, $\sum_{i=1}^{n} w_i = 1$ (in the general case they do not satisfy the condition of commutativity);
- weighted quasi-arithmetic means: $M_{\mathbf{w},g}(\mathbf{x}) = g^{-1}(\sum_{i=1}^{n} w_i g(x_i))$;

- ordered weighted averaging (Yager) $OWA_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}$;
- generalised ordered weighted averaging: $OWA_{\mathbf{w},g}(\mathbf{x}) = g^{-1}(\sum_{i=1}^{n} w_i g(x_{(i)}))$;
- weighted ordered weighted averaging ($WOWA$) aggregator (Torra), combines advantages of OWA operator and weighted means operator;
- other means (identric, logarithmic, …);
- median, weighted median, quasi-median;
- fuzzy integrals: Choquet**,** Sugeno, and particular cases; (the Choquet integral allows expressing interaction between criteria in multicriteria decision-making, and, for example, expressing (physician's) preferences [4]).

Other Aggregators, Not Conjunctive/Disjunctive Or Averaging

In that class of aggregating operators are uninorms, nulnorms, T-S operators, symmetric sums, and others operators.

COMPENSATORY AGGREGATORS

Fuzzy logic theory offers a multitude of connectives that can be used as aggregators to aggregate membership values representing uncertain information. These operators can be classified, as we have seen, into the following three general classes: conjunction, disjunction (Section 3.2.1), and compensation operators (Section 3.2.2). In the case of Zadeh's operators, *min* for conjunction and *max* for disjunction, used as aggregators, only inputs with extreme values affect the value of the output fuzzy set. However, both intuitive and formal criteria of human reasoning contain numerous requirements that are combined using models of simultaneity and substitutability (partial conjunction and partial disjunction), which set requirements for further development of fuzzy aggregators. In [5], logic operators based on continuous transition from conjunction to disjunction, were introduced, see also [6]. Results from [5] were strong contribution to development of aggregation as part of a soft computing. Those results, [6], improve Zadeh's approach in dealing with uncertain and vague information common in human reasoning.

So, any operator A, that, for example, applies to two arguments a_1 and a_2 from [0, 1], is compensatory operator if it satisfies the following:

$\min(a_1, a_2) \le A(a_1, a_2) \le \max(a_1, a_2)$.

After [5], others also dealt with this issue of compensatory operators, the review is given in [3; p.183].

The disjunction (union) operator provides full compensation, and the conjunction (intersection) operator does not allow compensation. The arithmetic mean is neutral in terms of disjunction and conjunction. It represents the midpoint between them and represents a special case of weighted averaging.

In [5] andness and orness were defined by Dujmović as the level of simultaneity and substitutability, respectively, of the aggregation. They are defined in terms of the similarity to minimum and maximum, respectively. Andness was introduced as a degree of conjunction, Orness was introduced as a degree of disjunction. A high orness permits that a bad criteria be compensated by a good one. On the other hand, a high andness requires both criteria to be satisfied to a great degree. Andness and orness are related and add up to one. So, andness-directed transition from conjunction to disjunction (introduced in 1973 to its current status [6]), is the history of an effort to interpret aggregation as a soft computing propositional calculus.

In some cases, we need to consider stronger functions in the sense that the outcome of an aggregation is less than the minimum or it is larger than the maximum. Fuzzy logic provides these type of operators, they are called t-norms and t-conorms, ($xy \le \min(x, y)$, product t-norm is still more conjunctive than minimum). Because of this relationship, while minimum has an andness equal to one, product t-norm has an andness that is larger than one. When operators are between minimum and maximum, andness is for any number of inputs in the range [0, 1]. Operators that can return values smaller than the minimum (as t-norms) or larger than the maximum (as t-conorms) will provide andness outside [0, 1], reaching the minimum and the maximum of the interval with drastic disjunction and drastic conjunction [6].

The resulting analytic framework is a graded logic [6], based on analytic models of graded simultaneity (various forms of conjunction), graded substitutability (various forms of disjunction) and complementing (negation).

Basic graded logic functions can be *conjunctive, disjunctive,* or *neutral*. Conjunctive functions have andness α greater than orness ω, $\alpha > \omega$. Similarly, disjunctive functions have orness greater than andness, $\alpha < \omega$, and neutral is only the arithmetic mean where $\alpha = \omega = ½$. Between the drastic conjunction and the drastic disjunction, we have andness-directed logic aggregators that are special cases of a fundamental logic function called *graded* *conjunction/disjunction* (GCD) [6]. GCD has the status of a logic aggregator, and it can be idempotent or nonidempotent, as well as *hard* (supporting annihilators) or *soft* (not supporting annihilators). The annihilator of hard conjunctive aggregators is 0, and the annihilator of hard disjunctive aggregators is 1.

The whole range of conjunctive aggregators is presented in Figure 2 [6].

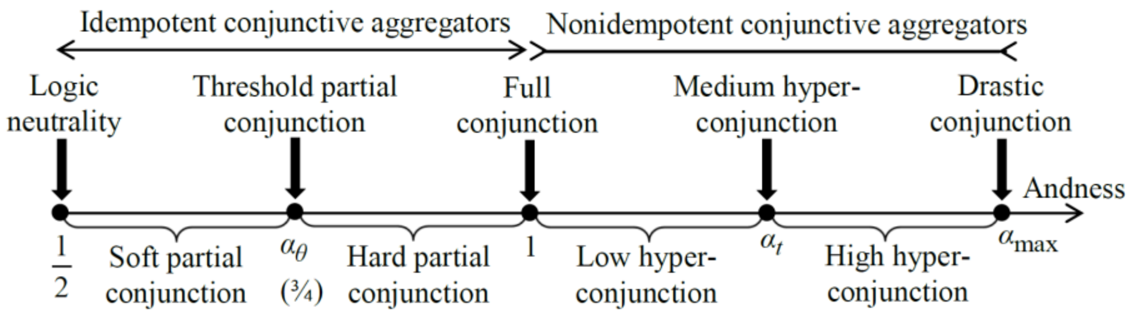

Figure 2. The range of conjunctive aggregators: border aggregators and aggregation segments [6].

A detailed classification of GCD aggregators, based on combinations conjunctive/ disjunctive, idempotent/nonidempotent, and hard/soft aggregators is presented in Table 1 [6].

Table 1. Classification of andness-directed graded logic (GL) functions and aggregators [6].

		Logic function/aggregator	I	T	A	Global andness (α)	
GRADED LOGIC FUNCTIONS	CONJUNCTIVE	Drastic conjunction	N	H	0	$\alpha = \alpha_{max} = n / (n-1)$	
		High hyperconjunction	N	H	0	$\alpha_t < \alpha < \alpha_{max}$	BASIC GL AGGREGATORS
		Medium hyperconjunction	N	H	0	$\alpha = \alpha_t = (n\,2^n - n - 1) / (n-1)\,2^n$	
		Low hyperconjunction	N	H	0	$1 < \alpha < \alpha_t$	
		Full conjunction	Y	H	0	$\alpha = 1$	
		Hard partial conjunction	Y	H	0	$\alpha_\theta \le \alpha < 1$; $½ < \alpha_\theta < 1$	
		Soft partial conjunction	Y	S	-	$½ < \alpha < \alpha_\theta$	
	Neutrality		Y	S	-	$\alpha = ½$	
	DISJUNCTIVE	Soft partial disjunction	Y	S	-	$1 - \alpha_\theta \le \alpha < ½$	
		Hard partial disjunction	Y	H	1	$0 < \alpha \le 1 - \alpha_\theta$	
		Full disjunction	Y	H	1	$\alpha = 0$	
		Low hyperdisjunction	N	H	1	$1 - \alpha_t < \alpha < 0$	
		Medium hyperdisjunction	N	H	1	$\alpha = 1 - \alpha_t$	
		High hyperdisjunction	N	H	1	$\alpha_{min} < \alpha < 1 - \alpha_t$	

		Drastic disjunction	N	H	1	$\alpha = \alpha_{min} = -1/(n-1)$
Columns: **I** = idempotent, **Y/N =** yes/no; **T =** type, **H/S** = hard/soft; **A** = annihilator						

All disjunctive aggregators can be realized as De Morgan duals of conjunctive aggregators, so, it is sufficient to analyse only the conjunctive aggregators.

In the case of using weighted conjunctive means or weighted disjunctive means as a aggregator, the value of the output fuzzy set is affected by all the inputs or by outputs of all rules in a case of a fuzzy rule-based system.

In addition, weighted conjunctive means and weighted disjunctive means enable continuous weighting of the influence of the output of individual rules on the total output fuzzy set.

Choosing aggregator

The aggregator is chosen on the basis of available data about modelled system and about application requirements of developed (fuzzy) intelligent system. Requirements are translated into mathematical properties: idempotency, neutral element, commutativity, and similar. These mathematical properties, in turn, define the class of aggregators. The data allows us to select specific members of the aggregator families that are best suited to the data.

In the case of GCD aggregator, the specification of requirements for aggregator consists of choosing features of an aggregator: *idempotent or nonidempotent*; *simultaneity or substitutability*; *hard or soft*; *the desired strength of simultaneity/substitutability* (andness/or-ness); *the degrees of importance.*

The andness-directed interpolative method for implementing GCD [6], consists of implementing the border aggregators shown in Figure 2 and then using interpolative aggregators in the range of andness between them. This method can be used to implement all logic GCD aggregators shown in Table 1.

The family of graded logic functions and similar aggregators, investigated in [6], includes GCD (introduced in 1973), various OWA aggregators (introduced in 1988), aggregators based on fuzzy integrals (introduced in 1974), and various means (introduced more than 2000 years ago).

CONCLUSIONS

Models of combining information are integral parts of the methods of implementation of artificial-intelligent systems. In many applications, and especially in the development of artificial-intelligent systems, there is a need to aggregate not only numerical, but also linguistic, qualitative, organized information.

Research in the field of aggregators includes purely theoretical studies (which include sophisticated mathematics), the development of practical aggregation tools (programming), as well as the applications of aggregators.

The process of aggregating information occurs in many applications related to the development of not only fuzzy systems but also other intelligent systems: neural networks, vision systems, robotics, multicriteria decision making systems in general, robotic networks (for example, platforms in smart cities [7], Self-Driving car networks [8]) and others. Aggregators represent a current research topic [9, 10]. For example, in [11] a new approach is proposed upon which a new theory of aggregation could be developed. The aggregation method dealing with so called order-2 fuzzy sets is considered in [12]. Work is also underway on the development of aggregators for aggregating arguments of various natures (numerical, qualitative, mixed), as well as on the systems for determining the parameters of aggregators (learning systems).

REFERENCES

[1] Russell, S. and Norvig, P.: *Artificial Intelligence: A Modern Approach.* Third Edition. Princeton Education, p.1132, 2010. Translated by: RAF, Beograd, p.1236, 2011,

[2] Zadeh, L.A.: *Fuzzy sets*. Information and Control **8**, 338-353, 1965, http://dx.doi.org/10.1016/S0019-9958(65)90241-X,

[3] Šaletić, D.Z.: *Fuzzy sets and fuzzy-logical systems.* In Serbian. Eduka, Beograd, p.458, 2012,

[4] Šaletić, D.Z.; Savić, M. and Zurovac, A.: *Improvement in Implementation of Fuzzy Decision Making in Determining the Severity of Respiratory Distress*. 8th International Symposium on Intelligent Systems and Informatics. IEEE, Subotica, pp.47-52, 2010,

[5] Dujmović, J.: *A generalization of some functions in continuous mathematical logic – Evaluation function and its applications*. In Serbo-Croatian.
Informatica Conference. Bled, paper d27, 1973,

[6] Dujmović, J. and Torra, V.: *Logic Aggregators and Their Implementations*.
20th International Conference on Modelling Decisions for Artificial Intelligence. Umeå, pp.1-40, 2023,

[7] Simon, J. and Mester, G.: *Critical Overview of the Cloud-Based Internet of Things Pilot Platforms for Smart Cities*.
Interdisciplinary Description of Complex Systems **16**(3-A), 397- 407, 2018,
http://dx.doi.org/10.7906/indecs.16.3.12,

[8] Pisarov, J. and Mester, G.: *The Future of Autonomous Vehicles*.
FME Transactions **49**(1), 29-35, 2021,
http://dx.doi.org/10.5937/fme2101029P,

[9] Grabisch, M.; Marichal, J.L.; Mesiar, R. and Pap, E.: *Aggregation Functions*.
Cambridge University Press, New York, p.460, 2009,

[10] Kishar, A.; Singh A.K.; Sonam, S. and Pal, N.R.: *A New Family of OWA Operators Featuring Constant Orness*,
IEEE Transactions on Fuzzy Systems **28**(9), 2263-2269, 2020,
http://dx.doi.org/10.1109/TFUZZ.2019.2928519,

[11] Perez-Fernandez, R. and De Baets, B.: *Aggregation Theory Revisited*.
IEEE Transactions on Fuzzy Systems **29**(4), 798-804, 2021,
http://dx.doi.org/10.1109/TFUZZ.2021.3066708,

[12] Pedrycz, W.; Gacek, A. and Wang, X.: *Aggregation of Order – 2 Fuzzy Sets*.
IEEE Transactions on Fuzzy Systems **29**(11), 3570-3575, 2021,
http://dx.doi.org/10.1109/TFUZZ.2020.3025659.