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A note on $\ell^p(G)$-linear independence and projections of uniqueness*

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Abstract. We study the problem of ℓ^p-linear independence of orbits of unitary dual integrable representations of countable discrete, not necessarily abelian groups. Under the assumption that the system is Bessel, we prove that for $p \in \langle 1, 2\rangle$ the system is $\ell^p(G)$-linearly independent precisely when the projection onto the kernel of the corresponding bracket operator is a projection of uniqueness for $\ell^p(G)$. The existence of such projections for any infinite countable discrete group is guaranteed by the result of Cecchini and Figà–Talamanca.

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1. Introduction

Translation, modulation and dilation are three basic operators in harmonic analysis, used for the construction of Gabor and wavelet systems. All these operators can be considered as actions of a certain group, for example $\mathbb{Z}^n$ or $\mathbb{R}^n$, on $L^2(\mathbb{R}^n)$. These are, nevertheless, just a few examples of reproducing function systems that can be represented as orbits of a group representation, and the concept of dual integrability plays an important role in the study of such systems. The concept was introduced in [8] in the abelian setting; for a locally compact abelian (LCA) group G, a unitary representation Π of G on a separable Hilbert space $\mathbb{H}$ is called *dual integrable* if there exists a function $[\cdot,\cdot] \equiv [\cdot,\cdot]_\Pi : \mathbb{H} \times \mathbb{H} \to L^1(\widehat{G}; d\xi)$ called the *bracket*, such that

$$\langle \varphi, \Pi(g)\psi\rangle = \int_{\widehat{G}} [\psi, \psi](\xi) e_{-g}(\xi) d\xi, \tag{1}$$

for all $g \in G$, and all $\varphi, \psi \in \mathbb{H}$. If we return to translation, modulation and dilation, considering each of these operators separately, the underlying group would be abelian; however, if we consider their combination, they in general do not commute (recall also that there are still important cases in which they do; see for instance [8], Section 4). In recent years, the concepts of dual integrable representations and bracket have been introduced and studied in the non-abelian setting for certain

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classes of groups, including discrete and compact groups, see [1], [2], [9], and for an overview of the subject, see [7]. Most recently, in our collaboration [18], definition has been extended to the setting of all locally compact groups. In this paper, due to the nature of the problem, we shall consider only the case of countable discrete groups, and refer only to the results of [2].

If Π is a dual integrable representation, then for any $\psi \in \mathbb{H}$ the properties of orbit

$$\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\} \tag{2}$$

can be analyzed in terms of $[\psi, \psi]$. While in the abelian case $[\psi, \psi]$ is a function on the dual group $\widehat{G}$, and the theory is based on the Fourier analysis on LCA groups, in the non-abelian case the study is based on the theory of non-commutative integration. In that case $[\psi, \psi]$ is no longer a function, but an operator (closed, densely defined, and generally unbounded) belonging to a certain non-commutative L^1 space, with the appropriate analogues holding for almost all of its main properties (see [2] and [22]).

Considering the systems of type (2), of particular importance from the application point of view are systems which allow redundancy. In infinite-dimensional Hilbert spaces, redundancy can be expressed through various levels of linear independence. The question of ℓ^p-linear independence for integer translates of a square integrable function for $p < 2$ and $p > 2$ was studied in [19] and [20], respectively, and in [21] for orbits of dual integrable representations of countable discrete abelian groups (see also the discussion in [14], 1.7). If Π represents $\mathbb{Z}$-translations on $L^2(\mathbb{R})$, then $[\psi, \psi] =: p_\psi$ is the periodization function. The set of zeroes of p_ψ plays an important role in the study of (2); for instance, condition $p_\psi > 0$ a.e. completely characterizes $\ell^2(\mathbb{Z})$-linear independence of $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ for $\psi \in L^2(\mathbb{R})$. In contrast with the $p = 2$ case, this condition no longer characterizes $\ell^p(\mathbb{Z})$-linear independence if $p < 2$, neither for integer translates, nor for orbits of any infinite countable discrete abelian group. More precisely, a characterization for Bessel systems was established in terms of ℓ^p-sets of uniqueness (see [19] and [21]), the subsets of $\widehat{G}$ which do not contain the support of any nonzero function with the Fourier transform in $\ell^p(G)$; we note, however, that the problem is still open in general.

The purpose of this paper is to formulate and prove an analogue of this result in the non-commutative setting. We prove that for $p \in [1, 2]$ the Bessel system $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$ is $\ell^p(G)$-linearly independent precisely when the projection onto the kernel of $[\psi, \psi]$ is a projection of uniqueness for $\ell^p(G)$. The existence of such projections for $p < 2$ follows from the result of Cecchini and Figà-Talamanca, [4].

2. Preliminaries

In this section, we introduce the setting. For more details concerning non-commutative L^p spaces, we refer to [16] and [25], and for the properties of operators and operator algebras, we refer to [10].

Let G be a locally compact group and consider the left Haar measure on G. For $p \in [1, +\infty\rangle$, we shall denote by $L^p(G)$ the standard Lebesgue spaces with respect to this measure; moreover, we shall use notation $\ell^p(G)$ if G is discrete. Recall that in that case for any $p, q \in [1, +\infty]$ we have

$$p \leqslant q \quad \Rightarrow \quad \ell^p(G) \subseteq \ell^q(G). \tag{3}$$

Let $\mathbb{H}$ be a separable Hilbert space, $\mathcal{B}(\mathbb{H})$ the algebra of bounded operators on $\mathbb{H}$ and $\mathcal{U}(\mathbb{H})$ the group of unitary operators on $\mathbb{H}$. A *unitary representation* Π of G on $\mathbb{H}$ is a homomorphism $\Pi : G \to \mathcal{U}(\mathbb{H})$, which is continuous with respect to the strong operator topology. As is well-known, the latter is equivalent to functions

$$g \mapsto \langle \Pi(g)\varphi, \psi \rangle$$

being continuous, for every $\varphi, \psi \in \mathbb{H}$. A closed subspace $\mathbb{V}$ of $\mathbb{H}$ is called Π-invariant if $\Pi(G)(\mathbb{V}) \subseteq \mathbb{V}$. The smallest nontrivial Π-invariant subspaces are of form

$$\langle \psi \rangle = \overline{\operatorname{span}\{\Pi(g)\psi : g \in G\}}; \tag{4}$$

we refer to these spaces as *cyclic subspaces* (generated by ψ).

On any locally compact group, we can consider the *left* and the *right* regular representation, $\lambda_G : G \to \mathcal{U}(L^2(G))$ and $\rho_G : G \to \mathcal{U}(L^2(G))$, defined by

$$\lambda_G(g)f(x) = f(g^{-1}x), \quad \rho_G(g)f(x) = \Delta^{1/2}(g)f(xg), \quad x \in G, \quad g \in G. \tag{5}$$

Here, Δ denotes the modular function on G; G is unimodular if $\Delta \equiv 1$ (recall that discrete groups are unimodular).

Suppose now that G is a countable discrete group. We denote by $\mathscr{L}(G)$ and $\mathscr{R}(G)$ the left and the right von Neumann algebra associated to G, defined as the closure with respect to weak operator topology of $\operatorname{span}\{\lambda(g) : g \in G\}$ and $\operatorname{span}\{\rho(g) : g \in G\}$, respectively. As in [2], we shall refer to the elements of $\operatorname{span}\{\lambda(g) : g \in G\}$ as *trigonometric polynomials.* We consider the standard normalized trace τ on $\mathscr{L}(G)$, defined by

$$\tau(F) = \langle F\delta_e, \delta_e \rangle, \quad F \in \mathscr{L}(G). \tag{6}$$

Here, $\{\delta_g\}$ denotes the canonical base on $\ell^2(G)$ and e denotes the identity element of G. Trace τ defined by (6) is normal, faithful and finite. For $1 \leqslant p < \infty$, $L^p(\mathscr{L}(G))$ space is defined as the completion of $\mathscr{L}(G)$ with respect to norm defined by

$$\|F\|_p = (\tau(|F|^p))^{\frac{1}{p}}, \quad F \in \mathscr{L}(G),$$

where $|F|$ is the selfadjoint operator defined as $|F| = (F^*F)^{1/2}$. Moreover, $L^\infty(\mathscr{L}(G))$ is identified with $\mathscr{L}(G)$, equipped with the operator norm. These spaces are Banach spaces; the space is Hilbert if $p = 2$, with the scalar product defined by

$$\langle F, H \rangle_2 = \tau(H^*F). \tag{7}$$

If $1 \leqslant r, p, q \leqslant \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $F \in L^p(\mathscr{L}(G))$ and $H \in L^q(\mathscr{L}(G))$, then $FH \in L^r(\mathscr{L}(G))$ and

$$\|FH\|_r \leqslant \|F\|_p \|H\|_q. \tag{8}$$

For $F \in \mathscr{L}(G)$, the *Fourier coefficients* are defined via

$$\widehat{F}(g) = \tau(\lambda(g)^* F), \quad g \in G, \tag{9}$$

and F acts on $\ell^2(G)$ as a convolution operator:

$$F\zeta(g') = \sum_{g \in G} \widehat{F}(g)\zeta(g^{-1}g'), \quad \forall \zeta \in \ell^2(G), \quad \forall g' \in G. \tag{10}$$

Since τ is finite, we have

$$1 \leqslant p \leqslant q \leqslant +\infty \quad \Rightarrow \quad L^q(\mathscr{L}(G)) \subseteq L^p(\mathscr{L}(G)). \tag{11}$$

Fourier coefficients are defined for any $F \in L^1(\mathscr{L}(G))$; we have $\widehat{F} \in \ell^\infty(G)$, and the theorem of uniqueness holds (see Lemma 2.1. in [2]). Moreover, since $\{\lambda(g) : g \in G\}$ forms an orthonormal basis for $L^2(\mathscr{L}(G))$, the analogue of Plancherel's theorem can be proved (Lemma 2.2 in [2]).

Following [2], a unitary representation $\Pi : G \to \mathcal{U}(\mathbb{H})$ will be called *dual integrable* if there exists a map $[\cdot,\cdot] : \mathbb{H} \times \mathbb{H} \to L^1(\mathscr{L}(G))$ such that

$$\langle \varphi, \Pi(g)\psi \rangle = \tau([\varphi, \psi]\lambda(g)^*), \tag{12}$$

for all $\varphi, \psi \in \mathbb{H}$, and all $g \in G$. For any $\psi \in \mathbb{H}$, $[\psi, \psi]$ is a positive operator which belongs to $L^1(\mathscr{L}(G))$ (thus, it is closed and densely defined, but generally not bounded), and $\|[\psi,\psi]\|_1 = \|\psi\|^2_{\mathbb{H}}$. The appropriate analogues of the main properties of $[\cdot,\cdot]$ hold in the non-abelian setting, except for property (ii) in Corollary 2.6, [8] (for more details, see [22]).

Throughout the paper, for a closed densely defined linear operator T, we shall denote by $N(T)$ and $R(T)$ its kernel and range, respectively. Let us denote by N_ψ the kernel of $[\psi,\psi]$. By a projection, we mean an orthogonal projection; if P is a projection onto some closed subspace K, we shall denote it by P_K. Projections can be compared using the relation $\leqslant$ between positive operators. We say that P is a *subprojection* of Q if $P \leqslant Q$, and this is equivalent to $R(P) \subseteq R(Q)$, as well as to $PQ = P$ and $QP = P$ (see, for instance, Proposition 2.5.2 in [10]). For a closed densely defined operator F, the *left support* is the smallest projection P which satisfies $PF = F$, and it is actually the projection onto $\overline{R(F)}$; analogously, the *right support* is the smallest projection such that $FP = F$, and it is the projection onto $N(F)^\perp$.

Since $[\psi,\psi]$ is a closed operator, N_ψ is a closed subspace of $L^2(G)$. Moreover, since $[\psi,\psi]$ is self-adjoint, $P_{\overline{R([\psi,\psi])}} = P_{N_\psi^\perp}$. Since $[\psi,\psi] \in L^1(\mathscr{L}(G))$, we know that $P_{N_\psi^\perp} \in \mathscr{L}(G)$ and N_ψ and $N_\psi^\perp$ are right-invariant subspaces of $L^2(G)$.

One of the important consequences of the dual integrability of Π is the existence of an isometric isomorphism $S_\psi : \langle \psi \rangle \to L^2(\mathscr{L}(G), [\psi,\psi])$ such that

$$S_\psi[\Pi(g)\psi] = \lambda(g), \quad \forall g \in G, \tag{13}$$

where $L^2(\mathscr{L}(G), [\psi,\psi])$ is defined as the completion of ${}^{\mathscr{L}(G)}/_{\mathcal{N}_\psi}$ with respect to the scalar product defined by

$$\langle F, H \rangle_{2,\psi} = \tau(H^* F[\psi,\psi]) = \langle F[\psi,\psi]^{\frac{1}{2}}, H[\psi,\psi]^{\frac{1}{2}} \rangle_2,$$

and $\mathcal{N}_\psi = \{F \in \mathscr{L}(G) : F[\psi,\psi]^{1/2} = 0\}$ (see, for instance, [2], Proposition 3.4 for the result in the context of discrete groups).

3. Main result

We first introduce the main concepts.

Definition 1. *Let $\mathcal{K}$ be a subset of the set of all sequences of complex numbers. We say that a sequence $\{x_n\}$ in a Hilbert space $\mathbb{H}$ is $\mathcal{K}$-linearly independent if*

$$(c_n) \in \mathcal{K}, \quad \sum_{n=1}^{\infty} c_n x_n = 0 \text{ in } \mathbb{H} \quad \Rightarrow \quad c_n = 0, \quad \forall n \in \mathbb{N}. \tag{14}$$

In our case, $\mathcal{K} = \ell^p(G)$, for $p \in [1,2]$. More precisely, we would like to characterize $\ell^p(G)$-linear independence of $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$. Observe that, in general, $\sum_{g\in G} c_g \Pi(g)\psi$ does not have to converge, unless $(c_g) \in \ell^1(G)$. However, it converges for every $(c_g) \in \ell^2(G)$ if the system $\mathcal{B}_\psi$ is Bessel, i.e., if there exists a constant $B > 0$ such that

$$\sum_{g\in G} |\langle \varphi, \Pi(g)\psi\rangle|^2 \leqslant B\|\varphi\|^2, \quad \forall \varphi \in \mathbb{H}. \tag{15}$$

Recall also that in this case, the series converges unconditionally (see, for example, Section 3.2 in [5]).

Remark 1. *By Theorem A (iii) in [2], it follows that $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$ is a Bessel system if and only if $[\psi,\psi] \leqslant B\mathbb{I}$, where $\mathbb{I}$ denotes the identity operator on $\ell^2(G)$; hence, $[\psi,\psi]$ is bounded. Moreover, by the fact that $[\psi,\psi] \in L^1(\mathscr{L}(G))$, it follows that $[\psi,\psi]$ is affiliated with the von Neumann algebra $\mathscr{L}(G)$, i.e., it commutes with all elements of the commutant of $\mathscr{L}(G)$ (which is actually $\mathscr{R}(G)$). It follows by the Double Commutant Theorem that $[\psi,\psi] \in \mathscr{L}(G)$ (see also [3], 2.3).*

In [4], Cecchini and Figà-Talamanca proved the following result, valid for all locally compact noncompact unimodular groups G such that that $\mathscr{L}(G)$ is not purely atomic, i.e., it is not generated, as a von Neumann algebra, by its minimal projections; this condition is fulfilled if G is discrete.

Theorem 1 ([4])**.** *Let $\mathscr{L}(G) = \mathfrak{A} \oplus \mathfrak{B}$, where $\mathfrak{B}$ is the von Neumann algebra generated by the minimal projections. Let $1 < p < 2$, $P \in \mathfrak{A}$, $m(P) > 0$. Then there exists $Q \in \mathfrak{A}$, $Q \leqslant P$ such that if $T \in L^1(\Gamma)$, $QT = T$ and $\widehat{T} \in L^p$, it follows that $T = 0$. Furthermore, Q can be chosen in such a way that $m(Q)$ is arbitrarily close to $m(P)$.*

The notation and methods used in [4] are based on the theory of integration with respect to a gage space developed by Segal in [17], and the Fourier transform on locally compact unimodular groups developed by Stinespring, [23] and Kunze, [12]. The latter is defined as an operator of left convolution $\lambda(f)$ by f acting on $L^2(G)$, for any measurable function f on G such that $\lambda(f)$ is closed and densely defined operator with certain additional properties. Space $L^1(\Gamma)$ is the L^1 space over the *dual gage space*, $\Gamma = (L^2(G), \mathscr{L}(G), m)$, where the *dual gage* m is the unique gage on $\mathscr{L}(G)$ (the von Neumann algebra generated by $\{\lambda(g) : g \in G\}$), such that whenever f is a continuous positive definite function in $L^1(G)$, then $\lambda(f) \in L^1(\Gamma)$ and $m(\lambda(f)) = f(e)$ (see Theorem 9.2 in [23]; see also Theorem 1.7 in [17] and the definition of m as in [4], p. 39). The norm on $L^1(\Gamma)$ is defined by $\|T\|_1 = m(|T|)$, and $T \mapsto \widehat{T}$, $\widehat{T}(x) = m(\lambda(x)^*T)$, is the inverse Fourier transform considered by Stinespring. Mapping $f \mapsto \lambda(f)$ is a unitary operator from $L^2(G)$ onto $L^2(\Gamma)$ (for $f \in L^2(G)$, $\lambda(f)$ is the closed operator on $L^2(G)$ defined by $\lambda(f)g = f * g$ for $g \in L^1(G) \cap L^2(G)$; see [4] and Corollary 9.3 in [23]), and $L^2(\Gamma)$ is the Hilbert space with the scalar product defined by $\langle T, S\rangle = m(TS^*)$. A different approach (related to our setting), in terms of measurability with respect to a trace φ_0 on a von Neumann algebra was developed by Nelson in [15], and instead of $L^q(\Gamma)$, we can consider L^q space over $\mathscr{L}(G)$ with respect to φ_0; see, for example, the discussion in [24], p. 548, and [25], pp. 23-24, and see also [13], 2.3. If we now return to our setting, described in Section 2, note that for τ defined by (6) we have $m(\lambda(f)) = f(e) = \tau(\lambda(f))$ (recall (6) and (10)), for all continuous positive definite functions in $L^1(G)$. The set $\lambda(f)$ of all such f is a self-adjoint algebra which is weakly dense in $\mathscr{L}(G)$ (recall also that the strong closure coincides with the weak closure; see [23], p. 19). Note that for $\Phi \in L^2(\mathscr{L}(G))$, $\widehat{\Phi}$ is actually the inverse Fourier transform in the sense of [23]. According to these remarks, we define the following concept.

Definition 2. *Let G be a countable discrete group and let $p \in [1,2]$. We say that $P \in \mathscr{L}(G)$ is a projection of uniqueness for $\ell^p(G)$ if*

$$\Phi \in L^1(\mathscr{L}(G)), \quad \widehat{\Phi} \in \ell^p(G), \quad P\Phi = \Phi \quad \Rightarrow \quad \Phi = 0. \tag{16}$$

There is another interpretation of the projections of uniqueness, as noted in [4]. Since $P \in \mathscr{L}(G)$, its range is a right-invariant subspace of $L^2(G)$, and the existence of projections of uniqueness for $\ell^p(G)$ means that there exist nontrivial closed right-invariant subspaces of $L^2(G)$ which do not contain any nontrivial element of $\ell^p(G)$. We note also that the result of [4] extends the result of Katznelson, [11] and Figà-Talamanca and Gaudry, [6].

Remark 2. *If G is discrete, then $\ell^p(G) \subseteq \ell^2(G)$ for all $p \in [1,2]$. Hence, it follows from $\widehat{\Phi} \in \ell^p(G)$ (by applying Lemma 2.1 and Lemma 2.2 in [2]) that $\Phi \in L^2(\mathscr{L}(G))$.*

We now state the main result.

Theorem 2. *Suppose that Π is a unitary dual integrable representation of a countable discrete group G on a separable Hilbert space $\mathbb{H}$. Let $1 \leqslant p \leqslant 2$ and $\psi \in \mathbb{H}$, $\psi \neq 0$ such that $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$ is a Bessel system. The following are equivalent:*

(i) $\mathcal{B}_\psi$ is $\ell^p(G)$-linearly independent,

(ii) the linear span of the right translates of $([\psi,\psi]^{\frac{1}{2}})^\wedge$ is dense in $\ell^q(G)$,

(iii) P_{N_ψ} is a projection of uniqueness for $\ell^p(G)$.

Lemma 1. *Let $1 \leqslant p \leqslant 2$ and $\psi \in \mathbb{H}$, $\psi \neq 0$. The system $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$ is $\ell^p(G)$-linearly dependent if and only if there exists $0 \neq \Phi \in L^2(\mathscr{L}(G))$ such that $\widehat{\Phi} \in \ell^p(G)$ and $\sum_{g\in G} \widehat{\Phi}(g)\lambda(g) = 0$ in $L^2(\mathscr{L}(G), [\psi,\psi])$.*

Proof. By definition, $\mathcal{B}_\psi$ is $\ell^p(G)$-linearly dependent if there exists a nontrivial $(c_g) \in \ell^p(G)$ such that $\sum_{g\in G} c_g \Pi(g)\psi = 0$ in $\mathbb{H}$. It follows from Lemma 2.2 in [2] and Remark 2 that this is equivalent to the existence of $0 \neq \Phi \in L^2(\mathscr{L}(G))$ such that $c_g = \widehat{\Phi}(g)$, for all $g \in G$ and

$$\sum_{g\in G} \widehat{\Phi}(g)\Pi(g)\psi = 0. \tag{17}$$

Since in (17) we actually consider the limit in $L^2(\mathscr{L}(G))$ of a sequence of trigonometric polynomials and the isometric isomorphism S_ψ maps $\sum_{g\in\Omega} c_g \Pi(g)\psi$, for any finite $\Omega \subseteq G$, into a trigonometric polynomial $\sum_{g\in\Omega} c_g\lambda(g)$, by continuity of S_ψ and S_ψ^{-1}, it follows that (17) is equivalent with

$$\sum_{g\in G} \widehat{\Phi}(g)\lambda(g) = 0 \tag{18}$$

in $L^2(\mathscr{L}(G), [\psi,\psi])$, which proves the claim. □

Proof. (Theorem 2) $(ii) \Rightarrow (i)$ Suppose that $\mathcal{B}_\psi = \{\Pi(g)\psi : g \in G\}$ is $\ell^p(G)$-linearly dependent. By Lemma 1, there exists a nontrivial $(c_g) \in \ell^p(G)$ such that

$$\sum_{g\in G} c_g \lambda(g)[\psi,\psi]^{\frac{1}{2}} = 0$$

in $L^2(\mathscr{L}(G))$. Specifically, for any $H \in L^2(\mathscr{L}(G)))$ we have

$$\Big\langle \sum_{g\in G} c_g\lambda(g)[\psi,\psi]^{\frac{1}{2}}, H\Big\rangle_2 = 0.$$

Hence, for any $g' \in G$ (take $H = \lambda(g') \in L^2(\mathscr{L}(G)))$, we have

$$\begin{aligned}\Big\langle \sum_{g\in G} c_g\lambda(g)[\psi,\psi]^{\frac{1}{2}}, \lambda(g')\Big\rangle_2 &= \sum_{g\in G} c_g \langle\lambda(g)[\psi,\psi]^{\frac{1}{2}}, \lambda(g')\rangle_2 \\ &= \sum_{g\in G} c_g \tau(\lambda(g')^*\lambda(g)[\psi,\psi]^{\frac{1}{2}}) \\ &= \sum_{g\in G} c_g \tau((\lambda(g^{-1})\lambda(g'))^*[\psi,\psi]^{\frac{1}{2}}) \\ &= \sum_{g\in G} c_g ([\psi,\psi]^{\frac{1}{2}})\widehat{\ }(g^{-1}g').\end{aligned}$$

Note that $([\psi,\psi]^{1/2})\widehat{\ } \in \ell^2(G) \subseteq \ell^q(G)$. Therefore, $(a_g) \in \ell^p(G) = (\ell^q(G))^*$, where $a_g = c_{g^{-1}}$, $g \in G$, annihilates $\operatorname{span}\{\rho(g)([\psi,\psi]^{\frac{1}{2}})\widehat{\ } : g \in G\}$ and, thus, this set is not dense is $\ell^q(G)$.

(iii) ⇔ *(i)* Suppose that $\mathcal{B}_\psi$ is $\ell^p(G)$-linearly dependent. By Lemma 1, there exists $0 \neq \Phi \in L^2(\mathscr{L}(G))$ such that $\widehat{\Phi} \in \ell^p(G)$ and $\sum_{g\in G} \widehat{\Phi}(g)\lambda(g)[\psi,\psi]^{\frac{1}{2}} = 0$ in $L^2(\mathscr{L}(G))$. Using Lemma 2.2 in [2], we first conclude that

$$\sum_{g\in G} \widehat{\Phi}(g)\lambda(g) = \Phi \text{ in } L^2(\mathscr{L}(G)). \tag{19}$$

Moreover, since $\mathcal{B}_\psi$ is Bessel, using Remark 1 and (8), we have

$$\sum_{g\in G} \widehat{\Phi}(g)\lambda(g)[\psi,\psi]^{1/2} = \Phi[\psi,\psi]^{1/2} \text{ in } L^2(\mathscr{L}(G)). \tag{20}$$

Hence, $\Phi[\psi,\psi]^{\frac{1}{2}} = 0$; consequently, $[\psi,\psi]^{1/2}\Phi^* = 0$. It follows that $\overline{R(\Phi^*)} \subseteq N_\psi$ (recall that $N_\psi = N([\psi,\psi]^{1/2})$; see, for example, [10], 2.5), that is, $P_{\overline{R(\Phi)^*}} \leqslant P_{N_\psi}$. Since $P_{\overline{R(\Phi)^*}}$ is the left support of Φ^*, we have

$$P_{N_\psi}\Phi^* = \Phi^*. \tag{21}$$

Using the definition of Fourier coefficients and the traciality of τ, it follows that

$$\widehat{\Phi^*}(g) = \tau(\Phi^*\lambda(g)^*) = \langle(\lambda(g)\Phi)^*\delta_e,\delta_e\rangle = \overline{\langle\lambda(g)\Phi\delta_e,\delta_e\rangle} = \overline{\widehat{\Phi}(g^{-1})},$$

for all $g \in G$; consequently, $\widehat{\Phi^*} \in \ell^p(G)$. We have, thus, proved that P_{N_ψ} is not a projection of uniqueness for $\ell^p(G)$.

Suppose now that P_{N_ψ} is not a projection of uniqueness for $\ell^p(G)$. Then there exists $0 \neq \Phi \in L^2(\mathscr{L}(G))$ such that $\widehat{\Phi} \in \ell^p(G)$ and $P_{N_\psi}\Phi = \Phi$. It follows that

$$N_\psi = R(P_{N_\psi}) \supseteq \overline{R(\Phi)}.$$

Passing to the orthogonal complements (recall that Φ is a closed, densely defined operator, and $[\psi,\psi]$ is, moreover, bounded), we have

$$N(\Phi^*) \supseteq N_\psi^\perp = \overline{R([\psi,\psi]^{1/2})}; \tag{22}$$

therefore, $\Phi^*[\psi,\psi]^{\frac{1}{2}} = 0$, with $\widehat{\Phi^*} \in \ell^p(G) \setminus \{0\}$. Using again Lemma 2.2 in [2] and (8), it follows that $\sum_{g\in G} \widehat{\Phi^*}(g)\lambda(g) = 0$ in $L^2(\mathscr{L}(G),[\psi,\psi]^{1/2})$; hence, $\mathcal{B}_\psi$ is $\ell^p(G)$-linearly dependent.

(ii) ⇒ *(iii)* Suppose that P_{N_ψ} is not a projection of uniqueness for $\ell^p(G)$, and take the corresponding $\Phi \in L^2(\mathscr{L}(G))$. Similarly as before, we conclude that $\Phi^*[\psi,\psi]^{1/2} = 0$, and hence, $\{\rho(g)([\psi,\psi]^{1/2}\widehat{)} : g \in G\}$ is not dense in $\ell^q(G)$ (the same arguments are used as in *(i)* ⇒ *(ii)*). □

Remark 3. *As in the abelian case, the theorem has the following consequences and extensions.*

(a) Since there are no nontrivial projections of uniqueness for $\ell^2(G)$*, Theorem 2 implies that the Bessel system* $\mathcal{B}_\psi$ *is* $\ell^2(G)$*-linearly independent precisely when* $N_\psi = \{0\}$.

(b) *Observe also that we could actually prove* $(iii) \Rightarrow (i)$ *similarly to* $(ii) \Rightarrow (i)$*, without using the assumption that* $\mathcal{B}_\psi$ *is Bessel; hence, this implication is true in the general case. The converse,* $(i) \Rightarrow (iii)$*, is much more difficult, and the problem is open even in the case of integer translates on* $L^2(\mathbb{R})$*.*

(c) *If* $p = 1$*, the series always converges; hence Theorem 2 holds for* $p = 1$ *without the assumption that* $\mathcal{B}_\psi$ *is Bessel.*

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