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Touchard Series for Solving Volterra Integral Equations Form of the Lane-Emdan Equations

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Abstract: This study is concerned by the Collocation method for solving Volterra integral equations form of the Lane-Emden type numerically through the so-called Touchard polynomials, which are essentially binomial polynomials. Where the system of miniature linear equations is solved numerically using the MATALEB program. Then some examples are presented to verify the reliability and effectiveness of the method in addition to the speed of its convergence.

Keywords: Collocation Method; Lane-Emden equation; Touchard polynomials; Volterra integral equations

1 INTRODUCTION

Let the Lane-Emden equation be in the following form [20]

$$y''+\frac{\mu}{x}y'+\beta f(x)g(y)=0,\ \mu\geq 0,\ 0<x\leq 1,\ \beta\geq 0. \tag{1}$$

With the following initial conditions

$$y(0)=a,\ y'(0)=0.$$

Where f and g are given functions of x and y respectively. In physics and astronomy, the above equation can represent many problems, depending on the change of β.

$$y''+\frac{\mu}{x}y'+f(x)g(y(x))=h(x),\ \mu\geq 0,\ x>0.$$

Where $y(x)$ represents temperature. In the steady state and when $\mu = 2$, we call this type the generalized Emden-Fowler equation. As we know, the Volterra integral equation has been raised in many scientific applications, and many numerical solutions to the integral equations and Volterra has been studied by [1, 6, 8, 10, 13, 15]. For methods using the square rule, collocation and interpolation, degenerate kernels, Chebyshev, and Euler series see [9, 11, 13, 15]. The variational iteration method, which is an efficient method, and the variational iteration method for solving Volterra integral form for lane-Emden and Emden Fowler problems in terms of initial and boundary value [20]. Where is the Lane-Emden equation converted to the equivalent Volterra integral form. In this work, we will study the solution to the integral Volterra equation, whose solution is also a solution to the Lane-Emden equation.

2 VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL FORM OF THE SINGULAR EMDEN-FOWLER TYPE DIFFERENTIAL EQUATION

Let Emden-Fowler equations be defined in Eq. (1).

$$\begin{aligned}&y''+\frac{\mu}{x}y'+\beta f(x)g(y)=0,\\&\mu>1,\ \beta>0,\ y(0)=a,\ y'(0)=0.\end{aligned} \tag{2}$$

Eq. (2) can be converted to the following form

$$y(x)=a-\frac{\beta}{\mu-1}\int_0^x t\left(1-\frac{t^{\mu-1}}{x^{\mu-1}}\right)h(t)g(y(t))\mathrm{d}t. \tag{3}$$

By differentiating Eq. (3) twice in a row, and depending on Leibniz's principle, we arrive in the form of the integro-differential equation of Emden-Fowler's equation

$$\begin{aligned}&y'(x)=-\beta\int_0^x\left(\frac{t^{\mu}}{x^{\mu}}\right)h(t)g(y(t))\mathrm{d}t,\\&y''(x)=\beta h(x)g(y(x))-\beta\int_0^x k\left(\frac{t^{\mu}}{x^{\mu+1}}\right)f(t)g(y(t))\mathrm{d}t.\end{aligned} \tag{4}$$

Therefore, for $\mu = 1$, the integral form becomes

$$y(x)=a-\beta\int_0^x t\ln\left(\frac{t}{x}\right)h(t)g(y(t))\mathrm{d}t.$$

Where we can get it, assuming that μ belongs to the ($\mu \to 1$) in Eq. (3). Based on the last results, we set the Volterra integral form of Emden-Fowler equations to the following mathematical form

$$y(x)=\begin{cases}a-\beta\displaystyle\int_0^x t\ln\left(\frac{t}{x}\right)h(t)g(y(t))\mathrm{d}t,\ \mu=1\\ a-\dfrac{\beta}{\mu-1}\displaystyle\int_0^x t\left(1-\frac{t^{\mu-1}}{x^{\mu-1}}\right)h(t)g(y(t))\mathrm{d}t,\ \mu>1\end{cases}$$

For the solution of the Eq. (3) In the complete function spaces, usually take it $C(\Omega)$, we choose a sequence finite dimensional subspace X_n, $n \geq 1$ having n basis functions $\{T_1, T_2, \ldots, T_n\}$ with dimension of $X_n = n$. Seeking the approximate function $y_n \in X$ of the function y given by

$$y_n=\sum_{k=1}^{n}\alpha_k T_k(x) \tag{5}$$

Where the expression Eq. (5) describes the truncated Touchard series of the solution of the Eq. (3), with the functions $\{T_k\}_{0\leq k\leq n}$ represent the Touchard polynomials and

$\{\alpha_k\}_{0\le k\le n}$ the coefficients to be determined. In other words, we can write Eq. (3)

$$y(x)=a-\frac{\beta}{\mu-1}\int_0^x t\left(1-\frac{t^{\mu-1}}{x^{\mu-1}}\right)h(t)g(y(t))\mathrm{d}t,$$
$$r_n(x)=y_n(x)+\frac{\beta}{\mu-1}\int_0^x t\left(1-\frac{t^{\mu-1}}{x^{\mu-1}}\right)h(t)g(y(t))\mathrm{d}t-a. \tag{6}$$

Change the variable $t\to\varepsilon x$ where the previous Volterra integral can be converted to the Fredholm integral form

$$y(x)=a-\frac{x^2}{\mu-1}\int_0^1 \varepsilon(1-\varepsilon^{\mu-1})h(\varepsilon x)g(y(\varepsilon x))\mathrm{d}\varepsilon. \tag{7}$$

3 COLLOCATION METHOD WITH TOUCHARD POLYNOMIALS

Choose a selection of distinct points $x_1, x_2, x_3, \ldots, x_n \in \Omega$, where $0 < x_1 < \cdots < x_n \le 1$.

And require:

$$r_n(x_j)=0, j=1, 2, ..., n \tag{8}$$

Then we can replace y with $y_n=\sum_{k=1}^{n}\alpha_k T_k(x)$ so Eq. (7) becomes. The condition Eq. (8) leads us to determine the coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ solution of the linear system:

$$\sum_{k=1}^{n}\alpha_k T_k(x)=$$
$$=a-\frac{x^2}{\mu-1}\int_0^1 \varepsilon(1-\varepsilon^{\mu-1})h(\varepsilon x)g\left(\sum_{k=1}^{n}\alpha_k T_k(\varepsilon x)\right)\mathrm{d}\varepsilon, \tag{9}$$
$$x\in[0, 1]$$

Placing x with the evaluation points of $x_1, x_2, x_3, \ldots, x_n$ in Eq. (9), we will arrive at the system of equations in the application description

$$\sum_{k=1}^{n}\alpha_k T_k(x_j)+$$
$$+\frac{(x_j)^2}{\mu-1}\int_0^1 \varepsilon(1-\varepsilon^{\mu-1})h(\varepsilon x_j)g\left(\sum_{k=1}^{n}\alpha_k T_k(\varepsilon x_j)\right)\mathrm{d}\varepsilon=a, \tag{10}$$
$$x\in[0, 1].$$

Using the quadrature formula with the coefficients (τ_l) and weights $(w_l)=1$ in the interval [0, 1] for numerically solving the integration in Eq. (10) Yields

$$\sum_{k=1}^{n}\alpha_k T_k(x_j)+$$
$$+\frac{(x_j)^2}{\mu-1}\sum_{k=1}^{n}\tau_l(1-\tau_l{}^{\mu-1})h(\varepsilon x_j)g\left(\sum_{k=1}^{n}\alpha_k T_k(\tau_l x_j)\right)w_l\mathrm{d}\varepsilon=a,$$
$$x\in[0, 1].$$

Then the values of $y(x)$ at any point of $x\in[0, 1]$ can be approximated by the equation

$$y_n=\sum_{k=1}^{n}\alpha_k T_k(x_j).$$

We transform Eq. (7) into the following form

$$(\lambda-A)y=f. \tag{11}$$

Where

$$Ay=\int_\Omega k(x, \varepsilon)g(y(\varepsilon x))$$

Therefore, Eq. (9) can be written in the following form

$$(\lambda-A)y_n=f. \tag{12}$$

Assume that A is a compact operator.

Lemma if Eq. (11) is uniquely solvable and $\|y-y_n\|\to 0$ then Eq. (12) is uniquely solvable.

Proof. See [12].

3.1 Properties of Touchard polynomials

The n^{th} Touchard polynomials $T_n(x)$ is defined by $T_0(x) = 1$ and the following recursion

$$T_n(n)=\sum_{k=0}^{n}\binom{n}{k}(x)^n$$

Noting that, the Touchard polynomial $T_n(x)$ is polynomials with rational coefficients.

Table 1 Rational coefficients of the Touchard polynomials.

n	T_n
0	1
1	$1 + x$
2	$1 + 2x + x^2$
3	$1 + 3x + 3x^2 + x^3$
4	$1 + 4x + 6x^2 + 4x^3 + x^4$
5	$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$
6	$1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
7	$1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$
8	$1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$
9	$1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$
10	$1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + 210x^6 + 120x^7 + 45x^8 + 10x^9 + x^{10}$

4 ERROR ANALYSIS

To illustrate the underlying ideas, validity, effectiveness, accuracy and performance of the proposed technique, we analyze several linear Lane-Emden or Emden-Fowler equations. Throughout the calculations, the absolute error between the exacted and approximated solution (error norm) is defined by $e_r=|y(x_i)-y_n(x_i)|$.

5 ILLUSTRATING EXAMPLES

Example 1. Consider the following linear homogeneous Lane-Emden equation

$$y''+\frac{2}{x}y'-(4x^2+6)y=0,\ 0<x\le 1$$

Subject to initial conditions

$$y(0)=1,\ y'(0)=0$$

The exact solution for this problem is

$$y(x)=\mathrm{e}^{x^2}.$$

This equation is equivalent to the following integral equation

$$y(x)=1+\int_0^x\left(\frac{xt-t^2}{x}\right)(4t^2+6)y(t)\mathrm{d}t,\ 0<x\le 1$$

Table 2 Approximate and exact solutions for Example 1

x	*Exact solution* y	*Approx solution* y_n	*Error* for n = 10
0.0	1.0000E+00	1.0000E+00	3.1086E–15
0.1	1.0101E+00	1.0101E+00	2.2542E–10
0.2	1.0408E+00	1.0408E+00	2.7722E–10
0.3	1.0942E+00	1.0942E+00	2.8983E–10
0.4	1.1735E+00	1.1735E+00	3.1393E–10
0.5	1.2840E+00	1.2840E+00	3.4539E–10
0.6	1.4333E+00	1.4333E+00	3.8359E–10
0.7	1.6323E+00	1.6323E+00	4.4465E–10
0.8	1.8965E+00	1.8965E+00	4.9153E–10
0.9	2.2479E+00	2.2479E+00	7.1219E–10
1	2.7183E+00	2.7183E+00	3.3111E–10

Example 2. Consider the Emden-Fowler equation

$$y''+\frac{2}{x}y'-2(2x^2+3)y=0,\ 0<x\le 1$$

Subject to initial conditions

$$y(0)=1,\ y'(0)=0$$

This problem has the exact solution

$$y(x)=\mathrm{e}^{x^2}.$$

This equation is equivalent to the following integral equation

$$y(x)=1+2\int_0^x\left(\frac{xt-t^2}{x}\right)(2t^2+3)y(t)\mathrm{d}t,\ 0<x\le 1$$

Table 3 Approximate and exact solutions for Example 2

x	*Exact solution* y	*Approx solution* y_n	*Error* for n = 10
0.0	1.0000E+00	1.0000E+00	3.1086E–15
0.1	1.0101E+00	1.0101E+00	2.2542E–10
0.2	1.0408E+00	1.0408E+00	2.7722E–10
0.3	1.0942E+00	1.0942E+00	2.8983E–10
0.4	1.1735E+00	1.1735E+00	3.1393E–10
0.5	1.2840E+00	1.2840E+00	3.4539E–10
0.6	1.4333E+00	1.4333E+00	3.8359E–10
0.7	1.6323E+00	1.6323E+00	4.4465E–10
0.8	1.8965E+00	1.8965E+00	4.9153E–10
0.9	2.2479E+00	2.2479E+00	7.1219E–10
1	2.7183E+00	2.7183E+00	3.3111E–10

Example 3. Consider the following Lane-Emden equation

$$y''+\frac{2}{x}y'+y^{\varepsilon}=0,\ 0<x\le 1$$

Subject to initial conditions

$$y(0)=1,\ y'(0)=0$$

For ε = 1 we have

$$y(x)=1-\int_0^x t\left(1-\frac{t}{x}\right)y(t)\mathrm{d}t,\ 0<x\le 1$$

This problem has the exact solution

$$y(x)=\frac{\sin(x)}{x}$$

Table 4 Approximate and exact solutions for Example 3

x	*Exact solution* y	*Approx solution* y_n	*Error* for n = 10
0.0	1.0000E+00	1.0000E+00	4.4409E–16
0.1	9.9833E–01	9.9833E–01	1.5210E–14
0.2	9.9335E–01	9.9335E–01	1.8097E–14
0.3	9.8507E–01	9.8507E–01	1.7986E–14
0.4	9.7355E–01	9.7355E–01	2.0983E–14
0.5	9.5885E–01	9.5885E–01	5.7065E–14
0.6	9.4107E–01	9.4107E–01	3.6671E–13
0.7	9.2031E–01	9.2031E–01	2.2352E–12
0.8	8.9670E–01	8.9670E–01	1.1019E–11
0.9	8.7036E–01	8.7036E–01	4.5200E–11
1	8.4147E–01	8.4147E–01	1.5982E–10

Example 4. Consider the linear Lane-Emden equation

$$y''+\frac{1}{x}y'-4(x^2+1)y=0,\ 0<x\le 1$$

Subject to initial conditions

$$y(0)=1,\ y'(0)=0$$

The exact solution for this problem is

$$y(x)=\mathrm{e}^{x^2}.$$

This equation is equivalent to the following integral equation

$$y(x)=1+4\int_0^x t\ln\left(\frac{t}{x}\right)(t^2+1)y(t)\mathrm{d}t,\ k\ =\ 1,\ 0<x\le 1$$

Table 5 Approximate and exact solutions for Example 4

x	*Exact solution* y	*Approx solution* y_n	*Error* for n = 10
0.0	1.0000E+00	1.0000E+00	1.2790E–13
0.1	1.0101E+00	1.0101E+00	2.3704E–10
0.2	1.0408E+00	1.0408E+00	3.9754E–10
0.3	1.0942E+00	1.0942E+00	4.8524E–10
0.4	1.1735E+00	1.1735E+00	5.7053E–10
0.5	1.2840E+00	1.2840E+00	6.5991E–10
0.6	1.4333E+00	1.4333E+00	7.6140E–10
0.7	1.6323E+00	1.6323E+00	8.9231E–10
0.8	1.8965E+00	1.8965E+00	1.0330E–09
0.9	2.2479E+00	2.2479E+00	1.3366E–09
1	2.7183E+00	2.7183E+00	6.6797E–10

6 CONCLUSION

In this paper, the Touchard series with the Collocation method was introduced in order to solve the Volterra integral equations obtained from the Lane-Emden equations, where we found that the approximate solutions are very close to the exact solutions, that is, the higher the degree n of the polynomial.

The four examples studied in this paper confirmed the effectiveness of the method and the speed of its convergence. The error obtained through the difference between exact solutions and approximate solutions confirms what was mentioned above.

On the other hand, the obtained solutions of the Volterra integral equations are also solutions of the Lane-Emden equations.

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