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The Edge Szeged Index of Product Graphs

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The edge Szeged index of a molecular graph G is defined as the sum of products $m_u(e|G)m_v(e|G)$ over all edges $e = uv$ of G, where $m_u(e|G)$ is the number of edges whose distance to vertex u is smaller than the distance to vertex v, and where $m_v(e|G)$ is defined analogously. The aim of this paper is to compute the edge Szeged index of the Cartesian product of graphs. As a consequence of our result, the edge Szeged index of Hamming graphs and C_4-nanotubes are computed.

INTRODUCTION

A graph G is defined as a pair G = (V,E), where V = V(G) is a non-empty set of vertices and E = E(G) is a set of edges. Throughout this paper the term »graph« means finite graph in which the set of vertices V is finite. Chemical graphs are just graph-based descriptions of molecules, with vertices representing the atoms and edges representing the bonds. A numerical invariant associated with a chemical graph, especially if it is of chemical significance and/or applicability, is called topological index.

The Wiener index W is the first topological index introduced by the chemist Harold Wiener for investigating boiling points of alkanes.[1] After Wiener, many topological indices were proposed by chemist and also by mathematicians. The Szeged index is a topological index related to W, introduced by one of the present authors,[2] denoted here by Sz_v. To define the Szeged index of a graph G, we assume that $e = uv$ is an edge connecting the vertices u and v. Suppose that $n_u(e|G)$ is the number of vertices of G lying closer to u than to v, and that $n_v(e|G)$ is the number of vertices of G lying closer to v than to u. Vertices equidistant to u and v are not taken into account. Then the Szeged index of the graph G is defined as $Sz_v(G) = \Sigma_{e=uv\in E(G)} n_u(e|G)n_v(e|G)$. For more information about the Szeged index and its mathematical properties one should consult the articles.[3–7]

Suppose that u and v are vertices of G. The distance $d(u,v)$ is defined as the length of a minimal path between u and v. If $e = xy$ is an edge of G then $d(u,e) = \mathrm{Min}\{d(u,x),d(u,y)\}$.

In the Ref. 8, an edge version of the Szeged index was introduced, named »*edge Szeged index*«. This new index is defined as $Sz_e(G) = \Sigma_e m_u(e|G)m_v(e|G)$ where $m_u(e|G)$ is the number of edges whose distance to vertex u is smaller than the distance to vertex v, and where $m_v(e|G)$ is defined analogously.

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Throughout this paper our notation is standard and taken mainly from the books.[9–11] Following Imrich and Klavžar,[11] the Cartesian product $G \times H$ of two graphs G and H is defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. The edge set $E(G \times H)$ is the set of all pairs $[(u,v),(x,y)]$ of vertices for which either $u = x$ and $[v,y] \in E(H)$ or $[u,x] \in E(G)$ and $v = y$. Thus $V(G \times H) = V(G) \times V(H)$ and
$E(G \times H) = \{[(u,v),(x,y)] \mid u = x, [v,y] \in E(H)$, or, $[u,x] \in E(G), v = y\}$. Moreover, K_n denotes a complete graph with n vertices and a hypercube of dimension n, Q_n, is the Cartesian product of n copies of K_2.

MAIN RESULTS AND DISCUSSION

For investigating the mathematical properties of a given topological index χ, it is important to compute $\chi(G \times H)$, for every graphs G and H. One should recall that numerous molecular graphs are of the form $G \times H$. In the papers[12–14] this problem was solved for the Wiener, vertex Szeged, and PI index. In this section we continue along the same lines, and compute the edge Szeged index of product graphs.

We first introduce a further »edge-vertex Szeged index« denoted here by Sz_{ev}. It is defined as $Sz_{ev}(G) = 1/2\Sigma_{uv=e\in E(G)}[n_u(e|G)m_v(e|G) + n_v(e|G)m_u(e|G)]$, where $n_u(e|G)$, $n_v(e|G)$, $m_u(e|G)$ and $m_v(e|G)$ are defined in Section 1.

Lemma 1. – Suppose G is an acyclic graph with exactly n vertices. Then $Sz_{ev}(G) = W(G) – n(n – 1)/2$.

Proof: By a result of Dobrynin and Gutman,[15] $Sz(G) = W(G)$ for acyclic graphs G. So,

$Sz_{ev}(G) =$

$1/2\Sigma_{uv=e\in E(G)}[n_u(e|G)m_v(e|G) + n_v(e|G)m_u(e|G)] =$

$1/2\Sigma_{uv=e\in E(G)}[n_u(e|G)(n_v(e|G)–1) + n_v(e|G)(n_u(e|G)–1)] =$

$1/2\Sigma_{uv=e\in E(G)}[2\, n_u(e|G)n_v(e|G) – (n_u(e|G)+n_v(e|G))] =$

$Sz_v(G) – n(n – 1)/2 = W(G) – n(n – 1)/2.$ ■

In order to prove our main result, we assume that $V(G) = \{u_1,u_2,\ldots,u_r\}$, $V(H) = \{v_1,v_2,\ldots,v_s\}$, $A_m = \{[(u_m,v_i),(u_m,v_j)] \mid v_iv_j \in E(H)\}$, $1 \le m \le r$, and $B_n = \{[(u_i,v_n),(u_j,v_n)] \mid u_iu_j \in E(G)\}$, $1 \le n \le s$. It is easy to see that the A_m's and B_n's are partitions of the edge set of $G \times H$. We also assume that $C_m = \{(u_m,v_i) \mid v_i \in V(H)\}$, $1 \le m \le r$, and $D_n = \{(u_i,v_n) \mid u_i \in V(G)\}$, $1 \le n \le s$. Clearly, the C_m's and also D_n's are two partitions of $V(G \times H)$.

The following Lemma is crucial for our main result.

Lemma 2. – With the above-specified notation we have:

(i) For every $[(u_p,v_k),(u_p,v_l)] \in A_p$,
$d_{G\times H}((u_p,v_k),[(u_t,v_i),(u_t,v_j)]) <$
$d_{G\times H}((u_p,v_l),[(u_t,v_i),(u_t,v_j)])$
if and only if $d_H(v_k,[v_i,v_j]) < d_H(v_l,[v_i,v_j])$.

(ii) $d_{G\times H}((u_p,v_k), (u_i,v_t)) < d_{G\times H}((u_p,v_l), (u_i,v_t))$
if and only if $d_H(v_k,v_t) < d_H(v_l, v_t)$.

Proof: (i) and (ii) are immediate consequences of Corollary 1.35 in the book.[11] ■

Lemma 3. – For every $[(u_p,v_k),(u_p,v_l)] \in A_p$,

(i) $m_{(u_p,v_k)}([(u_p,v_k),(u_p,v_l)]|G \times H) =$
$|V(G)|m_{v_k}([v_k,v_l]|H) + |E(G)|n_{v_k}([v_k,v_l]|H)$,

(ii) $n_{(u_p,v_k)}([(u_p,v_k),(u_p,v_l)]|G \times H = |V(G)|n_{v_k}([v_k,v_l]|H)$.

Proof: Directly from the definition, we have $m_{v_k}([v_k,v_l]|H = |\{[v_i,v_j] \in E(H)| d_H(v_k,[v_i,v_j]) < d_H(v_l,[v_i,v_j])\}|$ whereas by Lemma 2(i), $|\{[(u_t,v_i),(u_t,v_j)] \in A_t| d_{G\times H}((u_p,v_k),[(u_t,v_i),(u_t,v_j)]) < d_{G\times H}((u_p,v_l),[(u_t,v_i),(u_t,v_j)])\}| = m_{v_k}([v_k,v_l]|G)$. Since the A_i's are disjoint, $|\{ e \in \cup A_m | d_{G\times H}((u_p,v_k),e) < d_{G\times H}((u_p,v_l),e)\}| = |V(G)|m_{v_k}([v_k,v_l]|G)$. On the other hand, $n_{v_k}([v_k,v_l]|H) = |\{ v_t \in V(H)| d_H(v_k,v_t) < d_H(v_l,v_t)\}|$. Therefore By Lemma 2(ii), $|\{[(u_i,v_t),(u_j,v_t)] \in B_t| d_{G\times H}((u_p,v_k),[(u_i,v_t),(u_j,v_t)]) < d_{G\times H}((u_p,v_l), [(u_i,v_t),(u_j,v_t)])\}| = \deg_G(u_i)n_{v_k}([v_k,v_l]|H)$. Since the B_i's are disjoint, $|\{ e \in \cup B_n \mid d_{G\times H}((u_m,v_k),e) < d_{G\times H}((u_m,v_l),e)\}| = (1/2\Sigma_{l\le i\le r} \deg_G(u_i))n_{v_k}([v_k,v_l]|H)$. But $(\cup A_m) \cap (\cup B_n) = \varnothing$, so $m_{(u_p,v_k)}([(u_p,v_k),(u_p,v_l)]|G \times H) = |\{e \in (\cup A_m) \cup (\cup B_n)| d((u_p,v_k),e) < d((u_p,v_l),e)\}| = |V(G)|\ m_{v_k}([v_k,v_l]|H) + |E(G)|n_{v_k}([v_k,v_l]|H)$.

This completes the proof of (i). The proof of (ii) is similar and so it is omitted. ■

Theorem 1. – $Sz_e(G \times H) =$
$|V(G)|^3Sz_e(H) + 2|E(G)||V(G)|^2Sz_{ev}(H) +$
$|V(G)||E(G)|^2Sz_v(H) + |V(H)|^3Sz_e(G) +$
$2|E(H)||V(H)|^2Sz_{ev}(G) + |V(H)||E(H)|^2Sz_v(G)$.

Proof: Since $(\cup A_m) \cap (\cup B_n) = \varnothing$,
$Sz_e(G \times H) = \Sigma_{e\in\cup A_m}m_u(e|G \times H)\, m_v(e|G \times H) + \Sigma_{e\in\cup B_n}m_u(e|G \times H)m_v(e|G \times H)$. On the other hand, By Lemma 3

$\Sigma_{e\in A_m}m_u(e|G \times H)m_v(e|G \times H) =$
$\Sigma_{[v_i,v_j]\in E(H)}(|V(G)|^2m_{v_i}([v_i,v_j]|H)m_{v_j}([v_i,v_j]|H) +$
$|E(G)|^2n_{v_i}([v_i,v_j]|H)n_{v_j}([v_i,v_j]|H) +$
$|E(G)||V(G)|(m_{v_j}([v_i,v_j]|H)n_{v_i}([v_i,v_j]|H) +$
$m_{v_i}([v_i,v_j]|H)n_{v_j}([v_i,v_j]|H))$.

Then we have $\Sigma_{e\in\cup A_m}m_u(e|G \times H)m_v(e|G \times H) = |V(G)|^3Sz_e(H) + |E(G)|^2|V(G)|Sz_v(H) + 2|V(G)|^2|E(G)|Sz_{ev}(H)$. Since $H \times G \cong G \times H$, for computing $\Sigma_{e\in\cup B_n}m_u\ (e|G\times H)m_v(e|G\times H)$, it suffices to replace G and H in Eq. (1), which completes our proof. ■

Theorem 2. – $Sz_{ev}(G \times H) = |V(G)|^3Sz_{ev}(H) + |E(G)||V(G)|^2Sz_v(H) + |V(H)|^3Sz_{ev}(G) + |E(H)||V(H)|^2Sz_v(G)$.

Proof: By definition, we have:

$$Sz_{ev}(G\times H) = 1/2\Sigma_{e\in\cup A_m}[m_u(e|G\times H)n_v(e|G\times H) + m_v(e|G\times H)n_u(e|G\times H] + 1/2\Sigma_{e\in\cup B_n}[m_u(e|G\times H)n_v(e|G\times H) + m_v(e|G\times H)n_u(e|G\times H)] \qquad (2)$$

On the other hand, By Lemma 3,

$$\Sigma_{e\in\cup A_m}[m_u(e|G\times H)n_v(e|G\times H) + m_v(e|G\times H)n_u(e|G\times H)] = \Sigma_{w\in V(G)}\Sigma_{e=uv\in E(H)}[2|V(G)||E(G)|n_u(e|H)n_v(e|H) + |V(G)|^2(m_u(e|H)n_v(e|H) + m_v(e|H)n_u(e|H)] = |V(G)|^3Sz_{ev}(H) + |E(G)||V(G)|^2Sz_v(H).$$

Similarly, one can compute the second summation of Eq. (2). ■

Suppose $\otimes_{i=1}^{n} G_i$ denotes the Cartesian product of $G_1, G_2, \dots, G_n$. If $G_1 = G_2 = \dots = G_n = G$ then we write G^n as $\otimes_{i=1}^{n} G_i$. Also, $E^c_{n,i} = E(\otimes_{k=1,k\neq i}^{n} G_k)$ and $V^c_{n,i} = V(\otimes_{k=1,k\neq i}^{n} G_k)$. Suppose $E_i = E(G_i)$ and $V_i = V(G_i)$, $1 \le i \le n$. Then obviously $|V^c_{n,i}| = \prod_{j=1,j\neq i}^{n}|V_j|$ and by Corollary 1.35 of the book of Imrich and Klavžar,[11]

$$|E(G\times H)| = \frac{1}{2}\Sigma_{(a,b)\in V(G\times H)}\deg_{G\times H}((a,b)) = \frac{1}{2}\Sigma_{a\in V(G)}\Sigma_{b\in V(H)}[\deg_G(a) + \deg_H(b)] = |E(G)||V(H)| + |E(H)||V(G)|.$$

An inductive argument shows that

$$|E^c_{n,i}| = \Sigma_{j=1,j\neq i}^{n}\frac{|E_j|\cdot|V^c_{n,j}|}{|V_i|} \qquad (3).$$

Klavžar, Rajapakse, and Gutman[12] proved that $Sz_v(G\times H) = |V(G)|^3Sz_v(H) + |V(H)|^3Sz_v(G)$. Again, by means of mathematical induction one can see that $Sz_v(\otimes_{i=1}^{n} G_i) = \Sigma_{i=1}^{n}|V^c_{n,i}|^3\, Sz_v(G_i)$. Therefore, we arrive at the following generalizations of Theorems 1 and 2:

Theorem 3. –

(i) $Sz_{ev}(\otimes_{i=1}^{n} G_i) = \Sigma_{i=1}^{n}[|V^c_{n,1}|^3Sz_{ev}(G_i) + |E^c_{n,i}||V^c_{n,i}|^2Sz_v(G_i)]$,

(ii) $Sz_e(\otimes_{i=1}^{n} G_i) = \Sigma_{i=1}^{n}[|V^c_{n,i}|^3Sz_e(G_i) + 2|E^c_{n,i}||V^c_{n,i}|^2\, Sz_{ev}(G_i)] + |E^c_{n,i}|^2|V^c_{n,i}|Sz_v(G_i)$.

Proof: (I) In Theorem 2, we proved the case of $n = 2$. Suppose the result is valid for n.

$$Sz_{ev}(\otimes_{i=1}^{n+1} G_i) = Sz_{ev}(G_{n+1}\times\otimes_{i=1}^{n} G_i) =$$

$$|V_{n+1}|^3Sz_{ev}(\otimes_{i=1}^{n} G_i) + |E_{n+1}||V_{n+1}|^2Sz_v(\otimes_{i=1}^{n} G_i) +$$

$$|V^c_{n+1,n+1}|^3\, Sz_{ev}(G_{n+1}) + |E^c_{n+1,n+1}||V^c_{n+1,n+1}|^2\, Sz_v(G_{n+1}) =$$

$$\Sigma_{i=1}^{n}|V^c_{n+1,i}|^3\, Sz_{ev}(G_i) + |V^c_{n+1,n+1}|^3\, Sz_{ev}(G_{n+1}) +$$

$$\Sigma_{i=1}^{n}|E^c_{n,i}||V_{n+1}||V^c_{n+1,i}|^2\, Sz_v(G_i) +$$

$$\Sigma_{i=1}^{n}\frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|}\, Sz_v(G_i) +$$

$$|E^c_{n+1,n+1}||V^c_{n+1,n+1}|^2\, Sz_v(G_{n+1}) =$$

$$\Sigma_{i=1}^{n+1}|V^c_{n+1,i}|^3\, Sz_{ev}(G_i) +$$

$$\Sigma_{i=1}^{n}|V^c_{n+1,i}|^2\, Sz_v(G_i)\left(|V_{n+1}||E^c_{n,i}| + \frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|}\right) +$$

$$|E^c_{n+1,n+1}||V^c_{n+1,n+1}|^2\, Sz_v(G_{n+1})\,. \qquad (4)$$

Apply Eq. (3) and our formula for $|V^c_{n,i}|$, we have:

$$|V_{n+1}||E^c_{n,i}| + \frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|} =$$

$$|V_{n+1}|\,\Sigma_{j=1,j\neq i}^{n}\frac{|E_j|\cdot|V^c_{n,j}|}{|V_i|} + \frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|} =$$

$$\Sigma_{j=1,j\neq i}^{n}\frac{|E_j|\cdot|V_{n+1}|\cdot|V^c_{n,j}|}{|V_i|} + \frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|} =$$

$$\Sigma_{j=1,j\neq i}^{n}\frac{|E_j|\cdot|V^c_{n+1,j}|}{|V_i|} + \frac{|E_{n+1}|\cdot|V^c_{n+1,i}|}{|V_{n+1}|} =$$

$$\Sigma_{j=1,j\neq i}^{n+1}\frac{|E_j|\cdot|V^c_{n+1,j}|}{|V_i|}\,.$$

By this equation and Eq. (4), one can see that

$$Sz_{ev}(\otimes_{i=1}^{n+1} G_i) =$$

$$\Sigma_{i=1}^{n+1}\,[|V^c_{n+1,i}|^3\, Sz_{ev}(G_i) + |E^c_{n+1,i}||V^c_{n+1,i}|^2\, Sz_v(G_i)].$$

(II) In Theorem 1, we proved the case of $n = 2$ and so we can assume that the result is valid for $n > 2$. Then

$$Sz_e(G_{n+1}\times\otimes_{i=1}^{n} G_i) = |V_{n+1}|^3Sz_e(\otimes_{i=1}^{n} G_i) +$$

$$2|E_{n+1}||V_{n+1}|^2Sz_{ev}(\otimes_{i=1}^{n} G_i) +$$

$$|E_{n+1}|^2|V_{n+1}|Sz_v(\otimes_{i=1}^{n} G_i) + |V^c_{n+1,n+1}|^3Sz_e(G_{n+1}) +$$

$$2|E^c_{n+1,n+1}||V^c_{n+1,n+1}|^2\, Sz_{ev}(G_{n+1}) +$$

$$|E^c_{n+1,n+1}|^2|V^c_{n+1,n+1}|\, Sz_v(G_{n+1}) =$$

$$\Sigma_{i=1}^{n}|V^c_{n+1,i}|^3\, Sz_e(G_i) + |V^c_{n+1,n+1}|^3\, Sz_e(G_{n+1}) +$$

$$2\Sigma_{i=1}^{n}|E^c_{n,i}||V_{n+1}||V^c_{n+1,i}|^2\, Sz_{ev}(G_i) +$$

$$2\Sigma_{i=1}^{n}\frac{|E_{n+1}|\cdot|V^c_{n+1,i}|^3}{|V_{n+1}|}\, Sz_{ev}(G_i) +$$

$$2\,|E^c_{n+1,n+1}||V^c_{n+1,n+1}|^2\, Sz_{ev}(G_{n+1}) +$$

$$\Sigma_{i=1}^{n}|E_{n,i}^{c}|^2\,|V_{n+1}|^2\,|V_{n+1,i}^{c}|Sz_v(G_i) +$$
$$\Sigma_{i=1}^{n}\frac{|E_{n+1}|^2\cdot|V_{n+1,i}^{c}|^3}{|V_{n+1}|^2}Sz_v(G_i) +$$
$$2\Sigma_{i=1}^{n}|E_{n,i}^{c}||E_{n+1}||V_{n+1,i}^{c}|^2\,Sz_v(G_i) +$$
$$|E_{n+1,n+1}^{c}|^2\,|V_{n+1,n+1}^{c}|Sz_v(G_{n+1}) =$$
$$\Sigma_{i=1}^{n+1}|V_{n+1,i}^{c}|^3\,Sz_e(G_i) +$$
$$2\Sigma_{i=1}^{n}|V_{n+1,i}^{c}|^2\,Sz_{ev}(G_i)\left(|V_{n+1}||E_{n,i}^{c}|+\frac{|E_{n+1}|\cdot|V_{n+1,i}^{c}|}{|V_{n+1}|}\right)+$$
$$2|E_{n+1,n+1}^{c}||V_{n+1,n+1}^{c}|^2\,Sz_{ev}(G_{n+1}) +$$
$$\Sigma_{i=1}^{n}|V_{n+1,i}^{c}|Sz_v(G_i)\left(|V_{n+1}||E_{n,i}^{c}|+\frac{|E_{n+1}|\cdot|V_{n+1,i}^{c}|}{|V_{n+1}|}\right)+$$
$$|E_{n+1,n+1}^{c}|^2|V_{n+1,n+1}^{c}|\,Sz_v(G_{n+1}) =$$
$$\Sigma_{i=1}^{n+1}[|V_{n+1,i}^{c}|^3\,Sz_e(G_i) + 2|E_{n+1,i}^{c}||V_{n+1,i}^{c}|^2\,Sz_{ev}(G_i)] +$$
$$2\,|E_{n+1,i}^{c}|^2|V_{n+1,i}^{c}|\,Sz_v(G_i).$$

This proves the result. ■

As an immediate consequence of previous theorem, we have:

Corollary. –

(i) $Sz_{ev}(G^n) = n|V|^{3n-4}(|V|S_{ev}(G) + (n-1)\,|E|Sz_v(G))$,

(ii) $Sz_e(G^n) = n|V|^{3n-5}(|V|^2S_e(G) + 2(n-1)|V||E|Sz_{ev}(G) + (n-1)^2|E|^2Sz_v(G)$.

We now apply our result to compute the edge Szeged index of some well-known graphs. Following Imrich and Klavžar,[11] consider the graph G whose vertices are the r-tuples $b_1b_2\ldots b_r$ with $b_i \in \{0, 1, \ldots, n_i-1\}$, $n_i \geq 2$, and let two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a Hamming graph. It is well-known fact that a graph G is a Hamming graph if and only if it can be written in the form $G = \otimes_{i=1}^{N} K_{n_i}$. In the following example, the edge and edge-vertex Szeged indices of a Hamming graph are computed.

Example 1. – Consider the complete graph K_n. Then $Sz_v(K_n) = n(n-1)/2$, $Sz_e(K_n) = n(n-1)(n-2)^2/2$ and $Sz_{ev}(K_n) = n(n-1)(n-2)/2$.

Example 2. – Consider the hypercube Q_n. It is easy to see that $Sz_v(Q_n) = n2^{3n-3}$. Also, by Corollary of Theorem 3, $Sz_e(Q_n) = n(n-1)^2 2^{3n-5}$ and $Sz_{ev}(Q_n) = n(n-1)2^{3n-4}$.

Example 3. – In this example the edge Szeged index of a C_4 nanotube R is computed. By definition of Cartesian product of graphs $R = P_n \times C_m$, where P_n is a path on n vertices and C_m denotes the cycle on n vertices and m, n are positive integers. To compute the edge Szeged index of R, we first compute the verted, edge and edge-vertex Szeged indices of P_n and C_m. On can see that, $Sz_v(P_n) = \binom{n+1}{3}$, $Sz_e(P_n) = \binom{n+1}{3} - (n-1)^2$,

$$Sz_{ev}(P_n) = \binom{n+1}{3} - \binom{n+1}{2},$$

$$Sz_v(C_n) = \begin{cases} \dfrac{n^3}{4} & n \text{ is even} \\ \dfrac{n(n-1)^2}{4} & n \text{ is odd} \end{cases}$$

$$Sz_e(C_n) = \begin{cases} \dfrac{(n-4)(n-1)^2}{4} & n \text{ is even} \\ \dfrac{n(n-2)^2}{4} & n \text{ is odd} \end{cases} \quad \text{and}$$

$$Sz_{ev}(C_n) = \begin{cases} \dfrac{n^2(n-2)}{4} & n \text{ is even} \\ \dfrac{n(n-1)^2}{4} & n \text{ is odd} \end{cases}$$

Therefore by Theorem 3,

$$Sz_e(R) = \frac{1}{12}\times$$
$$\begin{cases} 16n^3m^3-26n^3m^2+27n^3m-12n^3- \\ 24n^2m^3+23nm^3-12m^3-16nm^2 & n \text{ is even} \\ 23nm^3-12n^2m-12m^3+3nm+12n^2m^2- \\ 26n^3m^2+21n^3m+16n^3m^3-24n^2m^3-22nm^2 & n \text{ is odd} \end{cases}$$

Example 4. – Suppose $H_{n_1,n_2,\ldots n_r}$ denotes the Hamming graph with parameters $n_1, n_2, \ldots, n_r$. By Theorem 3, one can compute a formula for computing edge Szeged index of $H_{n_1,n_2,\ldots n_r}$ For the edge-vertex Szeged index of this graph, we have Eq. (5)

$$Sz_{ev}(H_{n_1,n_2,\ldots n_r}) = \prod_{i=1}^{r} n_i^3\left(\frac{r}{4} - \frac{r+4}{4}\sum_{i=1}^{r}\frac{1}{n_i} + \frac{r+3}{4}\sum_{i=1}^{r}\frac{1}{n_i^2} + \frac{1}{4}\sum_{i,j=1}^{r}\frac{n_i}{n_j} - \frac{1}{4}\sum_{i,j=1}^{r}\frac{n_i}{n_j^2}\right). \tag{5}$$

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SAŽETAK

Bridni Segedinski indeks produkta grafova

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Bridni Segedinski indeks molekuskog grafa G je definiran kao zbroj produkata $m_u(e|G)m_v(e|G)$ preko svih bridova $e = uv$ grafa G, gdje je $m_u(e|G)$ broj bridova čija je udaljenost od vrha u manja nego udaljenost od vrha v, i gdje je $m_v(e|G)$ definiran analogno. U ovom radu određen je bridni Segedinski indeks za Kartezijev produkt grafova. Pomoću ovog rezultata izračunati su bridni Segedinski indeksi Hammingovih grafova te C_4-nanocijevi.