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Cartan calculus on the superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})^*$

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Abstract. By analogy with the classical case, noncommutative differential calculus on a quantum superspace can be extended to the Cartan calculus by introducing inner derivations and Lie derivatives. So, to give the Cartan calculus on the algebra of functions on quantum (2+1)-superspace $\mathbb{C}_q^{2|1}$, we first introduce two left-covariant differential calculi over $\mathcal{O}(\mathbb{C}_q^{2|1})$ and extend one of these calculi by adding inner derivations and Lie derivatives to the calculus. We also introduce tensor product realization of the wedge product of forms.

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1. Introduction

Noncommutative geometry continues to play an important role in different fields of mathematics and mathematical physics. The goal of this geometry is a differential calculus on an associative algebra. In the approach of Woronowicz [16], the quantum group is taken as noncommutative space, and the differential calculus on the group is derived from the properties of the group. Differential calculi on certain classes of quantum homogeneous spaces are described in [1, 2, 12]. The quantum plane and superplane are the simplest samples of noncommutative spaces. Following Woronowicz's approach, noncommutative differential calculi on some lower dimensional superspaces are presented in [3, 4].

In another approach initiated by Wess and Zumino [15], differential forms are defined by the differential and algebraic properties of quantum coordinates and quantum groups acting on them. The natural extension of their scheme to superspace [11] was introduced by many authors (for example, [5, 10, 14]).

Using the approach in [13], the extended calculus on the quantum plane was introduced in [9]. The extended calculus on the quantum superplane was introduced in [6].

In this paper, we investigate the noncommutative geometry of the algebra of functions on the quantum $(2 + 1)$-superspace denoted by $\mathcal{O}(\mathbb{C}_q^{2|1})$. In Section 4,

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we set up two left-covariant differential calculi, covariant under $\mathrm{GL}_q(2|1)$, on the quantum superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$. In Section 5, we extend one of these calculi by adding inner derivations and Lie derivatives to the calculus. In Section 6, we reformulate the results we got in the previous sections with an R-matrix and introduce the tensor product realization of the wedge product.

2. Preliminaries

In this section, we will briefly talk about some of the known basic concepts as necessary. Throughout the paper, we will fix a base field $\mathbb{C}$, the set of complex numbers. We write $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$.

A super vector space V over $\mathbb{C}$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ and we write $V = V_0 \oplus V_1$, where V_0 and V_1 are even and odd subspaces of V, respectively. The elements of V_0 and V_1 are called even and odd, respectively. The elements of $V_0 \cup V_1$ will be called homogeneous. For a homogeneous element v we write $p(v)$ for the parity or degree; if $v \in V_0$ (resp. V_1) we have $p(v) = 0$ (resp. 1).

A superalgebra (or $\mathbb{Z}_2$-graded algebra) $\mathcal{A}$ over $\mathbb{C}$ is a super vector space over $\mathbb{C}$ with a map $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$ for $i, j = 0, 1$. If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{Z}_2$-graded algebras, then the tensor product $\mathcal{A} \otimes \mathcal{B}$ exists. The following definition gives the product rule for the tensor product of $\mathbb{Z}_2$-graded algebras.

Definition 1. *If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{Z}_2$-graded algebras, their tensor product rule is defined by*

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{p(a_2)p(b_1)}(a_1 b_1 \otimes a_2 b_2),$$

where a_i's and b_i's are homogeneous elements of $\mathcal{A}$ and $\mathcal{B}$, respectively.

The elementary properties of a Hopf superalgebra are similar to the corresponding properties of ordinary Hopf algebras.

Definition 2. *A Hopf superalgebra (or a $\mathbb{Z}_2$-graded Hopf algebra) is a super vector space $\mathcal{H}$ over $\mathbb{C}$ with two algebra homomorphisms $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ called the coproduct, $\epsilon : \mathcal{H} \to \mathbb{C}$ called the counit and an algebra antihomomorphism $S : \mathcal{H} \to \mathcal{H}$ called the antipode, such that*

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta) \circ \Delta, \\
m \circ (\epsilon \otimes \mathrm{id}) \circ \Delta &= \mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta, \\
m \circ (S \otimes \mathrm{id}) \circ \Delta &= \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,
\end{aligned}$$

and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\epsilon(\mathbf{1}) = 1$, $S(\mathbf{1}) = \mathbf{1}$, where m is the multiplication map, id *is the identity map and $\eta : \mathbb{C} \longrightarrow \mathcal{H}$.*

Definition 3. *Let $\mathcal{X}$ be a superalgebra and $\mathcal{H}$ a Hopf superalgebra. Then the superalgebra $\mathcal{X}$ is called a left $\mathcal{H}$-comodule algebra if there exists an algebra homomorphism $\delta_L : \mathcal{X} \longrightarrow \mathcal{H} \otimes \mathcal{X}$ such that*

$$(\mathrm{id} \otimes \delta_L) \circ \delta_L = (\Delta \otimes \mathrm{id}) \circ \delta_L \quad \text{and} \quad (\epsilon \otimes \mathrm{id}) \circ \delta_L = \mathrm{id}.$$

Moreover, δ_L is called the left coaction of $\mathcal{H}$ on $\mathcal{X}$.

3. The algebra of polynomials on quantum superspace $\mathbb{C}_q^{2|1}$

Let x, y and θ be the elements of a superalgebra, where the generators x and y are of degree 0 (or even),and the generator θ is of degree 1 (or odd). Let $\mathcal{O}(\mathbb{C}^{2|1})$ be defined as the polynomial algebra $\mathbb{C}[x,y,\theta]$. It will sometimes be convenient and more illustrative to write a point (x,y,θ) of $\mathcal{O}(\mathbb{C}^{2|1})$ in the vector form $\mathbf{x} = (x,y,\theta) = (x_i)^t$.

Let $\mathbb{C}\langle x,y,\theta\rangle$ be a free algebra with the unit generated by x, y and θ, where $p(x) = 0 = p(y)$ and $p(\theta) = 1$. Also, let q be a nonzero complex number.

Definition 4 (see [7])**.** *Let I_q be the two-sided ideal of the algebra $\mathbb{C}\langle x,y,\theta\rangle$ generated by the elements $xy-yx$, $x\theta - q\theta x$, $y\theta - q\theta y$ and θ^2. The $\mathbb{Z}_2$-graded, associative, unital algebra*

$$\mathcal{O}(\mathbb{C}_q^{2|1}) = \mathbb{C}\langle x,y,\theta\rangle / I_q$$

is the algebra of polynomials on the $\mathbb{Z}_2$-graded quantum space $\mathbb{C}_q^{2|1}$.

This associative algebra over the complex number is known as the algebra of polynomials over the quantum (2+1)-superspace. In accordance with Definition 4, if $(x,y,\theta)^t \in \mathbb{C}_q^{2|1}$, then we have

$$xy = yx, \quad x\theta = q\theta x, \quad y\theta = q\theta y, \quad \theta^2 = 0. \tag{1}$$

If we consider the generators of the algebra $\mathcal{O}(\mathbb{C}_q^{2|1})$ as linear functionals, we can find many 3×3 matrix representations of these generators that preserve the relations (1):

Example 1. *There is a $\mathbb{C}$-linear homomorphism $\rho : \mathcal{O}(\mathbb{C}_q^{2|1}) \to M_3(\mathbb{C})$ defined by*

$$\rho(x) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & q^2-1 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad \rho(\theta) = \begin{pmatrix} 0 & 0 & q^2-1 \\ 0 & 0 & q^2-1 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponding to the coordinate functions satisfying the relations (1)*.*

Example 2. *There exists a representation $\rho : \mathcal{O}(\mathbb{C}_q^{2|1}) \to M_3(\mathbb{C})$ such that the matrices*

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}, \ \rho(y) = \begin{pmatrix} q^2 & 0 & 0 \\ 1-q^2 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}, \ \rho(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-q^2 & 1-q^2 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy the relations (1)*.*

Definition 5 (see [7])**.** *Let $\Lambda(\mathbb{C}_q^{2|1})$ be the algebra with the generators φ_1, φ_2 and z satisfying the relations*

$$\varphi_i\varphi_j = -q^{2(i-j)}\varphi_j\varphi_i, \quad \varphi_i z = q^{-1} z\varphi_i, \qquad (i,j = 1,2), \tag{2}$$

where the coordinates φ_i are of degree 1, and the coordinate z is of degree 0. We call $\Lambda(\mathbb{C}_q^{2|1})$ the exterior algebra of the $\mathbb{Z}_2$-graded space $\mathbb{C}_q^{2|1}$.

Of course, it is also possible to find $\mathbb{Z}_2$-graded $\mathbb{C}$-linear homomorphisms representing generators of the superalgebra $\Lambda(\mathbb{C}_q^{2|1})$:

Example 3. *As can be easily shown, the $\mathbb{Z}_2$-graded $\mathbb{C}$-linear homomorphism $\tilde{\rho}$: $\Lambda(\mathbb{C}_q^{2|1}) \to M_3(\mathbb{C})$ defined by*

$$\tilde{\rho}(\varphi_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\rho}(\varphi_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\rho}(z) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

represents the generators of $\Lambda(\mathbb{C}_q^{2|1})$ and preserves the relations (2).

In this section, we will finally talk about the superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ introduced in [8], which is sufficient to meet our needs.

Theorem 1 (see [8])**.** *The algebra $\mathcal{O}(\mathrm{M}_q(2|1))$ is the quotient of the free algebra $\mathbb{C}\langle a, b, c, e, f, \alpha, \beta, \gamma, \delta\rangle$ by two-sided ideal J_q generated by the relations*

$$\begin{array}{llll}
ab = q^2ba, & bc = q^{-2}cb, & ce = q^2ec, & e\alpha = \alpha e, \\
ac = ca, & be = eb, & c\alpha = q\alpha c, & e\beta = q\beta e, \\
a\alpha = q\alpha a, & b\alpha = q\alpha b, & c\beta = q\beta c, & e\gamma = q^{-1}\gamma e, \\
a\gamma = q\gamma a, & b\gamma = q^{-1}\gamma b, & c\gamma = q\gamma c, & e\delta = q\delta e, \\
f\alpha = \alpha f, & b\delta = q\delta b, & \alpha\gamma = -\gamma\alpha, & \beta\delta = -\delta\beta, \\
f\beta = q\beta f, & f\delta = q\delta f, & \alpha\delta = -\delta\alpha, & \gamma\delta = -\delta\gamma, \\
f\gamma = q\gamma f, & \alpha\beta = -q^{-2}\beta\alpha, & \beta\gamma = -\gamma\beta, & \mu^2 = 0, \\
ae = ea + q\lambda bc, & af = fa + \lambda\gamma\alpha, & a\beta = q^{-1}\beta a + \lambda\alpha c, & a\delta = q\delta a + \lambda\gamma b, \\
bf = fb + \lambda\delta\alpha, & b\beta = q^{-1}\beta b + \lambda\alpha e, & cf = fc + \lambda\gamma\beta, & ef = fe + \lambda\delta\beta, \\
c\delta = q\delta c + \lambda\gamma e, & & &
\end{array}$$

where $\mu \in \{\alpha, \beta, \gamma, \delta\}$ and $\lambda = q - q^{-1}$.

It will sometimes be convenient and more illustrative to write a point $(a, b, \ldots, \delta, f)$ of $\mathcal{O}(\mathrm{M}(2|1))$ in the matrix form, as a supermatrix, $T = (t_{ij}) = \begin{pmatrix} a & b & \alpha \\ c & e & \beta \\ \gamma & \delta & f \end{pmatrix}$. The quantum superdeterminant for the supermatrix T is given by

$$\mathrm{sdet}(T) = a(e - ca^{-1}b)[f - \gamma a^{-1}\alpha - (\delta - \gamma ca^{-1})(e - ca^{-1}b)^{-1}(\beta - \alpha ca^{-1})]^{-1},$$

where the formal inverses of the generators a, e and f exist. Using the quantum superdeterminant $\mathrm{sdet}(T)$ belonging to the algebra $\mathcal{O}(\mathrm{M}_q(2|1))$, we can define a superalgebra adding the inverse of $\mathrm{sdet}(T)$ to $\mathcal{O}(\mathrm{M}_q(2|1))$.

Definition 6. *The algebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ is the quotient of the algebra $\mathcal{O}(\mathrm{M}_q(2|1))$ by the two-sided ideal generated by the element $t \cdot \mathrm{sdet}(T) - 1$. In short, we write*

$$\mathcal{O}(\mathrm{GL}_q(2|1)) := \mathcal{O}(\mathrm{M}_q(2|1))[t]/\langle t \cdot \mathrm{sdet}(T) - 1\rangle.$$

The Hopf superalgebra structure of $\mathcal{O}(\mathrm{GL}_q(2|1))$ is given, as usual, in the following theorem.

Theorem 2 (see [8]). *There exists a unique Hopf superalgebra structure on the superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ with co-maps Δ, ϵ and S such that*

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad S(t_{ij}) = t_{ij}^{-1}.$$

Definition 7. *The Hopf superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ is called the coordinate algebra of the quantum supergroup $\mathrm{GL}_q(2|1)$.*

By Definition 3, we can consider the superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$ as a left comodule algebra with respect to the coproduct.

Theorem 3. *The algebra $\mathcal{O}(\mathbb{C}_q^{2|1})$ is a left comodule algebra of the Hopf superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ with left coaction δ_L such that*

$$\delta_L(x_i) = \sum_k t_{ik} \otimes x_k,$$

where $x_1 = x$, $x_2 = y$ and $x_3 = \theta$.

4. The quantum de Rham complex on $\mathcal{O}(\mathbb{C}_q^{2|1})$

The de Rham complex of a smooth manifold $\mathcal{X}$ is the cochain complex, which in degree $n \in \mathbb{N}$ has the vector space $\Omega^n(\mathcal{X})$ of n-degree differential forms on $\mathcal{X}$. Under the wedge product, the de Rham complex becomes a differential graded algebra. In this section, we set up two quantum de Rham complexes, two first order differential calculi, on the quantum superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$. It contains functions on $\mathbb{C}_q^{2|1}$ and their differentials as differential one-forms.

Definition 8. *Let $\mathcal{A}$ be an arbitrary superalgebra with unity and Ω a bimodule over $\mathcal{A}$. A first order $\mathbb{Z}_2$-graded differential calculus over $\mathcal{A}$ is a pair (Ω, d), where $d : \mathcal{A} \longrightarrow \Omega$ is a linear mapping such that the so-called $\mathbb{Z}_2$-graded Leibniz rule*

$$d(uv) = (du)\, v + (-1)^{p(u)} u\, (dv)$$

holds for any $u, v \in \mathcal{A}$, and Ω is the linear span of elements of the form $u \cdot dv \cdot w$ with $u, v, w \in \mathcal{A}$.

A $\mathbb{Z}_2$-graded differential algebra over $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra $\Omega = \bigoplus_{n \geq 0} \Omega^n$, with the linear map d of degree 1 such that $d \circ d := d^2 = 0$, and the $\mathbb{Z}_2$-graded Leibniz rule holds. Here we assume that $\Omega^0 := \mathcal{A}$ and $\Omega^{<0} = 0$.

To set up a quantum de Rham complex on the quantum superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$ we actually choose the cotangent superspace or differential 1-forms. Since one can multiply forms by functions from the left and the right, this must be an Ω-bimodule.

4.1. The relations between coordinates and differentials

We denote the quantum de Rham complex of the quantum superspace $\mathbb{C}_q^{2|1}$ by $\Omega(\mathbb{C}_q^{2|1})$ and assume the superalgebras $\mathcal{O}(\mathbb{C}_q^{2|1})$ and $\Lambda(\mathbb{C}_q^{2|1})$ as parts of the differential graded algebra $\Omega(\mathbb{C}_q^{2|1})$, as Wess and Zumino say [15]. For this reason, we introduce the first order differentials of the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ as $dx = \varphi_1$, $dy = \varphi_2$ and $d\theta = z$. Then the differential d is uniquely defined by the conditions in Definition 8, and the commutation relations between the differentials have the form

$$du \wedge du = 0, \quad dx \wedge dy = -q^{-2} dy \wedge dx, \quad d\theta \wedge du = q du \wedge d\theta \tag{3}$$

for $u \in \{x, y\}$.

In this case, the quantum de Rham complex $\Omega(\mathbb{C}_q^{2|1})$ is generated by the elements of the set $\{x, y, \theta, dx, dy, d\theta\}$ by adding the nine cross-commutation relations satisfied between the elements of $\mathcal{O}(\mathbb{C}_q^{2|1})$ and $\Lambda(\mathbb{C}_q^{2|1})$, which will be given in the following theorem, to relations (1) and (3).

We assume that the cross-commutation relations are of the following form:

$$x_i \cdot dx_j = \sum_{k,l} C_{kl}^{ij}\, dx_k \cdot x_l, \tag{4}$$

where $C = (C_{kl}^{ij})$ is a 9x9-matrix with constant entries. Here, there seems to be 81 indeterminate constants when the sum is explicitly written, but in fact, we have 41 indeterminate constants due to consistency and we can determine them in a few steps:

1. The differential d reduces to sixteen numbers of constants in relations (4).
2. The compatibility with the left coaction of $\mathcal{O}(\mathrm{GL}_q(2|1))$ leaves one free parameter and relations (4) are of the form:

$$\begin{aligned}
&x \cdot dx = [(q^2+1)r - 1]dx \cdot x, && y \cdot dx = q^2 r dx \cdot y + (r-1)dy \cdot x,\\
&x \cdot d\theta = (q^2 r - 1)dx \cdot \theta + qrd\theta \cdot x, && y \cdot dy = [(q^2+1)r - 1]dy \cdot y,\\
&x \cdot dy = (q^2 r - 1)dx \cdot y + rdy \cdot x, && y \cdot d\theta = (q^2 r - 1)dy \cdot \theta + qrd\theta \cdot y,\\
&\theta \cdot dx = -qrdx \cdot \theta + (1-r)d\theta \cdot x, && \theta \cdot dy = -qrdy \cdot \theta + (1-r)d\theta \cdot y,\\
&\theta \cdot d\theta = d\theta \cdot \theta.
\end{aligned}$$

3. The associativity of the graded differential algebra $\Omega(\mathbb{C}_q^{2|1})$ solves the parameter r as $r = 1$ or $r = q^{-2}$.

As a result, when we combine the above three steps, we have the following theorem.

Theorem 4. *There exist two left covariant $\mathbb{Z}_2$-graded first order differential calculi $\Omega(\mathbb{C}_q^{2|1})$ over $\mathcal{O}(\mathbb{C}_q^{2|1})$ with respect to the Hopf superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$ such that $\{dx, dy, d\theta\}$ is a free right $\mathcal{O}(\mathbb{C}_q^{2|1})$-module basis of $\Omega(\mathbb{C}_q^{2|1})$. The bimodule structures* *for these calculi are determined by the relations:*

$$\begin{aligned}
&x \cdot dx = q^2\, dx \cdot x, && y \cdot dx = q^2\, dx \cdot y,\\
&x \cdot dy = dy \cdot x + (q^2-1)\, dx \cdot y, && y \cdot dy = q^2\, dy \cdot y,\\
&x \cdot d\theta = q\, d\theta \cdot x + (q^2-1)\, dx \cdot \theta, && y \cdot d\theta = q\, d\theta \cdot y + (q^2-1)\, dy \cdot \theta,\\
&\theta \cdot dx = -q\, dx \cdot \theta, && \theta \cdot dy = -q\, dy \cdot \theta, \qquad \theta \cdot d\theta = d\theta \cdot \theta
\end{aligned} \tag{5a}$$

for Case I $(r = 1)$ *and,*

$$\begin{aligned}
&x \cdot dx = q^{-2}\, dx \cdot x, && y \cdot dx = dx \cdot y + (q^{-2}-1)\, dy \cdot x,\\
&x \cdot dy = q^{-2}\, dy \cdot x, && y \cdot dy = q^{-2}\, dy \cdot y,\\
&x \cdot d\theta = q^{-1}\, d\theta \cdot x, && y \cdot d\theta = q^{-1}\, d\theta \cdot y,\\
&\theta \cdot dx = -q^{-1}\, dx \cdot \theta + (1-q^{-2})\, d\theta \cdot x, && \theta \cdot d\theta = d\theta \cdot \theta\\
&\theta \cdot dy = -q^{-1}\, dy \cdot \theta + (1-q^{-2})\, d\theta \cdot y
\end{aligned} \tag{5b}$$

for Case II $(r = q^{-2})$.

As a note, in [7], a differential calculus on the Hopf superalgebra $\mathcal{F}(\mathbb{C}_q^{2|1})$ of functions on $\mathbb{C}_q^{2|1}$ was constructed using Woronowicz's approach. This calculus is right-covariant with respect to the Hopf superalgebra $\mathcal{F}((\mathbb{C}_q^{2|1})$ itself. Both calculi obtained above are left-covariant under the action of the quantum group $\mathrm{GL}_q(2|1)$, which is the symmetry group of the superspace $\mathbb{C}_q^{2|1}$. These calculi were obtained using Wess-Zumino's approach. However, one of the calculi that emerged in this work, namely relations (5b), coincides with the relations in [7].

Remark 1. *We know that a free right $\mathcal{O}(\mathbb{C}_q^{2|1})$-module basis of $\mathcal{O}(\mathbb{C}_q^{2|1})$-bimodule $\Omega(\mathbb{C}_q^{2|1})$ is the set $\{dx, dy, d\theta\}$ and relations* (1) *and* (3) *hold. We now consider a left module structure of $\mathcal{O}(\mathbb{C}_q^{2|1})$-bimodule $\Omega(\mathbb{C}_q^{2|1})$. The left product $dv \mapsto u \cdot dv$ is an endomorphism of the right module $\Omega(\mathbb{C}_q^{2|1})$. The ring of all endomorphisms of any free module of rank 3 is isomorphic to the ring of all 3×3 matrices. Since $\{dx, dy, d\theta\}$ is the homogeneous basis of $\Omega(\mathbb{C}_q^{2|1})$, there exists a map $\sigma : \mathcal{O}(\mathbb{C}_q^{2|1}) \longrightarrow M_3(\mathcal{O}(\mathbb{C}_q^{2|1}))$ defined by*

$$u \cdot dx_j = \sum_i dx_i \cdot \sigma_{ij}(u) \tag{6}$$

for all $u \in \mathcal{O}(\mathbb{C}_q^{2|1})$ and $x_1 = x$, $x_2 = y$ and $x_3 = \theta$. Indeed, one can see that relations (6) *equivalent to relations* (5a), *where*

$$\sigma(x) = \begin{pmatrix} q^2 x & (q^2-1)y & (q^2-1)\theta \\ 0 & x & 0 \\ 0 & 0 & qx \end{pmatrix}, \quad \sigma(y) = \begin{pmatrix} q^2 y & 0 & 0 \\ 0 & q^2 y & (q^2-1)\theta \\ 0 & 0 & qy \end{pmatrix},$$
$$\sigma(\theta) = \begin{pmatrix} -q\theta & 0 & 0 \\ 0 & -q\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}. \tag{7}$$

Theorem 5. *The map σ is a $\mathbb{C}$-linear homomorphism such that*

$$\sigma_{ij}(uv) = \sum_k \sigma_{ik}(u)\sigma_{kj}(v), \quad \forall u, v \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Remark 2. *It is easy to see that relations* (1) *are preserved under the action of the map σ.*

Remark 3. *We can also define a map $\tau : \mathcal{O}(\mathbb{C}_q^{2|1}) \to M_3(\mathcal{O}(\mathbb{C}_q^{2|1}))$ by the formulas*

$$dx_i \cdot u = \sum_j \tau_{ij}(u) \cdot dx_j, \quad \forall u, x_i \in \mathcal{O}(\mathbb{C}_q^{2|1}), \tag{8}$$

where

$$\tau(x) = \begin{pmatrix} q^{-2}x & 0 & 0 \\ (q^{-2}-1)y & x & 0 \\ (1-q^{-2})\theta & 0 & q^{-1}x \end{pmatrix}, \quad \tau(y) = \begin{pmatrix} q^{-2}y & 0 & 0 \\ 0 & q^{-2}y & 0 \\ 0 & (1-q^{-2})\theta & q^{-1}y \end{pmatrix},$$
$$\tau(\theta) = \begin{pmatrix} -q^{-1}\theta & 0 & 0 \\ 0 & -q^{-1}\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

for Case I.

Theorem 6. *The map τ is a $\mathbb{C}$-linear homomorphism such that*

$$\tau_{ij}(uv) = \sum_k \tau_{ik}(u)\tau_{kj}(v), \quad \forall u, v \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Remark 4. *It is easy to see from Theorem* 6 *that relations* (1) *are preserved under the action of the map τ.*

Remark 5. *The maps σ and τ are not the inverse of each other. However, they have the property*

$$\sum_k \sigma_{jk}(\tau_{ik}(u)) = u\delta_{ij}, \quad \forall u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

4.2. The action of the map σ on $\Omega(\mathcal{A})$

By taking the differential of both sides of (6), we get

$$du \wedge dx_j = \sum_i (-1)^{p(dx_i)}\, dx_i \wedge d\sigma_{ij}(u), \quad \forall u, x_j \in \mathcal{O}(\mathbb{C}_q^{2|1}). \tag{9}$$

From Remark 1, we know that the map σ acts on the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$. We wish to extend the map σ to the whole algebra $\Omega(\mathbb{C}_q^{2|1})$. For this, we define a map σ^Ω as follows:

$$\sigma^\Omega : \Omega(\mathbb{C}_q^{2|1}) \longrightarrow M_3(\Omega(\mathbb{C}_q^{2|1})), \quad \sigma^\Omega_{ij}(du) = d\sigma_{ij}(u), \qquad \forall\, du \in \Omega(\mathbb{C}_q^{2|1}),$$

where $\sigma^\Omega_{ij}(u) := \sigma_{ij}(u)$ for all $u \in \mathcal{O}(\mathbb{C}_q^{2|1})$. Then we have

Theorem 7. *The map σ^Ω is a $\mathbb{Z}_2$-graded $\mathbb{C}$-linear operator such that*

$$\begin{aligned}\sigma_{ij}^\Omega(du\cdot v) &= \sum_k \sigma_{ik}^\Omega(du)\cdot\sigma_{kj}(v),\\ \sigma_{ij}^\Omega(u\cdot dv) &= \sum_k (-1)^{p(x_i)+p(x_k)}\,\sigma_{ik}(u)\cdot\sigma_{kj}^\Omega(dv)\\ &= \sum_k (-1)^{p(u)+p(\sigma_{ik}(u))}\,\sigma_{ik}(u)\cdot\sigma_{kj}^\Omega(dv),\end{aligned}\tag{10}$$

for all $x_i, x_k\in\mathcal{O}(\mathbb{C}_q^{2|1})$ and $du, dv\in\Omega(\mathbb{C}_q^{2|1})$.

Proof. It is clear that σ^Ω is linear. To obtain the first equality in (10), we can use the identity $(du\cdot v)\wedge dx_j = du\wedge(v\cdot dx_j)$. Indeed, we write

$$(du\cdot v)\wedge dx_j = \sum_i (-1)^{p(dx_i)}\,dx_i\wedge\sigma_{ij}^\Omega(du\cdot v),\quad \forall u,v,x_i\in\mathcal{O}(\mathbb{C}_q^{2|1}),$$

according to (9). On the other hand, we have

$$\begin{aligned}du\wedge(v\cdot dx_j) &= \sum_k (du\wedge dx_k)\cdot\sigma_{kj}(v)\\ &= \sum_i (-1)^{p(dx_i)}\,dx_i\wedge\sum_k \sigma_{ik}^\Omega(du)\cdot\sigma_{kj}(v).\end{aligned}$$

We make use of the fact that $(u\cdot dv)\wedge dx_i = u\cdot(dv\wedge dx_i)$ to show that σ^Ω satisfies the first of the second equality in (10). According to (9), we can write

$$(u\cdot dv)\wedge dx_j = \sum_i (-1)^{p(dx_i)}\,dx_i\wedge\sigma_{ij}^\Omega(u\cdot dv),$$

for all $u, x_k\in\mathcal{O}(\mathbb{C}_q^{2|1})$ and $dv\in\Omega(\mathbb{C}_q^{2|1})$. On the other hand, we have

$$\begin{aligned}u\cdot(dv\wedge dx_j) &= \sum_k (-1)^{p(dx_k)}(u\cdot dx_k)\wedge\sigma_{kj}^\Omega(dv)\\ &= \sum_k (-1)^{p(dx_k)}\sum_i dx_i\wedge\sigma_{ik}(u)\cdot\sigma_{kj}^\Omega(dv)\\ &= \sum_i (-1)^{p(dx_i)}\,dx_i\wedge\sum_k (-1)^{p(dx_i)+p(dx_k)}\,\sigma_{ik}(u)\cdot\sigma_{kj}^\Omega(dv).\end{aligned}$$

When we compare these two results, considering that $p(du) = 1 + p(u)$ (mod2) for all $u\in\mathcal{O}(\mathbb{C}_q^{2|1})$, we get the first equality in (10). Finally, since

$$\begin{aligned}d\sigma_{ij}(u\cdot v) &= \sum_k [d\sigma_{ik}(u)\cdot\sigma_{kj}(v) + (-1)^{p(\sigma_{ik}(u))}\sigma_{ik}(u)\cdot d\sigma_{kj}(v)]\\ &= \sum_k [\sigma_{ik}^\Omega(du)\cdot\sigma_{kj}(v) + (-1)^{p(\sigma_{ik}(u))}\sigma_{ik}(u)\cdot\sigma_{kj}^\Omega(dv)],\\ \sigma_{ij}^\Omega(d(u\cdot v)) &= \sigma_{ij}^\Omega(du\cdot v + (-1)^{p(u)}u\cdot dv) = \sigma_{ij}^\Omega(du\cdot v) + (-1)^{p(u)}\sigma_{ij}^\Omega(u\cdot dv)\end{aligned}$$

for all i, j and $u, v \in \mathcal{O}(\mathbb{C}_q^{2|1})$, we can write

$$(-1)^{p(u)}\sigma_{ij}^{\Omega}(u \cdot dv) = \sum_k (-1)^{p(\sigma_{ik}(u))}\sigma_{ik}(u) \cdot \sigma_{kj}^{\Omega}(dv)$$

with the first equality in (10). □

Remark 6. *It is easy to see from Theorem* 7 *that relations* (5a) *and* (5b) *are preserved under the action of the map* σ^{Ω}.

The proof of the following corollary can be done using the fact that $(du \wedge dv) \wedge dx_j = du \wedge (dv \wedge dx_j)$ for all $du, dv, dx_j \in \Omega(\mathbb{C}_q^{2|1})$.

Corollary 1. *For all* $du, dv \in \Omega(\mathbb{C}_q^{2|1})$,

$$\sigma_{ij}^{\Omega}(du \wedge dv) = \sum_k (-1)^{p(dx_k)}\sigma_{ik}^{\Omega}(du) \wedge \sigma_{kj}^{\Omega}(dv).$$

We can obtain similar results for the map τ. Indeed, if we take the differential of both sides of (8), we get

$$dx_i \wedge du = (-1)^{p(dx_i)} \sum_j d\tau_{ij}(u) \wedge dx_j, \quad \forall x_i, u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

We now define a map τ^{Ω} as follows:

$$\tau^{\Omega} : \Omega(\mathbb{C}_q^{2|1}) \longrightarrow M_3(\Omega(\mathbb{C}_q^{2|1})), \quad \tau_{ij}^{\Omega}(du) = d\tau_{ij}(u), \qquad \forall\, du \in \Omega(\mathbb{C}_q^{2|1}),$$

where $\tau_{ij}^{\Omega}(u) := \tau_{ij}(u)$ for all $u \in \mathcal{O}(\mathbb{C}_q^{2|1})$. Then we have

Corollary 2. *For all* $u \in \mathcal{O}(\mathbb{C}_q^{2|1})$ *and* $dv \in \Omega(\mathbb{C}_q^{2|1})$,

$$\tau_{ij}^{\Omega}(u \cdot dv) = \sum_k (-1)^{p(x_i)+p(x_k)}\, \tau_{ik}(u) \cdot \tau_{kj}^{\Omega}(dv),$$

$$\tau_{ij}^{\Omega}(dv \cdot u) = \sum_k \tau_{ik}^{\Omega}(dv) \cdot \tau_{kj}(u).$$

It is easy to see from Corollary 2 that relations (5) are preserved under the action of the map τ^{Ω}.

4.3. The relations with partial derivatives

We will complete the calculus with the following two theorems. To obtain the commutation relations of the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ with derivatives, we first introduce the derivatives of the generators of the algebra. Since (Ω, d) is a left covariant differential calculus, for any element u in $\mathcal{O}(\mathbb{C}_q^{2|1})$ there are uniquely determined elements $\partial_k(u) \in \mathcal{O}(\mathbb{C}_q^{2|1})$ such that

$$du = dx\, \partial_x(u) + dy\, \partial_y(u) + d\theta\, \partial_\theta(u). \tag{11}$$

For consistency, the degree of the derivative ∂_θ should be 1.

Definition 9. *The linear mappings* $\partial_x, \partial_y, \partial_\theta : \mathcal{O}(\mathbb{C}_q^{2|1}) \to \mathcal{O}(\mathbb{C}_q^{2|1})$ *defined by* (11) *are called the partial derivatives of the calculus* (Ω, d).

The next theorem gives the relations between the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ and their partial derivatives.

Theorem 8. *The relations between the generators of* $\mathcal{O}(\mathbb{C}_q^{2|1})$ *and partial derivatives are as follows:*

$$\partial_j \cdot x_k = \delta_{jk} + (-1)^{p(x_k)} \sum_m \sigma_{jm}(x_k) \cdot \partial_m, \quad \forall x_k \in \mathcal{O}(\mathbb{C}_q^{2|1}). \tag{12}$$

The proof of the following theorem follows from the fact that $d^2 f = 0$ for a differentiable function f.

Theorem 9. *The partial derivatives satisfy the following commutation relations:*

$$\partial_x \partial_y = q^{-2} \partial_y \partial_x, \quad \partial_x \partial_\theta = q^{-1} \partial_\theta \partial_x, \quad \partial_y \partial_\theta = q^{-1} \partial_\theta \partial_y, \quad \partial_\theta^2 = 0. \tag{13}$$

Remark 7. *We can write relations* (13) *as (no summation)*

$$\partial_j \partial_k = Q_{jk} \partial_k \partial_j,$$

with a single formula, where $Q_{jj} = (-q^2)^{p(\partial_j)}$ *and* $Q_{jk} Q_{kj} = 1$.

4.4. A deformed Clifford superalgebra

Suppose that $\mathcal{A}$ is a unital $*$-algebra with the involution $x \mapsto x^+$ and the algebraic relation $uv = qvu$ holds for the generators $u, v \in \mathcal{A}$ and $q \in \mathbb{C}$. There are three important cases where this relation is invariant under involution. Case 1: u is unitary, v is hermitian and $q \in \mathbb{R}$; Case 2: $u = v^+$ and $q \in \mathbb{R}$; Case 3: u and v are hermitian and $|q| = 1$. All three cases arise from the definitions of real forms of quantum algebras. The corresponding $*$-algebra generated by u and v is the coordinate algebra of the real quantum plane.

In this subsection, we will introduce a deformation of the Clifford superalgebra pointing out relations (6) and (13). Let us consider the real quantum superspace, that is, $x^\star = x$, $y^\star = y$ and $\theta^\star = -\theta$. This request imposes the condition $\bar{q} = q^{-1}$ on the deformation parameter. Further, the involution of the partial derivatives should be of the form:

$$\partial_x^\star = -q^2 \partial_x, \quad \partial_y^\star = -\partial_y, \quad \partial_\theta^\star = \partial_\theta.$$

We can now define real momentum operators as

$$P_x = -\sqrt{-1}\, q^2 \partial_x, \quad P_y = -\sqrt{-1}\, \partial_y, \quad P_\theta = \partial_\theta.$$

Lemma 1. *The real momentum operators and the generators of* $\mathcal{O}(\mathbb{C}_q^{2|1})$ *satisfy the following commutation relations:*

$$P_j v_k = -(\sqrt{-1})^{p(P_j)+1} q^{2p(P_j)} \delta_{jk} + (-1)^{p(v_k)} \sum_i \sigma_{ji}(v_k) P_i, \qquad \forall v_k \in \mathcal{O}(\mathbb{C}_q^{2|1})$$

$$P_x P_y = q^{-2} P_y P_x, \quad P_u P_\theta = q^{-1} P_\theta P_u, \quad P_\theta^2 = 0,$$

where $u \in \{x, y\}$, *together with relations* (1).

To obtain the Clifford superalgebra, we define the super gamma matrices as follows:

$$c_1 = P_x, \quad c_2 = P_y, \quad c_3 = x, \quad c_4 = y; \quad \gamma_1 = P_\theta, \quad \gamma_2 = \theta.$$

Theorem 10. *The generators of the Clifford superalgebra satisfy the following commutation relations:*

$$\begin{aligned}
&c_1c_2 = q^{-2}c_2c_1, && c_1\gamma_1 = q^{-1}\gamma_1c_1, && c_3\gamma_1 = q^{-1}\gamma_1c_3, && \gamma_1^2 = 0,\\
&c_1c_4 = q^2c_4c_1, && c_1\gamma_2 = q\gamma_2c_1, && c_3\gamma_2 = q^{-1}\gamma_2c_3, && \gamma_2^2 = 0,\\
&c_2c_3 = c_3c_2, && c_2\gamma_1 = q^{-1}\gamma_1c_2, && c_4\gamma_1 = q^{-1}\gamma_1c_4, &&\\
&c_3c_4 = c_4c_3, && c_2\gamma_2 = q\gamma_2c_2, && c_4\gamma_2 = q^{-1}\gamma_2c_4, &&
\end{aligned}$$

$$c_1c_3 - q^2c_3c_1 = -\sqrt{-1}\,q^2\mathbf{1} + q^2(q^2-1)(c_4c_2 - \sqrt{-1}\,\gamma_2\gamma_1),$$
$$c_2c_4 - q^2c_4c_2 = -\sqrt{-1}\,\mathbf{1} + (1-q^2)\sqrt{-1}\,\gamma_2\gamma_1, \quad \gamma_1\gamma_2 + \gamma_2\gamma_1 = \mathbf{1}.$$

5. $\mathbb{Z}_2$-graded Cartan calculus on $\mathcal{O}(\mathbb{C}_q^{2|1})$

In this section, we will continue with relations (5a), (9) and (12). We know from Subsection 4.1 that the cotangent space $\Omega(\mathcal{T}^*\mathbb{C}_q^{2|1})$ is an $\mathcal{O}(\mathbb{C}_q^{2|1})$-bimodule spanned by the basis $\{dx, dy, d\theta\}$ with relations (5a) and from Subsection 4.3 that the tangent space $\Omega(\mathcal{T}\mathbb{C}_q^{2|1})$ is a $\mathcal{O}(\mathbb{C}_q^{2|1})$-bimodule spanned by the basis $\{\partial_x, \partial_y, \partial_\theta\}$ with relations (13). Therefore, we can define an inner product by analogy with the corresponding objects of the theory of ordinary manifolds. The general inner product between $\Omega(\mathcal{T}\mathbb{C}_q^{2|1})$ and $\Omega(\mathcal{T}^*\mathbb{C}_q^{2|1})$ is of the form:

$$\partial_j(dx_k) :=< \partial_j, dx_k >= \delta_{jk}.$$

5.1. The Cartan calculus in the classical geometry

We first briefly review the construction of a Cartan calculus in the classical differential geometry. Let $\mathcal{A}$ be a unital associative algebra over a field K, and $\Gamma(\mathcal{A})$ an $\mathcal{A}$-bimodule such that there exists a linear map $\mathbf{d} : \mathcal{A} \longrightarrow \Gamma(\mathcal{A})$ which obeys $\mathbf{d}(\mathbf{1}_\mathcal{A}) = 0$ and the Leibniz rule

$$\mathbf{d}(f \cdot g) = (\mathbf{d}f) \cdot g + f \cdot (\mathbf{d}g),$$

where $\mathbf{1}_\mathcal{A}$ is the unit in $\mathcal{A}$ and $f, g \in \mathcal{A}$. We now denote the differential algebra associated with $\mathcal{A}$ by $\Omega(\mathcal{A})$. This algebra is spanned by elements of the form $f_0 \cdot \mathbf{d}f_1 \wedge \mathbf{d}f_2 \wedge \cdots \wedge \mathbf{d}f_k$. Therefore, we can extend the linear map $\mathbf{d}$ to a linear map $\mathbf{d} : \Omega(\mathcal{A}) \longrightarrow \Omega(\mathcal{A})$ by requiring $\mathbf{d}(\mathbf{1}) = 0$, $\mathbf{d} \circ \mathbf{d} := \mathbf{d}^2 = 0$ and

$$\mathbf{d}(\alpha \wedge \beta) = (\mathbf{d}\alpha) \wedge \beta + (-1)^{p(\alpha)}\,\alpha \wedge (\mathbf{d}\beta) \equiv d\alpha \wedge \beta + (-1)^{p(\alpha)}\,\alpha \wedge d\beta, \tag{14}$$

where α and β are any differential forms in $\Omega(\mathcal{A})$. That is, $\mathbf{d}$ maps k-forms to $(k+1)$-forms (functions being 0-forms). Actually, we assume that the action of $\mathbf{d}$ on α (and then β) in (14) is the same as the differential of α, that is, $d\alpha$.

The exterior derivative $\mathbf{d}$ on the algebra $\mathcal{A}$ is given by

$$\mathbf{d}a \equiv \sum_j da_j\, \partial_{a_j}(u), \quad a, a_j \in \mathcal{A}, \tag{15}$$

so that it verifies (14) and the rule

$$\mathbf{d}\alpha = d\alpha + (-1)^{p(\alpha)}\, \alpha\mathbf{d},$$

where $\alpha \in \Omega(\mathcal{A})$. In particular,

$$\mathbf{d}a = da + a\mathbf{d}, \quad a \in \mathcal{A}. \tag{16}$$

The Cartan calculus contains inner derivations and Lie derivatives which act on $\Omega(\mathcal{A})$. An inner derivation is defined to be the contraction of a vector field with a differential form. Thus if X is a vector field on a manifold M, then the inner derivation, denoted by $\mathbf{i}_X$, is a linear operator which transforms k-forms to $(k-1)$-forms. The inner derivation is an anti-derivation of degree -1 on the exterior algebra and

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^{p(\alpha)}\, \alpha \wedge (\mathbf{i}_X\beta), \tag{17}$$

where α and β are any differential forms. The inner derivation $\mathbf{i}_X$ acts on 0- and 1-forms as follows:

$$\mathbf{i}_X(f) = 0, \quad \mathbf{i}_X(df) = X(f). \tag{18}$$

The anticommutativity of forms gives $\mathbf{i}_X \circ \mathbf{i}_Y = -\mathbf{i}_Y \circ \mathbf{i}_X$ and $\mathbf{i}_X \circ \mathbf{i}_X := \mathbf{i}_X^2 = 0$. From now on, unless stated otherwise, we will write $\mathbf{i}_X\mathbf{i}_Y$ instead of $\mathbf{i}_X \circ \mathbf{i}_Y$.

We know from the classical differential geometry that the Lie derivative of a vector field is a vector field and the Lie derivative of a k-form is a k-form, that is, the Lie derivative has degree 0. The Lie derivative of a smooth function f with respect to the vector field X corresponds to the action of derivatives: $\pounds_X f = X(f)$. Cartan's formula relates the Lie derivative to the inner derivation $\mathbf{i}_X$ and the exterior differential $\mathbf{d}$:

$$\pounds_X = \mathbf{i}_X \circ \mathbf{d} + \mathbf{d} \circ \mathbf{i}_X := \mathbf{i}_X\mathbf{d} + \mathbf{d}\mathbf{i}_X.$$

The Lie derivative has the following properties. If $\mathcal{F}(M)$ is the algebra of functions defined on a manifold M, then $\pounds_X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ is a linear derivation on the algebra $\mathcal{F}(M)$ and commutes the exterior derivative $\mathbf{d}$.

5.2. Extension of the Cartan calculus to the supergeometry

Let $\mathcal{A}$ be a unital $\mathbb{Z}_2$-graded associative algebra over a field K, that is, $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$. In this case, some formulas given above will undergo some changes. The linear map $\mathbf{d} : \mathcal{A} \longrightarrow \Gamma(\mathcal{A})$ will satisfy $\mathbf{d}(\mathbf{1}_\mathcal{A}) = 0$ and the $\mathbb{Z}_2$-graded Leibniz rule

$$\mathbf{d}(f \cdot g) = (\mathbf{d}f) \cdot g + (-1)^{p(f)} f \cdot (\mathbf{d}g),$$

where $f, g \in \mathcal{A}$. If $f \in \mathcal{A}$, then (16) will take the form

$$\mathbf{d}f = df + (-1)^{p(f)} f\mathbf{d}.$$

If X is a supervector field on $\mathcal{A}$, then the $\mathbb{Z}_2$-graded inner derivation $\mathbf{i}_X$ of the $\mathbb{Z}_2$-graded algebra $\Omega(\mathcal{A})$ preserves (18) and (17) and can be expressed as

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^{p(\mathbf{i}_X)p(\alpha)}\, \alpha \wedge (\mathbf{i}_X\beta),$$

where α and β are two differential forms in $\Omega(\mathcal{A})$. The map $\mathbf{i}_X : \Omega(\mathcal{A}) \longrightarrow \Omega(\mathcal{A})$ is K-linear and maps k-forms to $(k-1)$-forms. The $\mathbb{Z}_2$-graded Lie derivative $\pounds_X$ of the $\mathbb{Z}_2$-graded algebra $\Omega(\mathcal{A})$ acts on 0- and 1-forms as follows:

$$\pounds_X f = X(f), \quad \pounds_X df = dX(f),$$

for all smooth functions on $\mathcal{A} := \Omega^0(\mathcal{A})$. The map $\pounds_X : \Omega(\mathcal{A}) \longrightarrow \Omega(\mathcal{A})$ is K-linear and maps k-forms to k-forms such that

$$\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + (-1)^{p(\pounds_X)p(\alpha)}\, \alpha \wedge (\pounds_X\beta),$$

where α and β are two differential forms in $\Omega(\mathcal{A})$.

If X is a supervector field on $\mathcal{A}$, then the $\mathbb{Z}_2$-graded Cartan formula can be expressed as

$$\pounds_X = \mathbf{i}_X\mathbf{d} - (-1)^{p(\mathbf{i}_X)}\mathbf{d}\mathbf{i}_X.$$

The $\mathbb{Z}_2$-graded Lie derivative $\pounds_X$ supercommutes the exterior derivative $\mathbf{d}$:

$$\pounds_X\mathbf{d} = (-1)^{p(\pounds_X)}\mathbf{d}\pounds_X.$$

5.3. Commutation relations with inner derivations

In this and the next two subsections we will consider the vector fields as the partial derivatives of the generators and assume $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_\theta$.

We now wish to find the commutation relations of the generators of the superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$ with the inner derivations.

Theorem 11. *The relations of the generators with inner derivations are given by*

$$\mathbf{i}_{\partial_k} \cdot u = \sum_j \sigma_{kj}(u) \cdot \mathbf{i}_{\partial_j}, \quad u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Proof. The commutation relations (6) satisfied by the generators and their differentials allow us to write the possible relations of the generators with the inner derivations in the form:

$$\mathbf{i}_{\partial_k} \cdot u = \sum_j \tilde{\sigma}_{kj}(u) \cdot \mathbf{i}_{\partial_j}, \quad \forall u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Therefore, we need to determine the operator $\tilde{\sigma}$ for the proof. Recalling (18) and using (8), according to Remark 5 we can write:

$$\begin{aligned} 0 &= \mathbf{i}_{\partial_k}\left(dx_i \cdot u - \sum_j \tau_{ij}(u) \cdot dx_j\right) = \mathbf{i}_{\partial_k}(dx_i) \cdot u - \sum_{j,m} \tilde{\sigma}_{km}(\tau_{ij}(u)) \cdot \mathbf{i}_{\partial_m}(dx_j) \\ &= \delta_{ki} \cdot u - \sum_{j,m} \tilde{\sigma}_{km}(\tau_{ij}(u)) \cdot \delta_{mj} = \sum_j \left[\sigma_{ij}(\tau_{kj}(u)) - \tilde{\sigma}_{ij}(\tau_{kj}(u))\right]. \end{aligned}$$

So, it must be $\tilde{\sigma}_{ij} = \sigma_{ij}$ for all i, j. $\square$

Theorem 12. *The relations between differentials of the generators and inner derivations are given by the formulas:*

$$\mathbf{i}_{\partial_k} \cdot dx_i = \delta_{ki} + q^{-2} \sum_j (-1)^{p(\mathbf{i}_{\partial_k})}\, \sigma^{\Omega}_{kj}(dx_i) \cdot \mathbf{i}_{\partial_j}, \quad x_i \in \mathcal{O}(\mathbb{C}_q^{2|1}). \tag{19}$$

Proof. Let us assume that the possible relations of the differentials of the generators with inner derivations have the form:

$$\mathbf{i}_{\partial_k} \cdot dx_i = \mathbf{i}_{\partial_k}(dx_i) + \sum_j \hat{\sigma}_{kj}(dx_i) \cdot \mathbf{i}_{\partial_j}, \quad \forall x_i \in \mathcal{O}(\mathbb{C}_q^{2|1}). \tag{20}$$

Then using (9), we can write

$$\begin{aligned}
0 &= \mathbf{i}_{\partial_k}[dx_i \wedge dx_j - \sum_n (-1)^{p(dx_n)} dx_n \wedge \sigma^{\Omega}_{nj}(dx_i)] \\
&= [\delta_{ki} + \sum_m \hat{\sigma}_{km}(dx_i) \cdot \mathbf{i}_{\partial_m}] dx_j - \sum_n (-1)^{p(dx_n)} [\delta_{kn} + \sum_m \hat{\sigma}_{km}(dx_n) \cdot \mathbf{i}_{\partial_m}] \sigma^{\Omega}_{nj}(dx_i) \\
&= \delta_{ki} dx_j + \sum_m \hat{\sigma}_{km}(dx_i)[\delta_{mj} + \sum_s \hat{\sigma}_{ms}(dx_j) \cdot \mathbf{i}_{\partial_s}] - (-1)^{p(dx_k)} \sigma^{\Omega}_{kj}(dx_i) \\
&\quad - \sum_n (-1)^{p(dx_n)} \sum_m \hat{\sigma}_{km}(dx_n)[\mathbf{i}_{\partial_m}(\sigma^{\Omega}_{nj}(dx_i)) + \sum_s \hat{\sigma}_{ms}(\sigma^{\Omega}_{nj}(dx_i)) \mathbf{i}_{\partial_s}] \\
&= \delta_{ki} dx_j + \hat{\sigma}_{kj}(dx_i) - (-1)^{p(dx_k)} \sigma^{\Omega}_{kj}(dx_i) - \sum_{m,n} (-1)^{p(dx_n)} \mathbf{i}_{\partial_m}(\sigma^{\Omega}_{nj}(dx_i)) \hat{\sigma}_{km}(dx_n) \\
&\quad + \sum_{m,s} [\hat{\sigma}_{km}(dx_i) \wedge \hat{\sigma}_{ms}(dx_j) - \sum_n (-1)^{p(dx_n)} \hat{\sigma}_{km}(dx_n) \wedge \hat{\sigma}_{ms}(\sigma^{\Omega}_{nj}(dx_i))] \mathbf{i}_{\partial_s}.
\end{aligned}$$

There is a conclusion we can deduce from here:

$$\hat{\sigma}_{jk}(du) = q^{-2} (-1)^{p(\mathbf{i}_{\partial_j})} \sigma^{\Omega}_{jk}(du), \quad \forall u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Then, the last expression in the last equality above is:

$$\begin{aligned}
K &= \sum_m \left[\hat{\sigma}_{km}(dx_i) \wedge \hat{\sigma}_{ms}(dx_j) - \sum_n (-1)^{p(dx_n)} \hat{\sigma}_{km}(dx_n) \wedge \hat{\sigma}_{ms}(\sigma^{\Omega}_{nj}(dx_i)) \right] \\
&= q^{-4} (-1)^{p(dx_k)} \sum_m (-1)^{p(dx_m)} \sigma^{\Omega}_{km}(dx_i) \wedge \sigma^{\Omega}_{ms}(dx_j) \\
&\quad - q^{-4} (-1)^{p(dx_k)} \sum_n (-1)^{p(dx_n)} \sum_m (-1)^{p(dx_m)} \sigma^{\Omega}_{km}(dx_n) \wedge \sigma^{\Omega}_{ms}(\sigma^{\Omega}_{nj}(dx_i)) \\
&= q^{-4} (-1)^{p(dx_k)} \left[\sigma^{\Omega}_{ks}(dx_i \wedge dx_j) - \sum_n (-1)^{p(dx_n)} \sigma^{\Omega}_{ks}(dx_n \wedge \sigma^{\Omega}_{nj}(dx_i)) \right] \\
&= 0.
\end{aligned}$$

Thus the proof is complete. □

Remark 8. *This theorem is also valid for the algebra of functions on $\mathbb{C}_q^{m|n}$ excluding a factor q^{-2}. We found this factor for special matrices σ given by* (7).

Theorem 13. *The relations of the partial derivatives with the inner derivations are of the form (no summation):*

$$\mathbf{i}_{\partial_j}\partial_k = (-1)^{p(\partial_k)}\left[q^{-2H_{jk}}Q_{jk}\partial_k\mathbf{i}_{\partial_j} + (q^{-2}-1)(1-H_{jk})\partial_j\mathbf{i}_{\partial_k}\right], \tag{21}$$

where $Q_{jj} = (-q^2)^{p(\partial_i)}$ *and* $H_{jk} := H(j-k)$ *is the Heaviside step function.*

Proof. To find the commutation relations of the inner derivations with the partial derivatives, inspired by relations (5a), let us assume that they are simply (no summation)

$$\mathbf{i}_{\partial_j}\partial_k = a_{jk}\partial_k\mathbf{i}_{\partial_j} + b_{jk}\partial_j\mathbf{i}_{\partial_k},$$

where a_{jk} and b_{jk} are constants to be determined. By applying the inner derivations $\mathbf{i}_{\partial_j}$ from the left to relations (12), we determine all constants a_{jk} and b_{jk}. Then, when we make the necessary arrangements, we arrive at relations (21). □

The following lemma can be easily proved using the definition of $\mathbf{d}$ given by (15), and relations (19) and (21).

Lemma 2. *The partial derivatives in terms of the exterior derivative and the inner derivations are expressed below:*

$$\mathbf{i}_{\partial_k}\mathbf{d} = \partial_k + q^{-2}(-1)^{p(\mathbf{i}_{\partial_k})}\mathbf{d}\mathbf{i}_{\partial_k}. \tag{22}$$

Theorem 14. *The relations between the inner derivations are of the form:*

$$\mathbf{i}_{\partial_u}^2 = 0, \quad \mathbf{i}_{\partial_x}\mathbf{i}_{\partial_y} = -\mathbf{i}_{\partial_y}\mathbf{i}_{\partial_x}, \quad \mathbf{i}_{\partial_u}\mathbf{i}_{\partial_\theta} = q\mathbf{i}_{\partial_\theta}\mathbf{i}_{\partial_u},$$

where $\partial_u \in \{\partial_x, \partial_y\}$.

Proof. If it is assumed that the commutation relations between the inner derivations are of the form: $\mathbf{i}_{\partial_j}\mathbf{i}_{\partial_k} = Q'_{jk}\mathbf{i}_{\partial_k}\mathbf{i}_{\partial_j}$ for $j,k = 1,2,3$, and Theorem 13 and Lemma 2 are used, the desired relations can be deduced. For example, one has

$$\begin{aligned}
0 &= (\mathbf{i}_{\partial_x}\mathbf{i}_{\partial_y} - Q'_{12}\,\mathbf{i}_{\partial_y}\mathbf{i}_{\partial_x})\mathbf{d} = \mathbf{i}_{\partial_x}(\partial_y - q^{-2}\mathbf{d}\mathbf{i}_{\partial_y}) - Q'_{12}\mathbf{i}_{\partial_y}(\partial_x - q^{-2}\mathbf{d}\mathbf{i}_{\partial_x})\\
&= q^{-2}\partial_y\mathbf{i}_{\partial_x} + (q^{-2}-1)\partial_x\mathbf{i}_{\partial_y} - q^{-2}(\partial_x - q^{-2}\mathbf{d}\mathbf{i}_{\partial_x})\mathbf{i}_{\partial_y}\\
&\quad - Q'_{12}\left[\partial_x\mathbf{i}_{\partial_y} - q^{-2}(\partial_y - q^{-2}\mathbf{d}\mathbf{i}_{\partial_y})\mathbf{i}_{\partial_x}\right]\\
&= q^{-2}(1+Q'_{12})\left(\partial_y\mathbf{i}_{\partial_x} - q^2\partial_x\mathbf{i}_{\partial_y}\right) + q^{-4}\mathbf{d}(\mathbf{i}_{\partial_x}\mathbf{i}_{\partial_y} - Q'_{12}\mathbf{i}_{\partial_y}\mathbf{i}_{\partial_x}).
\end{aligned}$$

So it must be $1 + Q'_{12} = 0$. When performing general operations, it is seen that the constants Q'_{jk} must be

$$Q'_{jk} = (-1)^{p(\mathbf{i}_{\partial_j})+p(\mathbf{i}_{\partial_k})}\left[(1-q^{-2})H_{jk} - 1\right]q^{2H_{kj}}Q_{jk}, \quad Q'_{jj} = -q^{-2}Q_{jj}$$

for all j,k. □

Remark 9. *Using Lemma 2, one can easily see that*

$$\mathbf{d}\partial_u = (-1)^{p(\partial_u)}\,q^2\partial_u\mathbf{d},$$

for all $\partial_u \in \{\partial_x, \partial_y, \partial_\theta\}$.

5.4. $\mathbb{Z}_2$-graded Lie derivatives

We now will find the commutation rules of the Lie derivatives with the elements of the superalgebra $\mathcal{O}(\mathbb{C}_q^{2|1})$, their differentials, etc. The $\mathbb{Z}_2$-graded Cartan formula for the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ can be expressed as

$$\pounds_{\partial_u} = \mathbf{i}_{\partial_u}\mathbf{d} - (-1)^{p(\mathbf{i}_{\partial_u})}\mathbf{d}\mathbf{i}_{\partial_u}, \tag{23}$$

where $\partial_u \in \{\partial_x, \partial_y, \partial_\theta\}$.

Theorem 15. *The commutation relations of $\pounds_{\partial_u}$'s with the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ are as follows:*

$$\pounds_{\partial_j} x_k = \delta_{jk} + \sum_m \left[(-1)^{p(x_k)}\sigma_{jm}(x_k)\pounds_{\partial_m} + (q^{-2}-1)(-1)^{p(\mathbf{i}_{\partial_j})}\sigma^{\Omega}_{jm}(dx_k)\mathbf{i}_{\partial_m}\right],$$

for all $x_k \in \mathcal{O}(\mathbb{C}_q^{2|1})$.

Proof. Using (23) and relations (19) we can write

$$\begin{aligned}
\pounds_j x_k &= (\mathbf{i}_j\mathbf{d} - (-1)^{p(\mathbf{i}_j)}\mathbf{d}\mathbf{i}_j)x_k = \mathbf{i}_j(dx_k + (-1)^{p(x_k)}x_k\mathbf{d}) - (-1)^{p(\mathbf{i}_j)}\mathbf{d}(\mathbf{i}_j x_k)\\
&= \delta_{jk} + q^{-2}(-1)^{p(\mathbf{i}_j)}\sum_m \sigma^{\Omega}_{jm}(x_k)\mathbf{i}_m + (-1)^{p(x_k)}\sum_m \sigma_{jm}(x_k)\mathbf{i}_m\mathbf{d}\\
&\quad - (-1)^{p(\mathbf{i}_j)}\sum_m \mathbf{d}\sigma_{jm}(x_k)\cdot\mathbf{i}_m\\
&= \delta_{jk} + q^{-2}(-1)^{p(\mathbf{i}_j)}\sum_m \sigma^{\Omega}_{jm}(x_k)\mathbf{i}_m + (-1)^{p(x_k)}\sum_m \sigma_{jm}(x_k)\mathbf{i}_m\mathbf{d}\\
&\quad - (-1)^{p(\mathbf{i}_j)}\sum_m \left[\sigma^{\Omega}_{jm}(dx_k) + (-1)^{p(\sigma_{jm}(x_k))}\sigma_{jm}(x_k)\mathbf{d}\right]\mathbf{i}_m\\
&= \delta_{jk} + (-1)^{p(x_k)}\sum_m \sigma_{jm}(x_k)\pounds_m + (q^{-2}-1)(-1)^{p(\mathbf{i}_j)}\sum_m \sigma^{\Omega}_{jm}(dx_k)\mathbf{i}_m\\
&\quad + \sum_m \left[(-1)^{p(x_k)+p(\mathbf{i}_m)} - (-1)^{p(\mathbf{i}_j)+p(\sigma_{jm}(x_k))}\right]\sigma_{jm}(x_k)\mathbf{d}\mathbf{i}_m.
\end{aligned}$$

Since the last expression is equal to zero for all j,k, we have the desired result. □

Theorem 16. *The commutation relations of $\pounds_{\partial_u}$'s with the differentials of the generators of $\mathcal{O}(\mathbb{C}_q^{2|1})$ are as follows:*

$$\pounds_{\partial_k}\cdot du = q^{-2}(-1)^{p(du)+p(\pounds_{\partial_k})+1}\sum_j \sigma^{\Omega}_{kj}(du)\cdot\pounds_{\partial_j}, \quad u \in \mathcal{O}(\mathbb{C}_q^{2|1}).$$

Proof. Using (23) and relations (19) we can write

$$\begin{aligned}
\pounds_j\cdot du &= (\mathbf{i}_j\mathbf{d} - (-1)^{p(\mathbf{i}_j)}\mathbf{d}\mathbf{i}_j)du = (-1)^{p(du)}(\mathbf{i}_j\cdot du)\mathbf{d} - (-1)^{p(\mathbf{i}_j)}\mathbf{d}(\mathbf{i}_j\cdot du)\\
&= (-1)^{p(\mathbf{i}_j)}q^{-2}[(-1)^{p(du)}\sum_k \sigma^{\Omega}_{jk}(du)\cdot\mathbf{i}_k\mathbf{d} - \sum_k (-1)^{p(\sigma^{\Omega}_{jk}(du))}\sigma^{\Omega}_{jk}(du)\cdot\mathbf{d}\mathbf{i}_k]\\
&= (-1)^{p(du)+p(\pounds_j)+1}q^{-2}\sum_k \sigma^{\Omega}_{jk}(du)\cdot\pounds_k + M_j,
\end{aligned}$$

where

$$M_j = \sum_k \left[(-1)^{p(du)+p(\pounds_j)+p(\pounds_k)} - (-1)^{p(\sigma^{\Omega}_{jk}(du))}\right] \sigma^{\Omega}_{jk}(du) \cdot \mathbf{d}\mathbf{i}_k.$$

Since M_j is equal to zero for all j, k according to Theorem 7, we have completed the proof. □

Theorem 17. *The commutation relations between the Lie derivatives and partial derivatives are of the form:*

$$\pounds_{\partial_j}\partial_k = q^{2(1-H_{jk})} Q_{jk}\partial_k \pounds_{\partial_j} + (1-q^2)(1-H_{jk})\partial_j \pounds_{\partial_k}. \tag{24}$$

Proof. The desired relations can be easily deduced using Theorem 13 with Remark 9. Indeed,

$$\begin{aligned}
\pounds_{\partial_j}\partial_k &= (\mathbf{i}_{\partial_j}\mathbf{d} - (-1)^{p(\mathbf{i}_{\partial_j})}\mathbf{d}\mathbf{i}_{\partial_j})\partial_k \\
&= q^2\left[q^{-2H_{jk}} Q_{jk}\partial_k \mathbf{i}_{\partial_j} + (q^{-2}-1)(1-H_{jk})\partial_j \mathbf{i}_{\partial_k}\right]\mathbf{d} \\
&\quad - (-1)^{p(\mathbf{i}_{\partial_j})+p(\partial_k)}\mathbf{d}\left[q^{-2H_{jk}} Q_{jk}\partial_k \mathbf{i}_{\partial_j} + (q^{-2}-1)(1-H_{jk})\partial_j \mathbf{i}_{\partial_k}\right] \\
&= q^{2(1-H_{jk})} Q_{jk}\partial_k(\mathbf{i}_{\partial_j}\mathbf{d} - (-1)^{p(\mathbf{i}_{\partial_j})}\mathbf{d}\mathbf{i}_{\partial_j}) \\
&\quad + (1-q^2)(1-H_{jk})\partial_j(\mathbf{i}_{\partial_k}\mathbf{d} - (-1)^{p(\mathbf{i}_{\partial_k})}\mathbf{d}\mathbf{i}_{\partial_k}),
\end{aligned}$$

as expected. □

Theorem 18. *The relations between the Lie derivatives and the inner derivations are of the form:*

$$\pounds_{\partial_j}\mathbf{i}_{\partial_k} = (-1)^{p(\pounds_{\partial_j})} q^{2(1-H_{jk})} Q_{jk}\mathbf{i}_{\partial_k}\pounds_{\partial_j} + (-1)^{p(\mathbf{i}_{\partial_k})}(q^2-1)(1-H_{jk})\mathbf{i}_{\partial_j}\pounds_{\partial_k}. \tag{25}$$

Proof. We can do the proof using the equality given in (23), but it is quite long and a bit tedious. Therefore, we will use the equality given in (22) and hence relations (24). If we apply the operator $\pounds$ to both sides of the equality given in (22) and use relations (24), we write

$$\begin{aligned}
\pounds_j\mathbf{i}_k\mathbf{d} &= \pounds_j(\partial_k + q^{-2}(-1)^{p(\mathbf{i}_k)}\mathbf{d}\mathbf{i}_k) = \pounds_j\partial_k + q^{-2}(-1)^{p(\mathbf{i}_k)+p(\pounds_j)}\mathbf{d}\pounds_j\mathbf{i}_k \\
&= q^{2(1-H_{jk})} Q_{jk}\partial_k\pounds_j + (1-q^2)(1-H_{jk})\partial_j\pounds_k - q^{-2}(-1)^{p(\mathbf{i}_k)+p(\mathbf{i}_j)}\mathbf{d}\pounds_j\mathbf{i}_k \\
&= q^{2(1-H_{jk})} Q_{jk}[\mathbf{i}_k\mathbf{d} - q^{-2}(-1)^{p(\mathbf{i}_k)}\mathbf{d}\mathbf{i}_k]\pounds_j - q^{-2}(-1)^{p(\mathbf{i}_k)+p(\mathbf{i}_j)}\mathbf{d}\pounds_j\mathbf{i}_k \\
&\quad + (1-q^2)(1-H_{jk})[\mathbf{i}_j\mathbf{d} - q^{-2}(-1)^{p(\mathbf{i}_j)}\mathbf{d}\mathbf{i}_j]\pounds_k \\
&= q^{2(1-H_{jk})} Q_{jk}(-1)^{p(\pounds_j)}\mathbf{i}_k\pounds_j\mathbf{d} - q^{-2H_{jk}} Q_{jk}(-1)^{p(\mathbf{i}_k)}\mathbf{d}\mathbf{i}_k\pounds_j \\
&\quad + (1-q^2)(1-H_{jk})(-1)^{p(\pounds_j)}\mathbf{i}_j\pounds_k\mathbf{d} + (1-q^{-2})(1-H_{jk})(-1)^{p(\mathbf{i}_j)}\mathbf{d}\mathbf{i}_j\pounds_k.
\end{aligned}$$

Now, if we consider the parts where the operator $\mathbf{d}$ is on the left and on the right separately, we get relations (25). □

Theorem 19. *The relations between the Lie derivatives are of the form:*

$$\pounds_{\partial_j}\pounds_{\partial_k} = Q_{jk}\pounds_{\partial_k}\pounds_{\partial_j}.$$

Proof. Using (23) and relations (24) we can write

$$\begin{aligned}
\pounds_j \pounds_k &= \pounds_j[\mathbf{i}_k\mathbf{d} - (-1)^{p(\mathbf{i}_k)}\mathbf{d}\mathbf{i}_k] \\
&= [(-1)^{p(\pounds_j)} q^{2(1-H_{jk})} Q_{jk}\mathbf{i}_k\pounds_j + (-1)^{p(\mathbf{i}_k)}(q^2-1)(1-H_{jk})\mathbf{i}_j\pounds_k]\mathbf{d} \\
&\quad - \mathbf{d}[(-1)^{p(\mathbf{i}_k)} q^{2(1-H_{jk})} Q_{jk}\mathbf{i}_k\pounds_j + (-1)^{p(\pounds_j)}(q^2-1)(1-H_{jk})\mathbf{i}_j\pounds_k] \\
&= q^{2(1-H_{jk})}Q_{jk}[\mathbf{i}_k\mathbf{d} - (-1)^{p(\mathbf{i}_k)}\mathbf{d}\mathbf{i}_k]\pounds_j + (1-q^2)(1-H_{jk})[\mathbf{i}_j\mathbf{d} - (-1)^{p(\mathbf{i}_j)}\mathbf{d}\mathbf{i}_j]\pounds_k \\
&= q^{2(1-H_{jk})}Q_{jk}\pounds_k\pounds_j + (1-q^2)(1-H_{jk})\pounds_j\pounds_k
\end{aligned}$$

or

$$\pounds_j \pounds_k = \frac{q^{-2H_{jk}}Q_{jk}}{1+(q^{-2}-1)H_{jk}}\pounds_j\pounds_k.$$

On the other hand, since $1+(q^{-2}-1)H_{jk} = q^{-2H_{jk}}$ for all j, k, we get the desired result. □

6. The R-matrix formalism

We know from Theorem 4 that there exist left-covariant differential calculi over $\mathcal{O}(\mathbb{C}_q^{2|1})$ with respect to the Hopf superalgebra $\mathcal{O}(\mathrm{GL}_q(2|1))$. So, we can use the R-matrix of the quantum supergroup $\mathrm{GL}_q(2|1)$ to formulate the calculi. Here, we consider relations (5a).

6.1. Commutation relations of calculus

Using the commutation relations (5a), it is possible to find an R-matrix that obeys the $\mathbb{Z}_2$-graded Yang-Baxter equation. If it is assumed that an R-matrix is associated with the superspace $\mathbb{C}_q^{2|1}$, the relations of the coordinates with their differentials can be expressed as $x_j \cdot dx_k = \sum B_{mn}^{jk}\, dx_m \cdot x_n$. When we compare these relations with (5a), we see that relations (5a) can be expressed as

$$x_j \cdot dx_k = q\sum_{m,n}(-1)^{p(x_j)}\,\hat{R}_{mn}^{jk}\, dx_m \cdot x_n,$$

in terms of an R-matrix. Here the entries of the matrix $\hat{R}$ are $\hat{R}_{11}^{11} = q$, $\hat{R}_{12}^{12} = q-q^{-1}$, $\hat{R}_{21}^{12} = q^{-1}$, $\hat{R}_{13}^{13} = q-q^{-1}$, $\hat{R}_{31}^{13} = 1$, $\hat{R}_{12}^{21} = q$,$\hat{R}_{22}^{22} = q$, $\hat{R}_{23}^{23} = q-q^{-1}$, $\hat{R}_{32}^{23} = 1$, $\hat{R}_{13}^{31} = 1$, $\hat{R}_{23}^{32} = 1$ and $\hat{R}_{33}^{33} = -q^{-1}$, except the zero entries. The matrix R is given by $R = P\hat{R}$, where P is the $\mathbb{Z}_2$-graded permutation matrix. The matrix R satisfies the graded Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where $R_{12} = R \otimes I_3$, $R_{23} = I_3 \otimes R$ and $R_{13} = (P\otimes I_3)R_{23}(P\otimes I_3)$ with the 3×3 identity matrix I_3. The matrix $\hat{R}$ obeys braid relation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$ and the Hecke condition

$$\hat{R}^2 = (q-q^{-1})\hat{R} + I_9,$$

where I_9 is the 9×9 unit matrix.

As a note, the matrix R is skew-invertible, that is, there is a matrix B such that

$$\sum_{m,n} B_{lm}^{ni}R_{nj}^{km} = \delta_j^i\delta_l^k = \sum_{m,n} R_{jn}^{mk}B_{ml}^{in}.$$

The non-zero entries of this B matrix are $B^{11}_{11} = q^{-1}$, $B^{12}_{12} = q^{-1}$, $B^{13}_{13} = 1$, $B^{21}_{12} = q^{-2}(q^{-1} - 1)$, $B^{21}_{21} = q$, $B^{22}_{22} = q^{-1}$, $B^{23}_{23} = 1$, $B^{31}_{13} = q^{-2}(q^{-1} - 1)$, $B^{31}_{31} = 1$, $B^{32}_{23} = q^{-1} - q$, $B^{32}_{32} = 1$ and $B^{33}_{33} = q$.

The commutation rules of the generators of the function algebra on the superspace $\mathbb{C}_q^{2|1}$, with the matrix $\hat{R}$, can be expressed as follows:

$$\sum_{m,n} \hat{R}^{jk}_{mn} x_m x_n = q x_j x_k.$$

Using the $\hat{R}$-matrix, we can rewrite relations (12) and (13) as follows:

$$\partial_j x_k = \delta_{jk} + q \sum_{m,n} \hat{R}^{kn}_{jm}\, x_m \partial_n, \quad \partial_j \partial_k = q \sum_{m,n} (\hat{R}^{-1})^{nm}_{kj}\, \partial_m \partial_n.$$

6.2. Commutation relations of the Cartan calculus

Here, we give all cases formulated with the matrix R of the Cartan calculus presented in Section 5.

1. Relations involving the inner derivations:

$$\begin{aligned}
\mathbf{i}_{\partial_j} \cdot x_k &= q \sum_{m,n} (-1)^{p(x_k)}\, \hat{R}^{kn}_{jm}\, x_m \cdot \mathbf{i}_{\partial_n},\\
\mathbf{i}_{\partial_j} \cdot dx_k &= \delta_{jk} - q^{-1} \sum_{m,n} (-1)^{p(dx_m)+p(\mathbf{i}_{\partial_n})}\, \hat{R}^{kn}_{jm}\, dx_m \cdot \mathbf{i}_{\partial_n},\\
\mathbf{i}_{\partial_j} \partial_k &= q^{-1} \sum_{m,n} (-1)^{p(\partial_k)}\, (\hat{R}^{-1})^{nm}_{kj}\, \partial_m\, \mathbf{i}_{\partial_n},\\
\mathbf{i}_{\partial_j} \mathbf{i}_{\partial_k} &= q^{-1} \sum_{m,n} (-1)^{1+p(\mathbf{i}_{\partial_j})+p(\mathbf{i}_{\partial_m})}\, (\hat{R}^{-1})^{nm}_{kj}\, \mathbf{i}_{\partial_m}\, \mathbf{i}_{\partial_n}.
\end{aligned}$$

2. Relations involving the Lie derivatives:

$$\pounds_{\partial_j} \cdot x_k = \delta_{jk} + q \sum_{m,n} \hat{R}^{kn}_{jm} \left[x_m \cdot \pounds_{\partial_n} + (-1)^{p(dx_m)+p(\mathbf{i}_{\partial_n})}\, (1 - q^{-2})\, dx_m \cdot \mathbf{i}_{\partial_n} \right],$$

$$\pounds_{\partial_j} \cdot dx_k = q^{-1} \sum_{m,n} (-1)^{p(\pounds_{\partial_j})}\, \hat{R}^{kn}_{jm}\, dx_m \cdot \pounds_{\partial_n}, \quad \pounds_{\partial_j} \partial_k = q \sum_{m,n} (\hat{R}^{-1})^{nm}_{kj}\, \partial_m\, \pounds_{\partial_n},$$

$$\pounds_{\partial_j} \pounds_{\partial_k} = q \sum_{m,n} (\hat{R}^{-1})^{nm}_{kj}\, \pounds_{\partial_m}\, \pounds_{\partial_n}.$$

3. Mixed relations:

$$\pounds_{\partial_j} \mathbf{i}_{\partial_k} = q \sum_{m,n} (-1)^{1+p(\pounds_{\partial_j})+p(\mathbf{i}_{\partial_m})}\, (\hat{R}^{-1})^{nm}_{kj}\, \mathbf{i}_{\partial_m}\, \pounds_{\partial_n}.$$

6.3. Tensor product realization of the wedge product

The commutation relations of $\mathbf{i}_{X_j}$ with dx_k given by (20) can be used to define the wedge product $\wedge$ of forms as an antisymmetrized tensor product. Since $dx_i \otimes dx_j$ is an element in the tensor space of $\Omega(\mathcal{T}^*\mathbb{C}_q^{2|1})\otimes\Omega(\mathcal{T}^*\mathbb{C}_q^{2|1})$, we can define the product of two forms in terms of tensor products as:

$$\begin{aligned} dx_i \wedge dx_j &= dx_i \otimes dx_j + q^{-2}\sum_k (-1)^{p(dx_k)} dx_k \otimes \sigma^{\Omega}_{kj}(dx_i) \\ &= dx_i \otimes dx_j - \sum_{m,n} (-1)^{p(dx_i)+p(dx_m)} \Lambda^{ij}_{mn}\, dx_m \otimes dx_n, \end{aligned}$$

where $\Lambda = q^{-1}\hat{R}$. These equations give implicit commutation relations between the dx_k's. So we have

$$\langle \partial_i, dx_j \wedge dx_k \rangle = \delta_{ij}\, dx_k + \sum_{m,n} (-1)^{p(\partial_i)+p(dx_j)} \Lambda^{jk}_{mn}\, \delta_{im}\, dx_n.$$

We can define $\mathbf{i}_{X_j}$ to act on this product by contracting in the first tensor product space, that is,

$$\mathbf{i}_{\partial_i}(dx_j \wedge dx_k) = \delta_{ij}\, dx_k - \sum_{m,n} (-1)^{p(\mathbf{i}_{\partial_i})+p(dx_j)} \Lambda^{jk}_{mn}\, \delta_{im}\, dx_n.$$

Using the same method as for dx_k we can also obtain a tensor product decomposition of products of inner derivations as follows:

$$\mathbf{i}_{\partial_j} \wedge \mathbf{i}_{\partial_k} = \mathbf{i}_{\partial_j} \otimes \mathbf{i}_{\partial_k} - \sum_{m,n} (-1)^{p(\mathbf{i}_{\partial_j})+p(\mathbf{i}_{\partial_m})} \Lambda^{nm}_{kj}\, \mathbf{i}_{\partial_m} \otimes \mathbf{i}_{\partial_n}.$$

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