MATHEMATICAL COMMUNICATIONS 181
Math. Commun. **28**(2023), 181–201

Lattice paths inside a table

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Received July 4, 2022; accepted May 4, 2023

Abstract. A lattice path in $\mathbb{Z}^d$ is a sequence $\nu_1, \nu_2, \ldots, \nu_k \in \mathbb{Z}^d$ such that the steps $\nu_i - \nu_{i-1}$ lie in a subset $\mathbf{S}$ of $\mathbb{Z}^d$ for all $i = 2, \ldots, k$. Let $T_{m,n}$ be $m \times n$ table in the first area of the xy-axis and put $\mathbf{S} = \{(1,1), (1,0), (1,-1)\}$. Accordingly, let $\mathcal{I}_m(n)$ denote the number of lattice paths starting from the first column and ending at the last column of T. We will study the numbers $\mathcal{I}_m(n)$ and give explicit formulas for special values of m and n. As a result, we prove a conjecture of Alexander R. Povolotsky involving $\mathcal{I}_n(n)$. Finally, we present some relationships between the number of lattice paths and Fibonacci and Pell-Lucas numbers, and pose several problems and conjectures.
AMS subject classifications: 05A15, 11B37, 11B39

Keywords: Lattice path, Catalan numbers, directed animal, Fibonacci number, Motzkin number, Hankel matrix

1. Introduction

A *lattice path* L in $\mathbb{Z}^d$ is any sequence $\nu_1, \nu_2, \ldots, \nu_k$ of points of $\mathbb{Z}^d$ (see [36, 37]). The vectors $\nu_2 - \nu_1, \nu_3 - \nu_2, \ldots, \nu_k - \nu_{k-1}$ are called the *steps* of L. Lattice paths are studied by fixing a set of steps and an area $U \subseteq \mathbb{Z}^d$, where the paths live. A typical problem to carry out is to count possible lattice paths of a given length in the given area U with steps in a given set $\mathbf{S} \subseteq \mathbb{Z}^d$.

Lattice paths and more generally lattice animals have deep roots in physics and appear in the study of thermodynamic models, phase transitions, statistical physics, lattice gas models, river networks, etc. (for example, see [29]). A typical problem there is modeling physical phenomena, say motion of gas molecules, as paths inside a (triangular, square, hexagonal, etc.) lattice and study the behavior of the paths. The main question to address is to give exact formulas or asymptotic results for the number of lattice paths (animals) satisfying some constraints. For example, it is shown that the number a_n of directed animals of size n satisfies

$$a_n \sim \mu^n n^{-\theta}$$

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for some constants μ and θ in various models. For a thorough study of 2-dimensional lattices in physics we refer the interested reader to [1, 13, 14, 18, 19, 23, 25, 26, 30, 32, 41, 44, 53], and to [24, 33, 40, 43, 42, 41, 52, 53] for higher dimensions. We also refer to [2, 10, 12, 28, 39, 46, 48, 51] for further results. Gouyou-Beauchamps and Viennot [30] give a bijection between compact-rooted directed lattice animals on a two-dimensional square lattice with some lattice paths in the plane. Later, Bousquet-Mélou and Conway [15] and Corteel, Denise, and Gouyou-Beauchamps [21] give bijective proofs to obtain algebraic equations satisfied by the area generating function of directed lattice animals on infinite families of two-dimensional lattices. Recall that a *lattice animal* is a set of points in a lattice, which is a union of some lattice paths starting from a single point (or a set of points in some contexts).

Lattice paths also arise naturally in various problems in mathematics and are well-studied in the literature. The general theory studies the analytic behavior of the complex generating function of the paths and gives estimations of the number of paths of a given length, etc. (see e.g. [3, 27, 45]). Particular lattice paths have received much attention and have been studied extensively. Very important paths be mentioned are Dyck paths and Motzkin paths. A *Dyck path* is a lattice path in $\mathbb{Z}^2$ starting from $(0,0)$ and ending at a point $(2n,0)$ $(n \geqslant 0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ that never passes below the x-axis. The *Catalan numbers* $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$, a ubiquity in various combinatorial problems, count the number of Dyck paths of length $2n$ (for details, see [31, 34, 35, 47, 50]). Allowing the right steps $(1,0)$ in addition to those of a Dyck path, we get Motzkin paths starting from $(0,0)$ and ending at a point $(n,0)$ that never pass below the x-axis.

Throughout this paper, $T_{m,n}$ stands for the $m \times n$ table in the first quadrant composed of mn unit squares whose (x,y)-cell is located in the x^{th}-column from the left side and the y^{th}-row from the bottom side of $T_{m,n}$. Furthermore, for a set $\mathbf{S} \subseteq \mathbb{Z}^d$ of steps, $l((i,j) \to (s,t);\mathbf{S})$ denotes the number of all lattice paths in $T_{m,n}$ starting from the (i,j)-cell and ending at the (s,t)-cell with steps in $\mathbf{S}$, where $1 \leqslant i,s \leqslant n$ and $1 \leqslant j,t \leqslant m$.

The paths we shall study in this paper use the same set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$ of steps as Motzkin paths but live in a bounded rectangular area, which we may assume to be $T_{m,n}$. Notice that the number $l((1,1) \to (n,1);\mathbf{S})$ of all lattice paths in the table $T_{m,n}$ starting from the $(1,1)$-cell and ending at the $(n,1)$-cell using Motzkin steps is the n^{th}-*Motzkin number* provided that $m \geqslant n$. The number of all lattice paths starting from the first column, and ending at the last column of $T_{m,n}$ is denoted by $\mathcal{I}_m(n)$. Indeed,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l((1,i) \to (n,j);\mathbf{S}).$$

Figure 1 shows the number of all lattice paths for $m = 2$ and $n = 3$. Clearly, $l((1,i) \to (n,j)) = l((1,i') \to (n,j'))$ when $i + i' = m+1$ and $j + j' = m+1$.

We intend to evaluate $\mathcal{I}_m(n)$ for special cases of (m,n). In Section 2, we give general formulas for $\mathcal{I}_m(n)$ when $m \geqslant n-1$. In Section 3, we compute $\mathcal{I}_m(n)$ explicitly for small values of m, namely $m = 1,2,3,4$, and present some results for $\mathcal{I}_5(n)$. In Section 4, we use Fibonacci and Pell-Lucas numbers to prove some

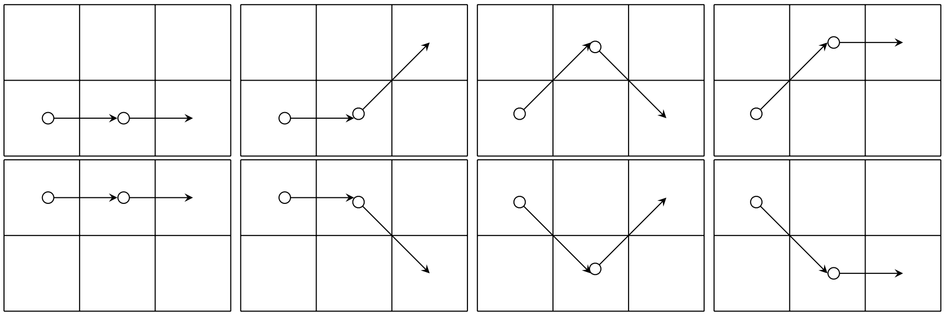

Figure 1: *All lattice paths in $T_{2,3}$.*

relations concerning lattice paths. Section 5 is devoted to the proof of the following conjecture of Povolotsky in OEIS sequence A081113. This identity first appeared in [11].

Conjecture 1 (Alexander R. Povolotsky, 2011)**.** *The following identity holds for the numbers $\mathcal{I}_n(n)$:*

$$(n+3)\mathcal{I}_{n+4}(n+4) = 27n\mathcal{I}_n(n) + 27\mathcal{I}_{n+1}(n+1) - 9(2n+5)\mathcal{I}_{n+2}(n+2) + (8n+21)\mathcal{I}_{n+3}(n+3).$$

Utilizing a recurrence relation for $\mathcal{C}_n(n,n)$ due to Michael Somos and the above conjecture involving $\mathcal{I}_n(n)$, we compute generating functions of these numbers as well. Finally, in the last section, we give some open problems for future research.

2. Formulas for $\mathcal{I}_m(n)$

Let $\mathbf{S} := \{(1,1),(1,0),(1,-1)\}$. For positive integers $1 \leqslant i,t \leqslant m$ and $1 \leqslant s \leqslant n$, the number of all lattice paths from the $(1,i)$-cell to the (s,t)-cell in the table $T = T_{m,n}$ is denoted by $\mathcal{C}^i_{m,n}(s,t)$, that is, $\mathcal{C}^i_{m,n}(s,t) = l((1,i) \to (s,t); \mathbf{S})$. Furthermore, we put $\mathcal{C}_{m,n}(0,t) = 1$ and

$$\mathcal{C}_{m,n}(s,t) = \sum_{i=1}^{m} \mathcal{C}^i_{m,n}(s,t)$$

for all $1 \leqslant s \leqslant n$ and $1 \leqslant t \leqslant m$. We usually use the notation $\mathcal{C}(s,t)$ for $\mathcal{C}_{m,n}(s,t)$ when there is no confusion. Also, we put $\mathcal{C}_n(s,t) := \mathcal{C}_{n,n}(s,t)$. Clearly, $\mathcal{C}_n(s,t)$ is the number of all lattice paths from first column to the (s,t)-cell of T. It is easy to see, for $n \geqslant 2$, that

$$\mathcal{C}_n(n,n) = \mathcal{C}_n(n-1,n) + \mathcal{C}_n(n-1,n-1),$$

where $\mathcal{C}_1(1,1) = 1, \mathcal{C}_2(2,2) = 2, \mathcal{C}_3(3,3) = 5, \mathcal{C}_4(4,4) = 13, \ldots$. The values of $\mathcal{C}_n(n,n)$ are given in the OEIS sequence A005773, where T is a square table. Notice that the diagram for $\mathcal{C}_4(4,4) = 13$ is

1 2 5 13
1 3 8 21
1 3 8 21'
1 2 5 13

where each entry is the sum of two or three entries in the preceding column.

Table 1 illustrates the values of $\mathcal{C}_6(6,t)$, for all $1 \leqslant t \leqslant 6$, where the number in the (s,t)-cell of T determines the number $\mathcal{C}_6(s,t)$. By symmetry of the table T, we have $\mathcal{C}_6(s,t) = \mathcal{C}_6(s,t')$ when $t+t'=7$.

					$\mathcal{C}_6(6,t)$
1	2	5	13	35	96
1	3	8	22	61	170
1	3	9	26	74	209
1	3	9	26	74	209
1	3	8	22	61	170
1	2	5	13	35	96

Table 1: *Values of* $\mathcal{C}_6(6,t)$

It is worth mentioning that the numbers $\mathcal{C}_n(n,n)$ coincide with the number of directed animals of size n starting from a single point (see [30]). The numbers $\mathcal{C}_n(n,n)$ appear is various other results, see e.g. [14, 16, 17, 20, 25]. Note also that Krattenthaler and Yaqubi [38] compute determinants of some Hankel matrices involving $\mathcal{C}_n(x,y)$, which is of independent interest.

Theorem 1. *Inside the $m \times n$ table we have*

$$I_m(n) = m3^{n-1} - 2\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{C}_{m,n}(s,1). \tag{1}$$

In particular,

$$I_m(n) = m3^{n-1} - 2\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{C}_{n}(s,1). \tag{2}$$

when $m \geqslant n-1$,

Proof. Let $T := T_{m,n}$. The number of all lattice paths from the first column to the last column is simply $m3^{n-1}$ if they are allowed to get out of T. Now we count all lattice paths that get out of T in some steps. First, observe that the number of lattice paths that leave T from the bottom row for the first time equals those that leave T from the top row. Suppose a lattice path gets out of T from the bottom row in the column s for the first time. The number of all partial lattice paths from the first column to the $(s-1,1)$-cell is simply $\mathcal{C}_{m,n}(s-1,1)$, and every such path continues in 3^{n-s} ways until it reaches the last column of T. Thus, we have $3^{n-s}\mathcal{C}_{m,n}(s-1,1)$ paths leaving the table T from the bottom in the column s for any $s=2,\ldots,n$. Hence, the number of lattice paths is simply

$$\begin{aligned}\mathcal{I}_m(n) &= m3^{n-1} - 2\sum_{s=2}^{n} 3^{n-s}\mathcal{C}_{m,n}(s-1,1)\\ &= m3^{n-1} - 2\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{C}_{m,n}(s,1).\end{aligned}$$

The second formula follows from the fact that $\mathcal{C}_{m,n}(s,1) = \mathcal{C}_n(s,1)$ for all $s < n$, provided that $m \geqslant n-1$. $\square$

Corollary 1. *Inside the $m \times n$ table ($m \geqslant n-1$) we have*

$$\mathcal{I}_{m+1}(n) = 3\mathcal{I}_m(n-1) - 2\mathcal{C}_{n-1}(n-1, n-1) + 3^{n-1}.$$

In particular,

$$\mathcal{I}_n(n) = 3\mathcal{I}_{n-1}(n-1) - 2\mathcal{C}_{n-1}(n-1, n-1) + 3^{n-1}.$$

Example 1. *Let T be the square 6×6 table. In Table 1, every cell represents the number of all lattice paths from the first column to that cell. Summing up the last column yields*

$$\mathcal{I}_6(6) = 96 + 170 + 209 + 209 + 170 + 96 = 950.$$

Now, utilizing Theorem 1, we calculate $\mathcal{I}_6(6)$ in another way as follows:

$$\begin{aligned}\mathcal{I}_6(6) =& 6 \cdot 3^{6-1} - 2\left(3^{6-1-1}\mathcal{C}_6(1,1) + 3^{6-2-1}\mathcal{C}_6(2,1) + 3^{6-3-1}\mathcal{C}_6(3,1)\right.\\ &\left.+3^{6-4-1}\mathcal{C}_6(4,1) + 3^{6-5-1}\mathcal{C}_6(5,1)\right)\\ =& 1458 - 2\left(3^4 \cdot 1 + 3^3 \cdot 2 + 3^2 \cdot 5 + 3^1 \cdot 13 + 3^0 \cdot 35\right) = 950.\end{aligned}$$

Theorem 2. *Inside the $m \times n$ table ($m \geqslant n-1$) we have that*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{C}(i,1)\mathcal{C}(n-i,1) = 3^{n-1}$$

is constant.

Proof. Consider the table $T := T_{m,n}$. We construct the table T' by adding a new row $m+1$ at the top of T. Now, to count the number of all lattice paths in T', it is sufficient to consider lattice paths that reach to the row $m+1$. Assume a lattice path reaches to the row $m+1$ at column i for the first time. Then its initial part from column 1 to column $i-1$ is a lattice path from the first column of T to the $(i-1, m)$-cell. Furthermore, its terminal part from column i to column n is a lattice path from the $(i, m+1)$-cell of T' to its last column, which is in one-to-one correspondence with a lattice path from the (i, m)-cell of T to its last column as $m \geqslant n$. Hence, the number of such paths is simply $\mathcal{C}(i-1, m)\mathcal{C}(n-i+1, m)$, which is equal to $\mathcal{C}(i-1, 1)\mathcal{C}(n-i+1, 1)$ by symmetry. Therefore,

$$\begin{aligned}\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) &= \sum_{i=1}^{n} \mathcal{C}(i-1,1)\mathcal{C}(n-i+1,1)\\ &= \sum_{i=0}^{n-1} \mathcal{C}(i,1)\mathcal{C}(n-i,1).\end{aligned}$$

The fact that $I_{m+1}(n) - I_m(n) = 3^{n-1}$ follows from equation (2) of Theorem 1. $\square$

Theorem 2 gives formulas for the (convolution) product of a specific row with itself. Regarding columns, we get the following (more) general result.

Theorem 3. *Inside the $m \times n$ table, we have*

$$\mathcal{I}_m(n) = \sum_{i=1}^{m} \mathcal{C}(a,i)\mathcal{C}(b,i)$$

for all $a, b \geqslant 1$ such that $a+b = n+1$. In other words, the inner product of columns a and b equals $\mathcal{I}_m(n)$. In particular, if $n = 2k-1$ is odd, then

$$\mathcal{I}_m(n) = \sum_{i=1}^{m} \mathcal{C}(k,i)^2.$$

Proof. Every lattice path crosses the column a at some row, say i. The number of such paths equals the number $\mathcal{C}(a,i)$ of paths from the first column to the (a,i)-cell multiplied by the number $\mathcal{C}(n-(a-1),i) = \mathcal{C}(b,i)$ of paths from the last column to that cell, from which the result follows. □

Let us recall that the number $l(1,1; n+1, 1 : \mathbf{S})$ of lattice paths in $\mathbb{Z}^2$ that never slide below the x-axis is the n^{th}-*Motzkin number* $\mathcal{M}_n$ ($n \geqslant 0$). Motzkin numbers begin with $1, 1, 2, 4, 9, 21, \ldots$ (see OEIS sequence $A001006$) and can be expressed in terms of binomial coefficients and Catalan numbers via

$$\mathcal{M}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \mathcal{C}_k$$

(see [22]). The *trinomial triangle* is defined by the same steps $(1,1)$, $(1,-1)$, and $(1,0)$ (in our notation) with no restriction by starting from a fixed cell. The number of ways to reach a cell is simply the sum of three numbers in the adjacent previous column. The k^{th}-entry of the n^{th} column is denoted by $\binom{n}{k}_2$, where columns start by 0. The middle entries of the trinomial triangle, namely $1, 1, 3, 7, 19, \ldots$ (see $A002426$) are studied by Euler. Analogously, the Motzkin triangle is defined by the recurrence sequence

$$\mathcal{T}(n,k) = \mathcal{T}(n-1,k-2) + \mathcal{T}(n-1,k-1) + \mathcal{T}(n-1,k),$$

for all $1 \leqslant k \leqslant n-1$ and satisfy

$$\mathcal{T}(n,n) = \mathcal{T}(n-1,n-2) + \mathcal{T}(n-1,n-1)$$

for all $n \geqslant 1$ (see $A026300$).

Table 2 illustrates initial parts of the above triangles with the Motzkin triangle in the left and trinomial triangle in the right. For a positive integer $1 \leqslant s \leqslant n$, each entry of the column $\mathcal{C}_s(s,1)$ is the sum of all entries in the s^{th}-row in the rotated Motzkin triangle, that is, $\mathcal{C}_s(s,1) = \sum_{i=1}^{s} \mathcal{T}(s,i)$. For example,

$$\mathcal{C}_4(4,1) = \mathcal{T}(4,1) + \mathcal{T}(4,2) + \mathcal{T}(4,3) + \mathcal{T}(4,4) = 4+5+3+1 = 13.$$

The entries in the first column of the rotated Motzkin triangle are indeed Motzkin numbers.

$\mathcal{C}_s(s,1)$						
1	1					
2	1	1				
5	2	2	1			
13	4	5	3	1		
35	9	12	9	4	1	
96	21	30	25	14	5	1

				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1

Table 2: *Motzkin triangle (left) and trinomial triangle (right) rotated by* 90° *clockwise*

Lemma 1. *Inside the square $n \times n$ table we have*

$$\mathcal{C}(s,1) = 3\mathcal{C}(s-1,1) - \mathcal{M}_{s-2},$$

for all $1 \leqslant s \leqslant n$.

Proof. Let $T := T_{n,n}$. By definition, $\mathcal{C}(s,1)$ is the number of all lattice paths from the first column to the $(s,1)$-cell. This number equals the number of lattice paths from the $(s,1)$-cell to the first column with reverse steps that lie inside the table T that is 3^{s-1} minus those paths that leave T at some point. Consider all those lattice paths starting from the $(s,1)$-cell with reverse steps leaving T at $(i,0)$ for the first time, where $1 \leqslant i \leqslant s-1$. Clearly, the number of such paths is $3^{i-1}\mathcal{M}_{s-i-1}$. Thus,

$$\mathcal{C}(s,1) = 3^{s-1} - \sum_{i=1}^{s-1} 3^{i-1}\mathcal{M}_{s-i-1}.$$

Since

$$\mathcal{C}(s-1,1) = 3^{s-2} - \sum_{i=1}^{s-2} 3^{i-1}\mathcal{M}_{s-i-1},$$

it follows that $\mathcal{C}(s,1) - 3\mathcal{C}(s-1,1) = -\mathcal{M}_{s-2}$, as required. □

Example 2. *Consider Table 2. Using Lemma 1 we can compute $\mathcal{C}_6(6,1)$ as*

$$\mathcal{C}_6(6,1) = 3\mathcal{C}_6(5,1) - \mathcal{M}_4 = 3 \cdot 35 - 9 = 96.$$

Corollary 2. *Inside the $n \times n$ table ($m \geqslant n-1$) we have*

$$\mathcal{I}_m(n) = (3m - 2n + 2)3^{n-2} + 2\sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k.$$

Proof. By Lemma 1,

$$\begin{aligned}\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{C}(s,1) &= \mathcal{C}(n-1,1) + \sum_{s=1}^{n-2} 3^{n-s-1}\mathcal{C}(s,1)\\ &= 2\cdot 3\mathcal{C}(n-2,1) + \sum_{s=1}^{n-3} 3^{n-s-1}\mathcal{C}(s,1) - \mathcal{M}_{n-3}\\ &= 3\cdot 3^2\mathcal{C}(n-3,1) + \sum_{s=1}^{n-4} 3^{n-s-1}\mathcal{C}(s,1) - 2\cdot 3\mathcal{M}_{n-4} - \mathcal{M}_{n-3}\\ &\vdots\\ &= (n-1)3^{n-2}\mathcal{C}(1,1) - \sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k\\ &= (n-1)3^{n-2} - \sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k.\end{aligned}$$

Now the result follows from equation (2) of Theorem 1. □

Lemma 2. *Inside the $n \times n$ table we have*

$$\mathcal{C}(n,k+2) - \mathcal{C}(n,k) = \sum_{i=1}^{n-1}\left(\mathcal{C}(i,k+3) - \mathcal{C}(i,k-1)\right)$$

for all $1 \leqslant k \leqslant n$.

Proof. For $n = 2$, the result is trivially true. For any $l < n$ we have

$$\begin{aligned}\mathcal{C}(l+1,k+2) &= \mathcal{C}(l,k+3) + \mathcal{C}(l,k+2) + \mathcal{C}(l,k+1)\\ \mathcal{C}(l+1,k) &= \mathcal{C}(l,k+1) + \mathcal{C}(l,k) + \mathcal{C}(l,k-1),\end{aligned}$$

which implies that

$$\mathcal{C}(l+1,k+2) - \mathcal{C}(l+1,k) = \mathcal{C}(l,k+3) - \mathcal{C}(l,k-1) + \left(\mathcal{C}(l,k+2) - \mathcal{C}(l,k)\right).$$

Thus,

$$\mathcal{C}(n,k+2) - \mathcal{C}(n,k) = \sum_{i=1}^{n-1}\left(\mathcal{C}(i,k+3) - \mathcal{C}(i,k-1)\right)$$

as $\mathcal{C}(1,k+2) - \mathcal{C}(1,k) = 0$. This completes the proof. □

3. Tables with few rows

In this section, we shall compute $\mathcal{I}_m(n)$ for $m = 1,2,3,4$ and arbitrary positive integers n.Furthermore, we obtain some properties of $\mathcal{I}_m(n)$ for $m = 5$. Some values of $\mathcal{I}_3(n)$ and $\mathcal{I}_4(n)$ are already given in $A001333$ and $A055819$, respectively.

Lemma 3. *$\mathcal{I}_1(n) = 1$ and $\mathcal{I}_2(n) = 2^n$ for all $n \geqslant 1$.*

Let x and y be arbitrary real numbers. By the binomial theorem, we have the following identity:

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k},$$

where $n \geqslant 1$. This identity can also be rewritten as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k}, \tag{3}$$

where $\binom{r}{-1} = 0$. Pell-Lucas sequence [35] is defined as $\mathcal{Q}_1 = 1$, $\mathcal{Q}_2 = 3$, and $\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2}$ for all $n \geqslant 3$. It can also be defined by the so-called *Binet formula* as $\mathcal{Q}_n = (\alpha^n + \beta^n)/2$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$.

Lemma 4. *For all $n \geqslant 1$ we have $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$.*

Proof. The number of lattice paths to cells in columns $n-2$, $n-1$, and n of $T_{3,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$2x+y$	$4x+3y$
x	$x+y$	$3x+2y$

which implies that $\mathcal{I}_3(n-2) = 2x + y$, $\mathcal{I}_3(n-1) = 4x + 3y$, and $\mathcal{I}_3(n) = 10x + 7y$. Thus, the following linear recurrence exists for $\mathcal{I}_3$:

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2). \tag{4}$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geqslant 1$, as required. □

Corollary 3. *Let n be a positive integer. Then*

$$\mathcal{I}_3(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-k+1}{k} + \binom{n-k}{k-1} \right] 2^{n-2k}.$$

Proof. It is sufficient to put $x = \alpha$ and $y = \beta$ in (3). □

The Fibonacci sequence $A000045$ starts with the integers 0 and 1, and every other term is the sum of the two preceding ones, that is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ for all $n \geqslant 2$. This recursion gives Binet's formula $\mathcal{F}_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 5. *For all $n \geqslant 1$ we have $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$.*

Proof. The number of lattice paths to cells in columns $n-2$, $n-1$, and n of $T_{4,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$2x+3y$
y	$x+2y$	$3x+5y$
y	$x+2y$	$3x+5y$
x	$x+y$	$2x+3y$

which implies that $\mathcal{I}_4(n-2) = 2x+2y$, $\mathcal{I}_4(n-1) = 4x+6y$, and $\mathcal{I}_4(n) = 10x+16y$. Hence, we get the following linear recurrence for $\mathcal{I}_4$:

$$\mathcal{I}_4(n) = 3\mathcal{I}_4(n-1) - \mathcal{I}_4(n-2). \tag{5}$$

On the other hand,

$$\begin{aligned} \mathcal{F}_{2n+1} &= \mathcal{F}_{2n} + \mathcal{F}_{2n-1} \\ &= 2\mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} \\ &= 3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} \\ &= 3\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1}. \end{aligned}$$

Now, since $\mathcal{I}_4(1) = 2\mathcal{F}_3$ and $\mathcal{I}_4(2) = 2\mathcal{F}_5$, it follows that $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$ for all $n \geqslant 1$. The proof is complete. □

Corollary 4. *For all $n \geqslant 1$ we have*

$$\mathcal{I}_4(n) = \sum_{k=0}^{n} (-1)^k \left[\frac{2n+1}{k}\binom{2n-k}{k-1}\right] 5^{n-k}. \tag{6}$$

Proof. It is sufficient to put $x = \varphi$ and $y = \psi$ in (3). □

In the sequel, we obtain some properties of $C_{m,n}(s,t)$ and $\mathcal{I}_m(n)$, when $m = 5$.

Proposition 1. *Inside the $5 \times n$ table we have*

$$\mathcal{C}(s+2,1) = \mathcal{I}_5(s) \quad \textit{and} \quad \mathcal{C}(s+2,3) = 2\mathcal{I}_5(s) - 1$$

for all $1 \leqslant s \leqslant n$.

Proof. From the table in Example 3, it follows simply that $\mathcal{I}_5(s) = \mathcal{C}(s+2,1)$ for all $s \geqslant 1$. From the table, it also follows that

$$2\mathcal{C}(s+1,1) - \mathcal{C}(s+1,3) = 2\mathcal{C}(s,1) - \mathcal{C}(s,3)$$

for all $s \geqslant 1$, that is, $2\mathcal{C}(s,1) - \mathcal{C}(s,3)$ is constant. Since $2\mathcal{C}(1,1) - \mathcal{C}(1,3) = 1$, we get $2\mathcal{C}(s+2,1) - \mathcal{C}(s+2,3) = 1$, from which the result follows. □

Proposition 2. *Inside the* $5 \times n$ *table we have*

$$\mathcal{C}(s,1) \times \mathcal{C}(s+t,3) - \mathcal{C}(s,3) \times \mathcal{C}(s+t,1) = \sum_{i=s}^{s+t-1} \mathcal{C}(i,2)$$

for all $1 \leqslant s,t \leqslant n$.

Proof. From Proposition 1, we know that $\mathcal{C}(s,3) = 2\mathcal{C}(s,1) - 1$ for all $1 \leqslant s \leqslant n$. Then

$$\begin{aligned}
&\mathcal{C}(s,1)\mathcal{C}(s+t,3) - \mathcal{C}(s,3)\mathcal{C}(s+t,1)\\
=&\mathcal{C}(s,1)(2\mathcal{C}(s+t,1) - 1) - (2\mathcal{C}(s,1) - 1)\mathcal{C}(s+t,1)\\
=&2\mathcal{C}(s,1)\mathcal{C}(s+t,1) - \mathcal{C}(s,1) - 2\mathcal{C}(s,1)\mathcal{C}(s+t,1) + \mathcal{C}(s+t,1)\\
=&\mathcal{C}(s+t,1) - \mathcal{C}(s,1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{C}(s+t,1) - \mathcal{C}(s,1) &= \mathcal{C}(s+t-1,1) + \mathcal{C}(s+t-1,2) - \mathcal{C}(s,1)\\
&= \mathcal{C}(s+t-2,1) + \mathcal{C}(s+t-2,2) + \mathcal{C}(s+t-1,2) - \mathcal{C}(s,1)\\
&\vdots\\
&= \sum_{i=s}^{s+t-1} \mathcal{C}(i,2) + \mathcal{C}(s,1) - \mathcal{C}(s,1)\\
&= \sum_{i=s}^{s+t-1} \mathcal{C}(i,2),
\end{aligned}$$

from which the result follows. □

4. Further results on lattice paths by using Fibonacci and Pell-Lucas numbers

In this section, we apply Fibonacci and Pell-Lucas sequences to obtain some further relations and properties for lattice paths inside a table.

Proposition 3. *Inside the* $4 \times n$ *table we have*

$$\mathcal{C}(s,1) = \mathcal{F}_{2s-1} \quad \textit{and} \quad \mathcal{C}(s,2) = \mathcal{F}_{2s}$$

for all $s \geqslant 1$. *As a result,*

$$\mathcal{C}(s,1) \times \mathcal{C}(s+t,2) - \mathcal{C}(s,2) \times \mathcal{C}(s+t,1) = \mathcal{C}(t,2).$$

for all $s,t \geqslant 1$.

Proof. Clearly $\mathcal{C}(1,1) = \mathcal{C}(1,2) = \mathcal{F}_1 = \mathcal{F}_2 = 1$. Now since

$$\begin{aligned}
\mathcal{C}(s,1) &= \mathcal{C}(s-1,1) + \mathcal{C}(s-1,2),\\
\mathcal{C}(s,2) &= 2\mathcal{C}(s-1,2) + \mathcal{C}(s-1,1),
\end{aligned}$$

we may prove by induction that $\mathcal{C}(s,1)=\mathcal{F}_{2s-1}$ and $\mathcal{C}(s,2)=\mathcal{F}_{2s}$ for all $s\geqslant 1$. The second claim follows from the fact that

$$\mathcal{F}_{2s-1}\mathcal{F}_{2s+2t}-\mathcal{F}_{2s}\mathcal{F}_{2s+2t-1}=\mathcal{F}_{2t}.$$

The proof is complete. □

Proposition 4. *Inside the $4\times n$ table we have*

$$\mathcal{I}_4(2s+1)=\frac{1}{4}\mathcal{I}_4(s+1)^2+\mathcal{C}(s,2)^2$$

for all $1\leqslant s\leqslant n$.

Proof. Following Lemma 5 and Proposition 3, it is enough to show that

$$2\mathcal{F}_{4s+3}=\mathcal{F}_{2s+3}^2+\mathcal{F}_{2s}^2.$$

First, observe that the equation $\mathcal{F}_{2n-1}=\mathcal{F}_n^2+\mathcal{F}_{n-1}^2$ yields $\mathcal{F}_{4s+1}=\mathcal{F}_{2s+1}^2+\mathcal{F}_{2s+2}^2$ and $\mathcal{F}_{4s+5}=\mathcal{F}_{2s+3}^2+\mathcal{F}_{2s+2}^2$. Now, by combining these two formulas, we obtain

$$\begin{aligned}\mathcal{F}_{2s+3}^2+\mathcal{F}_{2s}^2&=\mathcal{F}_{4s+5}+\mathcal{F}_{4s+1}-(\mathcal{F}_{2s+1}^2+\mathcal{F}_{2s+2}^2)\\&=\mathcal{F}_{4s+4}+\mathcal{F}_{4s+3}+\mathcal{F}_{4s+1}-\mathcal{F}_{4s+3}\\&=\mathcal{F}_{4s+3}+\mathcal{F}_{4s+2}+\mathcal{F}_{4s+1}\\&=2\mathcal{F}_{4s+3},\end{aligned}$$

as required. □

Pell numbers $\mathcal{P}_n$ are defined recursively as $\mathcal{P}_1=1$, $\mathcal{P}_2=2$, and $\mathcal{P}_n=2\mathcal{P}_{n-1}+\mathcal{P}_{n-2}$ for all $n\geqslant 3$. Binet's formula corresponding to $\mathcal{P}_n$ is $\mathcal{P}_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$.

Proposition 5. *Inside the $3\times n$ table we have*

$$\mathcal{C}(s,1)=\mathcal{P}_s\quad\textit{and}\quad\mathcal{C}(s,2)=\mathcal{Q}_s$$

for all $s\geqslant 1$. As a result,

$$\mathcal{C}(s,1)\times\mathcal{C}(s+t,2)-\mathcal{C}(s,2)\times\mathcal{C}(s+t,1)=(-1)^{s+1}\mathcal{C}(t,1),$$

for all $s,t\geqslant 1$.

Proof. From the table in Lemma 4, we observe that

$$\begin{aligned}\mathcal{C}(s,1)&=2\mathcal{C}(s-1,1)+\mathcal{C}(s-2,1),\\\mathcal{C}(s,2)&=2\mathcal{C}(s-1,2)+\mathcal{C}(s-2,2)\end{aligned}$$

for all $s\geqslant 3$. Now, since $\mathcal{C}(1,1)=\mathcal{P}_1=1$, $\mathcal{C}(2,1)=\mathcal{P}_2=2$, $\mathcal{C}(1,2)=\mathcal{Q}_1=1$, and $\mathcal{C}(2,2)=\mathcal{Q}_2=3$, one can show by induction that $\mathcal{C}(s,1)=\mathcal{P}_s$ and $\mathcal{C}(s,2)=\mathcal{Q}_s$ for all s. To prove the second claim, we use the following formula:

$$\mathcal{P}_s\mathcal{Q}_{s+t}-\mathcal{Q}_s\mathcal{P}_{s+t}=(-1)^{s+1}\mathcal{P}_t$$

that can be proved simply by using Binet's formulas. □

5. Alexander R. Povolotsky's conjecture and generating functions for $\mathcal{C}_n(n,n)$ and $\mathcal{I}_n(n)$

In this section, we prove Conjecture 1 of Alexander R. Povolotsky involving $\mathcal{I}_n(n)$. First, we need to find the generating function for $\mathcal{C}_n(n,n)$.

Theorem 4. *Th generating function of $\mathcal{C}_n := \mathcal{C}_n(n,n)$ is given by*

$$\sum_{n=1}^{\infty} \mathcal{C}_n x^n = \frac{1}{2}\sqrt{\frac{1+x}{1-3x}} - \frac{1}{2}.$$

Proof. Let $F(x) := \sum_{n=1}^{\infty} \mathcal{C}_n x^n$ be the generating function of $\mathcal{C}_n$. Theorem 2 yields

$$\mathcal{C}_n(n,1) = 3^{n-1} - \sum_{i=1}^{n-1} \mathcal{C}_n(i,1)\mathcal{C}_n(n-i,1)$$

Since $\mathcal{C}_k(k,k) = \mathcal{C}_k(k,1)$, we get $\mathcal{C}_k = \mathcal{C}_n(k,1)$ for all $k = 1,\dots,n$. Hence

$$\mathcal{C}_n = 3^{n-1} - \sum_{i=1}^{n-1} \mathcal{C}_i\mathcal{C}_{n-i},$$

for all $n \geqslant 1$. It follows that

$$\begin{aligned}
F^2(x) &= \sum_{n=2}^{\infty} \Big(\sum_{i=1}^{n-1} \mathcal{C}_i\mathcal{C}_{n-i}\Big)x^n \\
&= \sum_{n=2}^{\infty} (3^{n-1} - \mathcal{C}_n)x^n \\
&= 3x^2 \sum_{n=0}^{\infty} (3x)^2 + \mathcal{C}_1 - \sum_{n=1}^{\infty} \mathcal{C}_n x^n \\
&= \frac{3x^2}{1-3x} + x - F(x).
\end{aligned}$$

Therefore

$$F(x) = \sum_{n=1}^{\infty} \mathcal{C}_n x^n = \frac{1}{2}\sqrt{\frac{1+x}{1-3x}} - \frac{1}{2},$$

as required. □

Using Theorem 4, we can prove the following recurrence relation of Michael Somos for $\mathcal{C}_n(n,n)$ posted in OEIS sequence A005773.

Theorem 5 (Somos' relation)**.** *Inside the square $n \times n$ table we have*

$$\mathcal{C}_n(n,n) = 2n\mathcal{C}_n(n-1,n-1) + 3(n-2)\mathcal{C}_n(n-2,n-2).$$

Proof. Let
$$F(x)=\sum_{n=1}^{\infty}C_nx^n=\frac{1}{2}\sqrt{\frac{1+x}{1-3x}}-\frac{1}{2}$$
be the generating function of $\mathcal{C}_n:=\mathcal{C}_n(n,n)$ obtained in Theorem 4. Differentiation gives us
$$F'(x)=\sum_{n=1}^{\infty}nC_nx^{n-1}=\frac{1}{(1-3x)^{3/2}\sqrt{1+x}}.$$
One can easily see that
$$F'(x)(1-2x-3x^2)=F'(x)(1+x)(1-3x)=1+2F(x).$$
Finally, comparing the coefficients on both sides yields Somos' recurrence
$$nC_n=2nC_{n-1}+3(n-2)C_{n-2},$$
as required. □

Now, we are ready to prove Conjecture 1 of Alexander R. Povolotsky.

Proof of Conjecture 1. Put
$$\begin{aligned}A&=(n+3)\mathcal{I}_{n+4}(n+4),\\ B&=(8n+21)\mathcal{I}_{n+3}(n+3),\\ C&=9(2n+5)\mathcal{I}_{n+2}(n+2),\\ D&=27\mathcal{I}_{n+1}(n+1),\\ E&=27n\mathcal{I}_n(n).\end{aligned}$$
Using Theorem 1, we can write
$$\begin{aligned}A=&(3n+9)\mathcal{I}_{n+3}(n+3)+(n+3)3^{n+3}-(2n+6)\mathcal{C}_{n+3}(n+3,n+3)\\ =&(8n+21)\mathcal{I}_{n+3}(n+3)-(5n+12)\mathcal{I}_{n+3}(n+3)+(n+3)3^{n+3}\\ &-(2n+6)\mathcal{C}_{n+3}(n+3,n+3)\\ =&B+(n+3)3^{n+3}-(5n+12)\mathcal{I}_{n+3}(n+3)\\ &-(2n+6)\mathcal{C}_{n+3}(n+3,n+3).\end{aligned}\tag{7}$$
Utilizing Theorem 1 once more for $\mathcal{I}_{n+3}(n+3)$ and $\mathcal{I}_{n+2}(n+2)$ yields
$$\begin{aligned}A=&B+(n+3)3^{n+3}-(5n+12)3^{n+2}\\ &-(18n+45)\mathcal{I}_{n+2}(n+2)-(2n+6)\mathcal{C}_{n+3}(n+3,n+3)\\ &+(10n+24)\mathcal{C}_{n+2}(n+2,n+2)+(3n+9)\mathcal{I}_{n+2}(n+2)+(n+3)3^{n+3}\\ =&B-C-(5n+12)3^{n+2}-(2n+6)\mathcal{C}_{n+3}(n+3,n+3)\\ &+(10n+24)\mathcal{C}_{n+2}(n+2,n+2)+9n\mathcal{I}_{n+1}(n+1)\\ &+27\mathcal{I}_{n+1}(n+1)+(3n+9)3^{n+1}-(6n+18)\mathcal{C}_{n+1}(n+1,n+1).\end{aligned}$$

It can be easily shown that

$$\begin{aligned} A =& B - C + D \\ &+ (n+3)3^{n+3} - (2n+6)\mathcal{C}_{n+3}(n+3,n+3) - (5n+12)3^{n+2} \\ &+ (10n+24)\mathcal{C}_{n+2}(n+2,n+2) + 9n\mathcal{I}_{n+1}(n+1) \\ &+ (3n+9)3^{n+1} - (6n+18)\mathcal{C}_{n+1}(n+1,n+1). \end{aligned} \tag{8}$$

Replacing $9n\mathcal{I}_{n+1}(n+1)$ by $27n\mathcal{I}_n(n) + n3^{n+2} - 18n\mathcal{I}_n(n)$ in 8 gives

$$\begin{aligned} A = B - C + D + E \\ - (2n+6)\mathcal{C}_{n+3}(n+3,n+3) + (10n+24)\mathcal{C}_{n+2}(n+2,n+2) \\ - 18n\mathcal{C}_n(n,n) - (6n+18)\mathcal{C}_{n+1}(n+1,n+1). \end{aligned}$$

Since the coefficient of $\mathcal{C}_{n+3}(n+3,n+3)$ is $2(n+3)$, it follows from Theorem 5 that

$$\begin{aligned} A =& B - C + D + E - (4n+12)\mathcal{C}_{n+2}(n+2,n+2) - 18n\mathcal{C}_n(n,n) \\ &+ (10n+24)\mathcal{C}_{n+2}(n+2,n+2) - (6n+6)\mathcal{C}_{n+1}(n+1,n+1) \\ &- (6n+18)\mathcal{C}_{n+1}(n+1,n+1) \\ =& B - C + D + E - (4n+12)\mathcal{C}_{n+2}(n+2,n+2) \\ &- (6n+6)\mathcal{C}_{n+1}(n+1,n+1) + 18n\mathcal{C}_n(n,n) - 18n\mathcal{C}_n(n,n) \\ &- (12n+24)\mathcal{C}_{n+1}(n+1,n+1) + (6n+18)\mathcal{C}_{n+1}(n+1,n+1) \\ =& B - C + D + E, \end{aligned}$$

as required. □

Even though we have given a proof of Povolotsky's conjecture, it is still of interest to find a more intuitive proof not only of Povolotsky's recurrence relation but also of that of Somos.

Problem 1. *Give bijective proofs of Somos' and Povolotsky's recurrence relations.*

In Theorem 4, we have obtained a generating function for $\mathcal{C}_n := \mathcal{C}_n(n,n)$ to prove Somos' relation in Theorem 5. Actually, we can reverse the procedure and drive the generating function of $\mathcal{C}_n$ from Somos' relation as follows: Let $F(x) := \sum_{n=1}^{\infty} \mathcal{C}_n x^n$ be the generating function of $\mathcal{C}_n$. Somos' relation

$$n\mathcal{C}_n = 2n\mathcal{C}_{n-1} + 3(n-2)\mathcal{C}_{n-2}$$

yields

$$(n+2)\mathcal{C}_{n+2} = 2\mathcal{C}_{n+1} + 2(n+1)\mathcal{C}_{n+1} + 3n\mathcal{C}_n$$

for all $n \geqslant 1$. Thus,

$$\sum_{n=1}^{\infty}(n+2)\mathcal{C}_{n+2}x^{n+2} = 2x\sum_{n=1}^{\infty}\mathcal{C}_{n+1}x^{n+1} + 2x\sum_{n=1}^{\infty}(n+1)\mathcal{C}_{n+1}x^{n+1} + 3x^2\sum_{n=1}^{\infty}n\mathcal{C}_n x^n.$$

Now, the fact that $xF'(x) = \sum_{n=1}^{\infty} n\mathcal{C}_n x^n$ implies

$$-x - 4x^2 + xF'(x) = 2x(-x + F(x)) + 2x(-x + xF'(x)) + 3x^2 \cdot xF'(x)$$

for $\mathcal{C}_1 = 1$ and $\mathcal{C}_2 = 2$. Thus,

$$(3x^2 + 2x - 1)F'(x) + 2F(x) + 1 = 0.$$

Solving this linear differential equation yields

$$F(x) = \frac{1}{2}\sqrt{\frac{1+x}{1-3x}} - \frac{1}{2}.$$

An analogous argument applied to Povolotsky's recurrence relation shows that the generating function $F(x)$ of $\mathcal{I}_n(n)$ satisfies the differential equation

$$(27x^5 - 18x^3 + 8x^2 - x)F'(x) + (27x^3 - 9x^2 - 3x + 1)F(x) + 9x^3 - x^2 = 0,$$

from which we obtain an explicit formula for $F(x)$.

Theorem 6. *The generating function of $\mathcal{I}_n := \mathcal{I}_n(n)$ is given by*

$$\sum_{n=1}^{\infty} \mathcal{I}_n x^n = \frac{-x}{1-3x}\sqrt{\frac{1+x}{1-3x}} + \frac{x}{1-3x} + \frac{x}{(1-3x)^2}.$$

6. Further work

We end our paper by posing few open problems of the determinant of matrices arising from lattice paths as well as strings encoding them.

First, consider the $m \times n$ table T with $2n \geqslant m$. For suitable positive integers $\ell_1, \ell_2, \ldots, \ell_{\lceil \frac{m}{2} \rceil}$, we can write $\mathcal{I}_m(n)$ as

$$\mathcal{I}_m(n) = \ell_1 \mathcal{I}_m(n-1) + \ell_2 \mathcal{I}_m(n-2) + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_m(n - \lceil \frac{m}{2} \rceil).$$

Furthermore, for $0 \leqslant s \leqslant \lceil \frac{m}{2} \rceil$ and a suitable positive integer $k_{1,s}, k_{2,s}, \ldots, k_{\lceil \frac{m}{2} \rceil, s}$, we can write

$$\mathcal{I}_m(n-s) = k_{1,s}x_1 + k_{2,s}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, s} x_{\lceil \frac{m}{2} \rceil},$$

where $x_t = \mathcal{C}(n - \lceil \frac{m}{2} \rceil, t) = \sum_{i=1}^{m} \mathcal{C}^i(n - \lceil \frac{m}{2} \rceil, t)$ is the number of all lattice paths from the first column to the $(n - \lceil \frac{m}{2} \rceil, t)$-cell of T, for each $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \lceil \frac{m}{2} \rceil$. Utilizing the above notation, we observe that

$$\begin{aligned}
\mathcal{I}_m(n) =& k_{1,0}x_1 + k_{2,0}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 0} x_{\lceil \frac{m}{2} \rceil} \\
=& \ell_1 \mathcal{I}_{n-1} + \ell_2 \mathcal{I}_{n-2} + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_{n - \lceil \frac{m}{2} \rceil} \\
=& \ell_1 (k_{1,1}x_1 + k_{2,1}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 1} x_{\lceil \frac{m}{2} \rceil}) \\
& + \ell_2 (k_{1,2}x_1 + k_{2,2}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 2} x_{\lceil \frac{m}{2} \rceil}) \\
& \vdots \\
& + \ell_{\lceil \frac{m}{2} \rceil} (k_{1,\lceil \frac{m}{2} \rceil}x_1 + k_{2,\lceil \frac{m}{2} \rfloor}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil} x_{\lceil \frac{m}{2} \rceil}).
\end{aligned} \tag{9}$$

From (9), we obtain the following system of linear equations:

$$\begin{cases} k_{1,1}\ell_1 & + \cdots + & k_{1,\lceil \frac{m}{2}\rceil}\ell_{\lceil \frac{m}{2}\rceil} & = & k_{1,0}, \\ k_{2,1}\ell_1 & + \cdots + & k_{2,\lceil \frac{m}{2}\rceil}\ell_{\lceil \frac{m}{2}\rceil} & = & k_{2,0}, \\ \vdots & \vdots \ddots \vdots & \vdots & \vdots & \vdots \\ k_{\lceil \frac{m}{2}\rceil,1}\ell_1 & + \cdots + & k_{\lceil \frac{m}{2}\rceil,\lceil \frac{m}{2}\rceil}\ell_{\lceil \frac{m}{2}\rceil} & = & k_{\lceil \frac{m}{2}\rceil,0}. \end{cases} \tag{10}$$

Now, consider the following coefficient matrix A of system (10)

$$A = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,\lceil \frac{m}{2}\rceil} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,\lceil \frac{m}{2}\rceil} \\ \vdots & \vdots & \ddots & \vdots \\ k_{\lceil \frac{m}{2}\rceil,1} & k_{\lceil \frac{m}{2}\rceil,2} & \cdots & k_{\lceil \frac{m}{2}\rceil,\lceil \frac{m}{2}\rceil} \end{bmatrix},$$

which we call the *coefficient matrix* of the table T and denote it by $\mathcal{C}(T)$.

Conjecture 2. *For a given $m \times n$ table T $(2n \geqslant m)$, we have* $\det(\mathcal{C}(T)) = -2^{\lfloor \frac{m}{2} \rfloor}$.

Example 3. *Let T be a $5 \times n$ table. The columns $n-3$, $n-2$, $n-1$, and n of T are given by*

$n-3$	$n-2$	$n-1$	n
x_1	x_1+x_2	$2x_1+2x_2+x_3$	$4x_1+6x_2+3x_3$
x_2	$x_1+x_2+x_3$	$2x_1+4x_2+2x_3$	$6x_1+10x_2+6x_3$
x_3	$2x_2+x_3$	$2x_1+4x_2+3x_3$	$6x_1+12x_2+7x_3$
x_2	$x_1+x_2+x_3$	$2x_1+4x_2+2x_3$	$6x_1+10x_2+6x_3$
x_1	x_1+x_2	$2x_1+2x_2+x_3$	$4x_1+6x_2+3x_3$

from which it follows that

$$\begin{aligned} \mathcal{I}_5(n-3) &= 2x_1+2x_2+x_3, \\ \mathcal{I}_5(n-2) &= 4x_1+6x_2+3x_3, \\ \mathcal{I}_5(n-1) &= 10x_1+16x_2+9x_3, \\ \mathcal{I}_5(n) &= 28x_1+44x_2+25x_3. \end{aligned}$$

Clearly,

$$\mathcal{I}_5(n) = \ell_1\mathcal{I}_5(n-1) + \ell_2\mathcal{I}_5(n-2) + \ell_3\mathcal{I}_5(n-3)$$

for some ℓ_1, ℓ_2, ℓ_3, and the coefficient matrix of the table T is $\mathcal{C}(T) = \begin{bmatrix} 10 & 4 & 2 \\ 16 & 6 & 2 \\ 9 & 3 & 1 \end{bmatrix}$. *It is obvious that* $\det(\mathcal{C}(T)) = -2^{\lfloor \frac{5}{2} \rfloor} = -4$.

Our second problem is to compute the determinant of special *Hankel matrices*. Recall that a Hankel matrix (or catalecticant matrix) of a numerical sequence $\mathcal{C} = \{c_i\}$, named after Hermann Hankel, is a matrix defined as follows:

$$H_n^t(\mathcal{C}) = \begin{bmatrix} c_t & c_{t+1} & c_{t+2} & \cdots & c_{t+n-1} \\ c_{t+1} & c_{t+2} & c_{t+3} & \cdots & c_{t+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{t+n-1} & c_{t+n} & c_{t+n+1} & \cdots & c_{t+2n-2} \end{bmatrix}.$$

In [38, theorems 3 and 4], the authors use a sequence of ideas to reduce the problem to a previous paper of Cigler and Krattenthaler [9] (the first paper of this series), which describes the Hankel determinants $\det H_n^1(\mathcal{C})$ and $\det H_n^2(\mathcal{C})$ of some similar sequences $\mathcal{C}$. Now, consider the sequence $\mathcal{C} = \{\mathcal{C}_n(n,n)\}$ with elements $1, 1, 2, 5, 13, 35, 96, \ldots$. In what follows, we suggest the values of the determinant of the Hankel matrix $H_n^0(\mathcal{C})$

Conjecture 3. *For positive integers n, consider the Hankel matrix*

$$H_n^0(\mathcal{C}) = \begin{bmatrix} 1 & 1 & 2 & 5 & \ldots & c_n \\ 1 & 2 & 5 & 13 & \ldots & c_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ c_n & c_{n+1} & c_{n+2} & c_{n+3} & \ldots & c_{2n} \end{bmatrix}.$$

Then

$$\det H_n^0(\mathcal{C}) = \begin{cases} 1, & n \equiv 1,2 \pmod 6, \\ 0, & n \equiv 0,3 \pmod 6, \\ -1, & n \equiv 4,5 \pmod 6. \end{cases}$$

In the rest of this section we give some general questions concerning lattice paths inside a table.

Most lattice paths can be stated in terms of strings over a given alphabet, say $\{-1, 0, 1\}$ in our case, admitting some constraints like forbidden patterns (see e.g. [7]). It is natural to ask the same question for lattice paths studied in this paper.

Problem 2. *Which are the strings encoding the paths inside the $m \times n$ table?*

In [4, 8], the authors present a Gray code for the q-ary k-generalized Fibonacci strings and particular regular languages. If the answer to the above problem is positive, one may ask if a Gray code exists for the given strings.

Problem 3. *Is there a possibility to list the encoded paths (strings over $\{-1, 0, 1\}$) in a Gray code sense?*

Lattice paths using the same steps as ours are studied extensively in the literature. In [5, 6], the authors study prefixes of such paths, namely Dyke, Motzking, and Schröder paths, and give algorithms for their generation and enumeration. The same problem arises in relation to our lattice paths as well.

Problem 4. *Enumerate the prefixes of certain length of lattice paths inside a table.*

Acknowledgement

The work of the third author is supported by the Swiss National Science Foundation project 200020-169022 “Lift and Project Methods for Machine Scheduling Through Theory and Experiments”. The authors would also like to thank the referees for many suggestions and corrections. Also, the authors thanks to Michael Somos for introducing his identity and suggesting a direct proof.

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