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Gaussian limit theorem for posterior distribution in the problem of conflicting expert opinions

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Abstract. Suppose we have n experts who have their prior opinions about the unknown probability q in the experiment with a binary outcome. It is known that expert opinions are in conflict with each other. To model "conflicting" expert opinions a prior distribution based on Selberg's integral is constructed. We prove a theorem regarding the limiting properties of the posterior distribution. Also, differential entropy and the Kullback-Leibler (KL) divergence of such posterior are studied.

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1. Introduction

When conducting a clinical trial, a researcher might be pursuing competing objectives, one of them is to maximize the statistical power, another is to maximize the number of patients responding to the treatment. For entropy-based designs, e.g. in a setting used in [7], the experiment starts with the treatment arm that minimises the proposed asymptotic criteria based on the prior Beta distribution. In the case of the rare disease Phase II clinical trials, the number of patients can be limited. To improve the operating characteristics of the designs expert opinions can be collected beforehand to specify the prior distribution of the parameters under study. Then several opinions must be combined to be used to calibrate the design characteristics. This process of synthesis of authorities opinions is called expert elicitation. Generally, this approach provides a way to quantify uncertainty. A popular method for quantifying uncertainty is the Bayesian approach.

In the setting of this work, we are talking about one treatment with a binary response with an unknown probability q. Several (n) experts have opinions on the distribution of q. We also assume that their responses are correlated with each other: they can either agree or disagree with each other.

The goal of this research is to construct a prior, which takes into consideration additional information on the degree of conflict in expert opinions. This is a continuation and generalization of some results obtained in [8]. Note that a treatment

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can possibly have multiple responses, hence in a classical setting with no conflict a prior would take a form of multivariate generalization of the Beta distribution, or the Dirichlet distribution. Also, the experts can be asked about several treatments at once. These topics will be the scope of further research.

Let there be m conditionally independent random variables $\xi_i, i = 1, \ldots, m$, identically distributed with $\mathcal{B}ernoulli(q)$. Let x_i be the value obtained in the i-th experiment, $S_m = \xi_1 + \xi_2 + \ldots + \xi_m$, and $x = \sum_{i=1}^{m} x_i$.

Consider a function

$$f(z_1, z_2, \ldots, z_n) = \frac{1}{S_n(\alpha, \beta, \gamma)} \prod_{i=1}^{n} z_i^{\alpha-1} (1 - z_i)^{\beta-1} \prod_{1 \le i < j \le n} |z_i - z_j|^{2\gamma}, \tag{1}$$

where $S_n(\alpha, \beta, \gamma)$ is the Selberg integral [10] defined as follows:

$$\begin{aligned} S_n(\alpha, \beta, \gamma) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{\alpha-1} (1 - t_i)^{\beta-1} \prod_{1 \le i < j \le n} |t_i - t_j|^{2\gamma} \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(\alpha + \beta + (n + j - 1)\gamma)\Gamma(1 + \gamma)}, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}, \alpha > 0, \beta > 0, \gamma > -\min\left(\frac{1}{n}, \frac{\alpha}{n-1}, \frac{\beta}{n-1}\right)$.

It is known that $f(z_1, z_2, \ldots, z_n)$ is a probability density function called the multivariate Selberg-Beta distribution of the first type [9]. In the context of a Bayesian setting, this distribution can be regarded as a *conflicting prior distribution* $\pi(\mathbf{q})$ (see Figure 1), where γ shows to what degree the experts are in conflict, and $\mathbf{q} \sim MSBeta1(\alpha, \beta, \gamma, n)$ is random vector of the unknown response probability $\mathbf{q}$ considered by each of the n experts. If $\gamma = 0$, hence there is no conflict, the distribution is equivalent to n independent Beta priors with parameters α, β. If $\gamma < 0$, the experts tend to agree with each other.

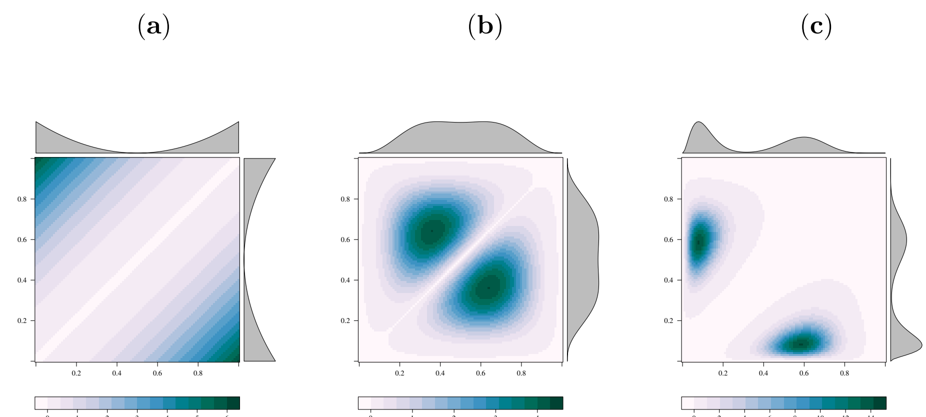

Figure 1: *Joint and marginal distribution functions for certain conflicting priors*: **(a)** $MSBeta1(1,1,1,2)$, **(b)** $MSBeta1(4,4,0.5,2)$, **(c)** $MSBeta1(3,9,4,2)$.

Let n experts have the joint prior distribution given by the density function f. Hence, by Bayes' theorem, the posterior probability density that after m experiments we observe x successes takes the form of MSBeta1$(\alpha + x, \beta + m - x, \gamma, n)$:

$$\begin{aligned}\pi(\mathbf{q} \mid S_m = x) &= f_m\left(z_1, z_2, \ldots, z_n \mid S_m = x\right)\\ &= \frac{f\left(z_1, z_2, \ldots, z_n\right) \prod_{i=1}^{n} z_i^x \left(1 - z_i\right)^{m-x}}{\int_0^1 \ldots \int_0^1 \prod_{i=1}^{n} t_i^x \left(1 - t_i\right)^{m-x} f\left(t_1, t_2, \ldots, t_n\right) \mathrm{d}t_1 \ldots \mathrm{d}t_n}\\ &= \frac{1}{S_n(\alpha + x, \beta + m - x, \gamma)} \prod_{i=1}^{n} z_i^{x+\alpha-1} \left(1 - z_i\right)^{m-x+\beta-1} \prod_{1 \le i < j \le n} \left|z_i - z_j\right|^{2\gamma}.\end{aligned}$$

By Aomoto's integral formula [1], the mean value predicted by the k-th expert equals:

$$\begin{aligned}\mathbb{E}(q_k) &= \frac{1}{S_n(\alpha, \beta, \gamma)} \int_0^1 \cdots \int_0^1 z_k \prod_{i=1}^{n} z_i^{\alpha-1} \left(1 - z_i\right)^{\beta-1} \prod_{1 \le i < j \le n} \left|z_i - z_j\right|^{2\gamma} \mathrm{d}z_1 \cdots \mathrm{d}z_n\\ &= \frac{\alpha + (n-1)\gamma}{\alpha + \beta + 2(n-1)\gamma}.\end{aligned}$$

Following the results obtained in [9], the mean and the moments of the posterior distribution of an unknown parameter q considered by the k-th expert are given as:

$$\begin{aligned}\mathbb{E}_m\left(q_k\right) &= \frac{\alpha + x + (n-1)\gamma}{\alpha + \beta + m + (2n-2)\gamma},\\ \mathbb{E}_m\left(q_k^2\right) &= \frac{[\alpha + x + 1 + 2(n-1)\gamma]\mathbb{E}_m\left(q_k\right) - (n-1)\gamma\mathbb{E}_m\left(q_1 q_2\right)}{\alpha + \beta + m + 1 + 2(n-1)\gamma},\\ \mathbb{E}_m\left(q_k q_l\right) &= \prod_{j=1}^{2} \frac{\alpha + x + (n-j)\gamma}{\alpha + \beta + m + (2n-1-j)\gamma}.\end{aligned}$$

Figure 2 illustrates that if a conflict parameter $\gamma > 0$, then the correlation between the opinions is negative, and vice versa. If $\gamma = 0$, then the correlation is zero.

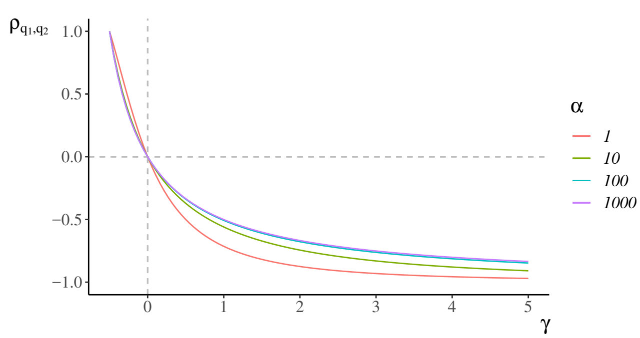

Figure 2: *Correlation of q_1 and q_2 in the conflicting prior distribution $MSBeta1(\alpha, 1, \gamma, 2)$ with $\alpha = \{1, 10, 100, 1000\}$ depending on a conflict parameter γ.*

2. Gaussian limit theorem

Consider a random variable $Z^m_{\kappa_m}$ with the density f_{m,κ_m}, which has the form (1), for $x = x(m) = \kappa_m m$, $0 < \kappa_m < 1$. Hence, κ_m represents the observed proportion of successes. By the Law of Large Numbers $\kappa_m \to \kappa$ a.s., the true probability of success, as $m \to \infty$. So, κ is the mean of binomial r.v., and $\frac{\kappa(1-\kappa)}{m}$ is its variance. To formulate the Gaussian limit theorem $Z^m_{\kappa_m}$ should be normalized. Thus, we introduce a new random variable $\widetilde{Z}_m$, which is expressed in terms of $Z^m_{\kappa_m}$, and denote its density as $\widetilde{f}_m = \widetilde{f}_{m,\kappa}$.

$$\begin{aligned}\widetilde{Z}_m &= A\left(Z^m_{\kappa_m} - \overline{\kappa}_m\right)^\top \\ &= \left(\frac{\sqrt{m}\left(Z^m_{\kappa_m,1} - \kappa_m\right)}{\sqrt{\kappa_m(1-\kappa_m)}}, \frac{\sqrt{m}\left(Z^m_{\kappa_m,2} - \kappa_m\right)}{\sqrt{\kappa_m(1-\kappa_m)}}, \dots, \frac{\sqrt{m}\left(Z^m_{\kappa_m,n} - \kappa_m\right)}{\sqrt{\kappa_m(1-\kappa_m)}}\right)^\top,\end{aligned}$$

where $A = \operatorname{diag}\left(\sqrt{\frac{m}{\kappa_m(1-\kappa_m)}}, \dots, \sqrt{\frac{m}{\kappa_m(1-\kappa_m)}}\right)$.

Our goal is to demonstrate that the correlation in the prior washes out in the large sample limit $m \to \infty$. Hence, the theorem can be formulated in the following way.

Theorem 1. *$\widetilde{Z}_m$ weakly converges in distribution to Z as $m \to \infty$:*

$$\widetilde{Z}_m \Rightarrow Z, \tag{2}$$

where $Z \sim \mathcal{N}(\vec{0}, \mathbb{I})$.

Proof. To prove the weak convergence in distribution of $\widetilde{Z}_m$ to Z in (2) we use the method of characteristic functions. The characteristic function of the standard multivariate normal distribution has the form:

$$e^{-\frac{1}{2}\sum\limits_{j=1}^{n} t_j^2}.$$

Let $\varphi_m(\mathbf{t})$ be the characteristic function of $\widetilde{Z}_m$. We should prove that as $m \to \infty$ the following holds:

$$\varphi_m(\mathbf{t}) = e^{-\frac{1}{2}\sum\limits_{j=1}^{n} t_j^2} + O\left(\frac{1}{m}\right).$$

In view of Slutsky's theorem, it is enough to prove the statement for non-random $\kappa = \lim_{m\to\infty} \kappa_m$ as the correction term tends to 1. Using the definition of $\varphi_m(\mathbf{t})$, after expanding the brackets and simplifications, we get:

$$\begin{aligned}\varphi_m(\mathbf{t}) &= \mathbb{E}\left[e^{i\left(t_1\widetilde{Z}^m_{\kappa,1} + \ldots + t_n\widetilde{Z}^m_{k,n}\right)}\right] = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} e^{it_j\frac{\sqrt{m}(z_j-\kappa)}{\sqrt{\kappa(1-\kappa)}}} f_{m,\kappa}\mathrm{d}z_1 \ldots \mathrm{d}z_n \\ &= \frac{1}{S_n(\alpha + \kappa m, \beta + (1-\kappa)m, \gamma)} \prod_{j=1}^{n} e^{it_j\frac{\sqrt{m}(-\kappa)}{\sqrt{\kappa(1-\kappa)}}} I(m)\end{aligned}$$

where

$$I(m) = \int_0^1 \dots \int_0^1 \prod_{j=1}^{n} e^{it_j \frac{\sqrt{m} z_j}{\sqrt{\kappa(1-\kappa)}}} z_j^{\kappa m+\alpha-1} (1-z_j)^{(1-\kappa)m+\beta-1} \times \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma} \,\mathrm{d}z_1 \dots \mathrm{d}z_n. \tag{3}$$

We introduce two functions:

$$S(z_1,\dots,z_n) = \sum_{j=1}^{n} it_j \frac{z_j}{\sqrt{m\kappa(1-\kappa)}} + \kappa \ln(z_j) + (1-\kappa)\ln(1-z_j),$$

$$g(z_1,\dots,z_n) = \prod_{j=1}^{n} z_j^{\alpha-1}(1-z_j)^{\beta-1} \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma}.$$

Note that $\mathrm{Re}(S(z))$ has a sharp maximum for sufficiently large m. Hence, (3) can be rewritten as:

$$I(m) = \int_0^1 \dots \int_0^1 e^{mS(z_1,\dots,z_n)} g(z_1,\dots,z_n)\,\mathrm{d}z_1 \dots \mathrm{d}z_n.$$

To find the asymptotics of $I(m)$ we use the saddle point method [5]:

$$I(m) = e^{mS(z^*)} \sqrt{\frac{(2\pi)^n}{m^n \det(-S''(z^*))}} \left(g(z^*) + O\left(\frac{1}{m}\right)\right),$$

where $S''(z^*)$ is the Hessian and z^* is a saddle point:

$$z^* = \left(\kappa + \frac{it_1\sqrt{\kappa(1-\kappa)}}{\sqrt{m}} + O\left(\frac{1}{m}\right), \dots, \kappa + \frac{it_n\sqrt{\kappa(1-\kappa)}}{\sqrt{m}} + O\left(\frac{1}{m}\right)\right).$$

After substituting the value of z^* we get:

$$g(z^*)\,e^{mS(z^*)} = \prod_{j=1}^{n} e^{it_j \frac{\sqrt{m}\kappa}{\sqrt{\kappa(1-\kappa)}}} e^{-t_j^2} \left(z_j^*\right)^{\kappa m+\alpha-1} \left(1-z_j^*\right)^{(1-\kappa)m+\beta-1} \prod_{1\le i<j\le n} \left|z_i^* - z_j^*\right|^{2\gamma}.$$

Thus, we derive a representation for the characteristic function of $\widetilde{Z}_\kappa^m$:

$$\varphi_m(\mathbf{t}) = \frac{\prod_{j=1}^{n} \left(z_j^*\right)^{\kappa m+\alpha-1} \left(1-z_j^*\right)^{(1-\kappa)m+\beta-1} \prod_{1\le i<j\le n} \left|z_i^* - z_j^*\right|^{2\gamma}}{S_n(\alpha+\kappa m, \beta+(1-\kappa)m, \gamma)} \times e^{-\sum_{j=1}^{n} t_j^2} \left(\frac{2\pi}{m\kappa(1-\kappa)}\right)^{\frac{n}{2}} \left(1 + O\left(\frac{1}{m}\right)\right).$$

Using the Stirling formula and the Taylor series for the logarithm we obtain:

$$\frac{\prod_{j=1}^{n}\left(z_j^*\right)^{\kappa m+\alpha-1}\left(1-z_j^*\right)^{(1-\kappa)m+\beta-1}\prod_{1\le i<j\le n}\left|z_i^*-z_j^*\right|^{2\gamma}}{S_n(\alpha+\kappa m,\beta+(1-\kappa)m,\gamma)}$$
$$=\frac{\prod_{j=1}^{n} e^{(\kappa m+\alpha-1)\ln\left(z_j^*\right)+((1-\kappa)m+\beta-1)\ln\left(1-z_j^*\right)}\prod_{1\le i<j\le n} e^{2\gamma\ln\left|z_i^*-z_j^*\right|}}{S_n(\alpha+\kappa m,\beta+(1-\kappa)m,\gamma)}$$
$$=e^{\frac{1}{2}\sum_{j=1}^{n}t_j^2}\left(\frac{m\kappa(1-\kappa)}{2\pi}\right)^{\frac{n}{2}}\left(1+O\left(\frac{1}{m}\right)\right).$$

Finally,

$$\varphi_m(\mathbf{t})=e^{\frac{1}{2}\sum_{j=1}^{n}t_1^2}\left(\frac{m\kappa(1-\kappa)}{2\pi}\right)^{\frac{n}{2}}e^{-\sum_{j=1}^{n}t_j^2}\left(\frac{2\pi}{m\kappa(1-\kappa)}\right)^{\frac{n}{2}}\left(1+O\left(\frac{1}{m}\right)\right)$$
$$=e^{-\frac{1}{2}\sum_{j=1}^{n}t_i^2}+O\left(\frac{1}{m}\right)$$

implies the weak convergence in distribution of the r.v. $\widetilde{Z}_m$ to Z and completes the proof of Theorem 1. □

Theorem 2. *The differential entropy $\widetilde{Z}_m$ converges to the differential entropy of Z as $m\to\infty$:*

$$h\left(\widetilde{f}_{m,\kappa}\right)\longrightarrow\frac{1}{2}\log\left((2\pi e)^n\right)=h(\phi),$$

where $\phi(\mathbf{z})=(2\pi)^{-n/2}\exp\left(-\frac{1}{2}\mathbf{z}^\top\mathbf{z}\right)$ is the PDF of Z.

Proof. Consider differential Shannon (4) and Rényi (5) entropies [4] for a r.v. with the density function $f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)$:

$$h\left(f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)\right)$$
$$=-\int_0^1\ldots\int_0^1 f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)\log\left(f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)\right)\mathrm{d}z_1\ldots\mathrm{d}z_n,\quad(4)$$
$$H_\nu\left(f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)\right)$$
$$=\frac{1}{1-\nu}\log\left(\int_0^1\ldots\int_0^1\left(f_{m,\kappa}\left(z_1,z_2,\ldots,z_n\right)\right)^\nu\mathrm{d}z_1\ldots\mathrm{d}z_n\right),\quad(5)$$

where $\nu\ge 0,\nu\ne 1$. Note that Shannon's entropy is bounded from ∞ because two moments are finite, and from $-\infty$ because the PDF (1) is log-concave, cf. the appendix in [3].

Before proceeding with the proof of Theorem 2, we need to find

$H_\nu\left(f_{m,\kappa}\left(z_1, z_2, \ldots, z_n\right)\right)$:

$$\begin{aligned}
&(1-\nu) H_\nu\left(f_{m,\kappa}\left(z_1, z_2, \ldots, z_n\right)\right) \\
&\quad= \log\left(\int_0^1 \cdots \int_0^1 \left(f_{m,\kappa}\left(z_1, z_2, \ldots, z_n\right)\right)^\nu \mathrm{d}z_1 \ldots \mathrm{d}z_n\right) \\
&\quad= \log\left(\frac{1}{S_n(\alpha+x, \beta+m-x, \gamma)^\nu} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\nu(x+\alpha-1)} \left(1-z_i\right)^{\nu(m-x+\beta-1)}\right. \\
&\qquad \left. \times \prod_{1 \le i<j \le n} \left(z_i - z_j\right)^{2\gamma\nu} \mathrm{d}z_1 \ldots \mathrm{d}z_n\right) \\
&\quad= \log\left(\frac{S_n(\nu(x+\alpha-1)+1, \nu(m-x+\beta-1)+1, \nu\gamma)}{S_n(\alpha+x, \beta+m-x, \gamma)^\nu}\right) \\
&\quad= \sum_{i=0}^{n-1} \log\left(\frac{\Gamma(\nu(x+\alpha-1)+1+i\nu\gamma)\Gamma(\nu(m-x+\beta-1)+1+i\nu\gamma)\Gamma(1+(i+1)\nu\gamma)}{\Gamma(x+\alpha+i\gamma)^\nu \Gamma(m-x+\beta+i\gamma)^\nu \Gamma(1+(i+1)\gamma)^\nu}\right. \\
&\qquad \left. \times \frac{\Gamma(\alpha+\beta+m+(n+i-1)\gamma)^\nu \Gamma(1+\gamma)^\nu}{\Gamma(\nu(x+\alpha-1)+\nu(m-x+\beta-1)+2+(n+i-1)\nu\gamma)\Gamma(1+\nu\gamma)}\right)
\end{aligned}$$

Using the properties of g Gamma functions, we obtain an expression for $H_\nu\left(f_{m,\kappa}\left(z_1, z_2, \ldots, z_n\right)\right)$:

$$\begin{aligned}
&H_\nu\left(f_{m,\kappa}\right) \\
&\quad= \sum_{i=0}^{n-1} \frac{1}{1-\nu}\left(\log\left(\frac{\Gamma(\nu(x+\alpha-1+i\gamma))}{\Gamma(x+\alpha-1+i\gamma)^\nu}\right) + \log\left(\frac{\Gamma(\nu(m-x+\beta-1+i\gamma))}{\Gamma(m-x+\beta-1+i\gamma)^\nu}\right)\right. \\
&\qquad \left. + \log\left(\frac{\Gamma(\nu(i+1)\gamma)}{\Gamma((i+1)\gamma)^\nu}\right) + \log\left(\frac{\Gamma(\gamma)^\nu}{\Gamma(\nu\gamma)}\right) + \log\left(\frac{\Gamma(\alpha+\beta+m-2+(n+i-1)\gamma)^\nu}{\Gamma(\nu(\alpha+\beta+m-2+(n+i-1)\gamma))}\right)\right) \\
&\qquad + \log\left(\frac{(x+\alpha-1+i\gamma)(m-x+\beta-1+i\gamma)(i+1)}{\alpha+\beta+m-2+(n+i-1)\gamma}\right) \\
&\qquad + \frac{1}{1-\nu} \log\left(\frac{\nu(\alpha+\beta+m-1+(n+i-1)\gamma)^\nu}{1+\nu(\alpha+\beta+m-2+(n+i-1)\gamma)}\right).
\end{aligned}$$

Now we can find $h\left(f_{m,\kappa}\right)$ by taking a limit:

$$\begin{aligned}
h\left(f_{m,\kappa}\right) &= \lim_{\nu \to 1} H_\nu\left(f_{m,\kappa}\right) \\
&= \sum_{i=0}^{n-1} \log(\Gamma(x+\alpha-1+i\gamma)) - (x+\alpha-1+i\gamma)\psi(x+\alpha-1+i\gamma)\log(e) \\
&\quad + \log(\Gamma(m-x+\beta-1+i\gamma)) - (m-x+\beta-1+i\gamma)\psi(m-x+\beta-1+i\gamma)\log(e) \\
&\quad + \log(\Gamma((i+1)\gamma)) - (i+1)\gamma\psi((i+1)\gamma)\log(e) - \log(\Gamma(\gamma)) + \gamma\psi(\gamma)\log(e) \\
&\quad - \log(\Gamma(\alpha+\beta+m-2+(n+i-1)\gamma))
\end{aligned}$$

$$+(\alpha+\beta+m-2+(n+i-1)\gamma)\psi(\alpha+\beta+m-2+(n+i-1)\gamma)\log(e)$$
$$-\frac{\log(e)}{\alpha+\beta+m-1+(n+i-1)\gamma}-\log(\alpha+\beta+m-1+(n+i-1)\gamma)$$
$$+\log\left(\frac{(x+\alpha-1+i\gamma)(m-x+\beta-1+i\gamma)(i+1)}{\alpha+\beta+m-2+(n+i-1)\gamma}\right),$$

where ψ stands for the digamma function.

Using the asymptotics of gamma and digamma functions [6], we derive the asymptotics for $h\left(f_{m,\kappa}\right)$:

$$\begin{aligned}
h\left(f_{m,\kappa}\right) &= \sum_{i=0}^{n-1}(n-2i-1)\gamma\log(e)+\frac{1}{2}\log(2\pi e) \\
&\quad+\left(\frac{1}{2}\log\left(\frac{(x+\alpha-1+i\gamma)(m-x+\beta-1+i\gamma)(i+1)}{\alpha+\beta+m-2+(n+i-1)\gamma}\right)\right. \\
&\quad\left.-\log(\alpha+\beta+m-1+(n+i-1)\gamma)\right)\left(1+O\left(\frac{1}{m}\right)\right) \\
&= \frac{1}{2}\log\left(\prod_{i=0}^{n-1}\frac{2\pi e(x+\alpha-1+i\gamma)(m-x+\beta-1+i\gamma)}{(\alpha+\beta+m-2+(n+i-1)\gamma)(\alpha+\beta+m-1+(n+i-1)\gamma)^2}\right) \\
&\quad\times\left(1+O\left(\frac{n(\log(n)-1)}{m}\right)\right).
\end{aligned}$$

Thus, $h\left(f_{m,\kappa}\right)$ takes the form:

$$\begin{aligned}
h\left(f_{m,\kappa}\right) &= \frac{1}{2}\log\left(\prod_{i=0}^{n-1}\frac{2\pi e(\kappa m+\alpha-1+i\gamma)((1-\kappa)m+\beta-1+i\gamma)}{(\alpha+\beta+m-2+(n+i-1)\gamma)(\alpha+\beta+m-1+(n+i-1)\gamma)^2}\right) \\
&\quad\times\left(1+O\left(\frac{n(\log(n)-1)}{m}\right)\right).
\end{aligned}$$

Note that for $n=1,\gamma=0,\alpha=\beta=1$ this result is similar to the results obtained in [8]. Using the properties of differential entropy, we can find

$$\begin{aligned}
h\left(\widetilde{f}_{m,\kappa}\right) &= h\left(f_{m,\kappa}\right)+\log(|\det(A)|) \\
&= \left(\frac{1}{2}\log\left(\prod_{i=0}^{n-1}\frac{2\pi e(\kappa m+\alpha-1+i\gamma)((1-\kappa)m+\beta-1+i\gamma)}{(\alpha+\beta+m-2+(n+i-1)\gamma)(\alpha+\beta+m-1+(n+i-1)\gamma)^2}\right)\right. \\
&\quad\left.+\frac{n}{2}\log\left(\frac{m}{\kappa(1-\kappa)}\right)\right)\left(1+O\left(\frac{n(\log(n)-1)}{m}\right)\right).
\end{aligned}$$

Thus, for $m\to\infty$,

$$h\left(\widetilde{f}_{m,\kappa}\right)\longrightarrow\frac{n}{2}\log\left(\frac{2\pi e\kappa(1-\kappa)}{m}\right)+\frac{n}{2}\log\left(\frac{m}{\kappa(1-\kappa)}\right)=\frac{1}{2}\log\left((2\pi e)^n\right)=h(\phi),$$

which concludes the proof of Theorem 2. □

Theorem 3. *The Kullback-Leibler (KL) divergence between $\widetilde{f}_{m,\kappa}$ and ϕ converges to zero as $m \to \infty$:*

$$\mathrm{KL}\left(\widetilde{f}_{m,\kappa}\|\phi\right) \longrightarrow 0,$$

where the Kullback-Leibler divergence between distributions π_1 and π_2 is defined as

$$\mathrm{KL}\left(\pi_1\|\pi_2\right) = \int_{\mathbb{R}^n} \pi_1(\mathbf{x}) \log \frac{\pi_1(\mathbf{x})}{\pi_2(\mathbf{x})} \mathrm{d}\mathbf{x}.$$

Proof. Using the results obtained in Theorem 2, we calculate the Kullback - Leibler divergence between $\widetilde{f}_{m,\kappa}$ and ϕ:

$$\begin{aligned}
\mathrm{KL}\left(\widetilde{f}_{m,\kappa}\|\phi\right) &= h\left(\widetilde{f}_{m,\kappa}, \phi\right) - h\left(\widetilde{f}_{m,\kappa}\right) \\
h\left(\widetilde{f}_{m,\kappa}, \phi\right) &= \int_0^1 \dots \int_0^1 \widetilde{f}_{m,\kappa}\left(z_1, z_2, \dots, z_n\right) \log\left(\phi\left(z_1, z_2, \dots, z_n\right)\right) \mathrm{d}z_1 \dots \mathrm{d}z_n \\
&= \frac{1}{2}\log\left((2\pi)^n\right) + \frac{1}{2}\log(e) \sum_{i=1}^{n} \int_0^1 \dots \int_0^1 \widetilde{f}_{m,\kappa}\left(z_1, z_2, \dots, z_n\right) z_i^2 \mathrm{d}z_1 \dots \mathrm{d}z_n.
\end{aligned}$$

By the results of Theorem 1, one can check that the second moment of $\widetilde{Z}_\kappa$ converges to one as $m \to \infty$. Hence,

$$\mathrm{KL}\left(\widetilde{f}_{m,\kappa}\|\phi\right) \to -\frac{1}{2}\log\left((2\pi e)^n\right) + \frac{1}{2}\log\left((2\pi)^n\right) + \frac{1}{2}\log(e)n = 0,$$

which concludes the proof of Theorem 3. Note that this result implies the weak convergence established in Theorem 1, cf. [2]. □

We have shown that the random variable $\widetilde{Z}_m$ obtained from $Z^m_{\kappa_m}$ is asymptotically normally distributed as $m \to \infty$. In future research, one could consider other types of relationship for $x = x(m)$ and study its properties. The problem we have analyzed has a natural extension in the context of weighted differential entropies [11, 12].

3. Conclusions

We have presented the conflicting prior distribution based on Selberg's integral considered in a setting of an expert elicitation problem and the Gaussian limit theorem for its posterior distribution. Applications of this prior distribution rely on the assumption that in a setting similar to [7], such a prior with an additional conflict parameter in the prior distribution would allow to achieve faster convergence to the truth than selecting one of experts' opinions at random.

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