Math. Commun. **28**(2023), 317–336

# Some representations of unlimited natural numbers

Djamel Bellaouar[1,*] and Abdelmadjid Boudaoud[2]

[1] *Department of Mathematics, University 08 Mai 1945 Guelma, B. P. 401-Guelma, 24 000, Guelma, Algeria*
[2] *Laboratory of Pure and Applied Mathematics (LMPA), University of M'sila, Msila 28 000, Algeria*

Received June 7, 2023; accepted September 10, 2023

**Abstract.** Based on the authors' article [5] and the work of Hrbáček [11], we prove that every unlimited natural number $\omega$ is of the form $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ in at least $k$ different ways ($k \geq 1$ is limited), where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i/\omega_j$ is appreciable for $1 \leq i, j \leq 4$. Other similar representations of unlimited natural numbers are also presented.

**AMS subject classifications**: 26E35, 03H05, 03H15, 11A51

**Keywords**: factoring of integers, nonstandard analysis, unlimited integers

## 1. Introduction

The study of which integers are represented by a given quadratic form is one of the most celebrated in the theory of numbers. In Guy [10, D4, p. 229], Waring's problem is that of representation of positive integers as a sum of a fixed number $s$ of nonnegative $k$-th powers, i.e., whether for a given $k$ there is any fixed $s = s(k)$ such that

$$n = x_1^k + x_2^k + \cdots + x_s^k$$

is solvable for any $n$. In 1640, Fermat stated his conjecture that every prime number $p \equiv 1 (\text{mod } 4)$ can be written in the form $p = x^2 + y^2$. A century later, Euler proved Fermat's conjecture and worked seriously on related problems and generalizations. In 1770, Lagrange and Euler (see, e.g., Adler [1, Theorem 8.22, p. 234]) proved that every positive integer is a sum of four squares. In 1798, Legendre and Gauss ([1, Theorem 8.25, p. 236]) classified the integers that could be represented as a sum of three squares. More precisely, they proved that a positive integer can be represented as a sum of three squares if and only if it is not of the form $4^m\,(8k + 7)$. This result is deeper and more difficult than either of the two-square or four-square theorems. Motivated by Lagrange's result, it is natural to ask about the collection of quadratic forms that represent all positive integers, or more generally, to fix in advance a collection $S$ of integers, and ask about quadratic forms that represent all numbers in $S$. In this context, Iwaniec [12] considered a more general problem of the number of representations of an integer $n$ by a positive definite quadratic form $Q(x_1, \ldots, x_s)$.

*Corresponding author. *Email addresses:* `bellaouar.djamel@univ-guelma.dz` (D. Bellaouar), `boudaoudab@yahoo.fr` (A. Boudaoud)

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For example, in [1, p. 259], it is shown that each nonnegative integer is either of the form $x^2+y^2+z^2$ or of the form $x^2+y^2+2z^2$, where $x, y$ and $z$ are positive integers.

In the context of nonstandard analysis [6], we shall need the following definition and principle which are used throughout this paper.

**Definition 1.** *Two positive real numbers $x$ and $y$ are of the same order, written $x \sim y$, if $x/y$ is appreciable. Or, equivalently, there exist standard real numbers $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 < x/y < r_2$.*

**Principle 1.** [Cauchy's principle [6, p. 19]] *No external set is internal.*

For details about internal and external sets, one can see [3, definitions 2.2, 2.3] and [6, pp. 5,6]. Furthermore, we explain here how to apply this principle. Let $\omega$ be unlimited. The set $\{n \in \mathbb{N} : \ \omega > n\}$ is internal and contains all limited positive integers. By Cauchy's principle, $\omega > n_0$ for some unlimited positive integer $n_0$.

As a continuation of our previous works [3, 4, 5] and Hrbáček's work [11], we prove in the present paper that every unlimited positive integer $n$ can be written in the form:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 \\ \omega_i \sim \omega_j \text{ for } 1 \leq i, j \leq 4, \end{cases} \tag{A$_2$}$$

where $\omega_i \in \mathbb{N}$ for $1 \leq i \leq 4$. Note that the second condition of (A$_2$) implies that each $\omega_i$ is unlimited. As a consequence, if $k \geq 2$ is a limited positive integer, then we can generalize the above form as follows:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 + \cdots + \omega_{2k-1} \cdot \omega_{2k}, \\ \\ \omega_i \sim \omega_j \text{ for } 1 \leq i, j \leq 2k. \end{cases} \tag{A$_k$}$$

Moreover, we present some families of unlimited positive integers which can be represented as in (A$_2$) by giving the values of $\omega_i$ $(1 \leq i \leq 4)$ in terms of $n$. Other similar types of representation of unlimited natural numbers are also discussed.

To start with our main results, we need the following lemmas:

**Lemma 1.** *Let $a, b, c, d \in \mathbb{R}_+$.*

*(1) $a \sim a$. If $a \sim b$, then $b \sim a$. If $a \sim b$ and $b \sim c$, then $a \sim c$.*

*(2) If $a \sim b$ and $r, s \in \mathbb{R}^+$ are appreciable, then $r \cdot a \sim s \cdot b$.*

*(3) If $a \sim c$ and $b \sim d$, then $a + b \sim c + d$.*

*(4) If $a \sim c$ and $b \sim d$, then $a \cdot b \sim c \cdot d$.*

*(5) If $a \sim b$ and $n \in \mathbb{N}^+$ is standard, then $a^n \sim b^n$ and $\sqrt[n]{a} \sim \sqrt[n]{b}$.*

**Proof**. Proof of (*3*). We have $r_1 \cdot c < a < r_2 \cdot c$ and $s_1 \cdot d < b < s_2 \cdot d$ for some standard $r_1, r_2, s_1, s_2 \in \mathbb{R}^+$. Hence $u_1 \cdot (c+d) \leq r_1 \cdot c + s_1 \cdot d < a+b < r_2 \cdot c + s_2 \cdot d \leq u_2 \cdot (c+d)$ for $u_1 = \min\{r_1, s_1\}$ and $u_2 = \max\{r_2, s_2\}$. □

To state the second lemma, we need the result known as Bertrand's postulate: For every $n \in \mathbb{N}$, $n \geq 2$, there is a prime $p$ such that $n < p < 2n$.

**Lemma 2.** *For every $x \in \mathbb{R}$, $x \geq 2$, there is a prime $p$ such that $x < p < 2x$.*

**Proof**. Recall that $[x]$ denotes the integer part of the real number $x$. There is a prime $p$ such that $[x] < p < 2\,[x]$. Then $x < [x] + 1 \leq p < 2\,[x] \leq 2x$. □

## 2. Unlimited integers of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$

One of the main results is the following:

**Theorem 1.** *Every unlimited $\omega \in \mathbb{N}$ can be written in the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \sim \sqrt{\omega}$ and $\omega_i > 0$ for $1 \leq i \leq 4$.*

**Proof**. By Bertrand's postulate, there is a prime number $p_1$ such that $\frac{\sqrt{\omega}}{2} < p_1 < \sqrt{\omega}$ and a prime number $p_2$ such that $\frac{\sqrt{\omega}}{4} < p_2 < \frac{\sqrt{\omega}}{2}$.

The Diophantine equation $p_1 \cdot x + p_2 \cdot y = \omega$ has a particular solution $x_0, y_0$ in integers (Euclid's algorithm) since $\gcd(p_1, p_2) = 1$. Moreover, all solutions are given by $x_t = x_0 + t \cdot p_2$ and $y_t = y_0 - t \cdot p_1$, where $t$ is an arbitrary integer. Now, we can choose $t$ so that

$$\frac{\sqrt{\omega}}{4} < x_t < \frac{3\sqrt{\omega}}{4}. \tag{1}$$

In fact, let $t^*$ be the largest integer for which $x_{t^*} \leq \sqrt{\omega}/4$. Then clearly $x_{t^*+1} > \sqrt{\omega}/4$ and since $x_{t^*+1} = x_{t^*} + p_2$, it follows that

$$x_{t^*+1} - \frac{\sqrt{\omega}}{4} \leq x_{t^*+1} - x_{t^*} = p_2 < \frac{\sqrt{\omega}}{2},$$

and so $x_{t^*+1} < \frac{\sqrt{\omega}}{4} + \frac{\sqrt{\omega}}{2} = \frac{3\sqrt{\omega}}{4}$. Thus, we let $t = t^* + 1$. This proves (1). For this $t$ we get $\frac{\omega}{8} < p_1 \cdot x_t < \frac{3\omega}{4}$ and hence $\omega/4 < p_2 \cdot y_t = \omega - p_1 \cdot x_t < 7\omega/8$. It follows that $\frac{\sqrt{\omega}}{2} < y_t < \frac{7\sqrt{\omega}}{2}$. We let $\omega_1 = p_1$, $\omega_2 = x_t$, $\omega_3 = p_2$ and $\omega_4 = y_t$. This completes the proof. □

We now consider the basic question: Can every unlimited natural number $n$ be represented in the form $n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \sim \omega_j$ holds for all $1 \leq i, j \leq 4$ in at least $k$ different ways ($k \geq 1$ limited)? For the answer, fix a standard $k$. By Bertrand's postulate, there is a prime number $p_1$ such that $\frac{\sqrt{\omega}}{2k} < p_1 < \frac{\sqrt{\omega}}{k}$ and a prime number $p_2$ such that $\frac{\sqrt{\omega}}{4k} < p_2 < \frac{\sqrt{\omega}}{2k}$, so $p_1 \sim \sqrt{\omega}$ and $p_2 \sim \sqrt{\omega}$. The Diophantine equation $p_1 \cdot x + p_2 \cdot y = \omega$ has a solution $x_0, y_0$ in integers. Moreover, every solution is of the form $x_t = x_0 + t \cdot p_2$, $y_t = y_0 - t \cdot p_1$ for some $t \in \mathbb{Z}$. We can now choose $t$ so that $\frac{\sqrt{\omega}}{4k} < x_t < \frac{3\sqrt{\omega}}{4k}$, so $x_t \sim \sqrt{\omega}$. For this $t$ we get $\frac{\omega}{8k^2} < p_1 \cdot x_t < \frac{3\omega}{4k^2}$ and hence

$$\frac{\left(4k^2 - 3\right)\omega}{4k^2} < p_2 \cdot y_t = \omega - p_1 \cdot x_t < \frac{\left(8k^2 - 1\right)\omega}{8k^2}.$$

It follows that $\frac{\left(4k^2-3\right)\sqrt{\omega}}{2k} < y_t < \frac{\left(8k^2-1\right)\sqrt{\omega}}{2k}$. Different values of $k$ give different values of the quadruple $p_1, p_2, x_t, y_t$.

**Proposition 1.** *Let $k \geq 1$ be limited. Every unlimited positive integer $\omega$ can be represented as $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ in at least $k$ different ways with the same values of $\omega_1$, $\omega_3$ for all $k$, where $\omega_i \in \mathbb{N}$ is unlimited for $1 \leq i \leq 4$.*

**Proof**. Let $p_1, p_2, p_3$ be distinct unlimited primes such that $\omega \geq p_1 p_2 p_3$ (such prime numbers exist by Cauchy's principle and the fact that there are infinitely many primes, since $\omega$ is greater than any product of three standard prime numbers). Since $\gcd(p_1, p_2) = 1$, we conclude that there exist integers $x_0$ and $y_0$ such that $p_1 \cdot x_0 + p_2 \cdot y_0 = 1$. Therefore, the integer solutions of $p_1 \cdot x + p_2 \cdot y = \omega$ are given by $x_t = \omega x_0 - p_2 t$ and $y_t = \omega y_0 + p_1 t$, where $p_1 \cdot x_0 + p_2 \cdot y_0 = 1$ and $t \in \mathbb{Z}$. Thus, this equation has positive solutions if $\omega x_0 > p_2 t$ and $\omega y_0 > -p_1 t$, from which it follows that

$$\frac{-\omega y_0}{p_1} < t < \frac{\omega x_0}{p_2}. \tag{2}$$

Now let $k \geq 1$ be limited. Since $\omega > p_1 p_2 k$, or equivalently $\omega(p_1 x_0 + p_2 y_0) > p_1 p_2 k$, we conclude that

$$\frac{-\omega y_0}{p_1} < \left[\frac{-\omega y_0}{p_1}\right] + k < \frac{\omega x_0}{p_2}. \tag{3}$$

Therefore, inequalities (2) hold for at least $k$ different values of $t$ with $t = [-\omega y_0/p_1] + i$ for $1 \leq i \leq k$.

Next, note that $x_t$ and $y_t$ are not both limited; otherwise $p_3 \leq \frac{x_t}{p_2} + \frac{y_t}{p_1} \cong 0$, which is a contradiction. In fact, without loss of generality, assume that $x_t$ is unlimited with $x_0 > 0$, i.e., $y_0 < 0$ and we show that $y_t$ is also unlimited.

Let $a \geq 1$ be limited. Since $\omega(p_1 x_0 + p_2 y_0) > a p_2$, we deduce that $p_2(a - \omega y_0) < p_1 \omega x_0$. Moreover, as in the proof of (3), we can prove that $\frac{a - \omega y_0}{p_1} + t' < \frac{\omega x_0}{p_2}$ for every limited $t' \geq 1$. Indeed, the last inequality holds since $\omega(p_1 x_0 + p_2 y_0) > p_2(a + t' p_1)$, and so the following inequalities:

$$\frac{a - \omega y_0}{p_1} < t < \frac{\omega x_0}{p_2} \tag{4}$$

hold at least for $k$ different values of $t$. It follows from the left-hand side of (4) that $p_1 t > a - \omega y_0$. Thus, $y_t = \omega y_0 + p_1 t > a$, which shows that $y_t$ is unlimited. We let $\omega_1 = p_1$, $\omega_2 = x_t$, $\omega_3 = p_2$ and $\omega_4 = y_t$, which are unlimited positive integers. This completes the proof. □

**Remark 1.** *One can give a proof of Proposition 1 as follows: By Bertrand's postulate there exist prime numbers $p_1$ and $p_2$ such that $\frac{\sqrt[3]{\omega}}{2} < p_1 < \sqrt[3]{\omega}$ and $\frac{\sqrt[3]{\omega}}{4} < p_2 < \frac{\sqrt[3]{\omega}}{2}$. The solutions of the equation $p_1 x + p_2 y = \omega$ are of the form $x_t = x_0 - t p_2$ and $y_t = y_0 + t p_1$, where $t$ is an integer. Fix $t$ so that $(\sqrt[3]{\omega})^2 - \sqrt[3]{\omega} < y_t < (\sqrt[3]{\omega})^2$. If $k \geq 0$ is standard, then $y_{t+k} = y_t + k p_1$, so $y_{t+k}$ is unlimited and $y_{t+k} < (\sqrt[3]{\omega})^2 + k\sqrt[3]{\omega}$, so that $p_1 x_{t+k} = \omega - p_2 y_{t+k} > \omega - \sqrt[3]{\omega}\left((\sqrt[3]{\omega})^2 + k\sqrt[3]{\omega}\right)/2 > \omega/4$ and $x_{t+k} > (\sqrt[3]{\omega})^2/4$ is also unlimited. We can let $\omega_1 = p_1$, $\omega_2 = x_{t+k}$, $\omega_3 = p_2$, $\omega_4 = y_{t+k}$ and $k \geq 0$.*

**Corollary 1.** *Let $k \geq 2$ be a standard natural number. Every unlimited $\omega \in \mathbb{N}$ can be written in the form $(\mathrm{A}_k)$.*

**Proof**. By induction. Note that $\omega_{2k-1}\cdot\omega_{2k}\sim\omega$, so Theorem 1 enables the inductive step by writing $\omega_{2k-1}\cdot\omega_{2k}=\omega'_{2k-1}\cdot\omega'_{2k}+\omega_{2k+1}\cdot\omega_{2k+2}$ with $\omega'_{2k-1},\omega'_{2k},\omega_{2k+1},\omega_{2k+2}\sim\sqrt{\omega}$. □

**Lemma 3.** *Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\omega_1^2\cdot\omega_3+\omega_4\cdot\eta$, where $\omega_1,\omega_3,\omega_4\sim\sqrt[3]{\omega}$ and $\eta\sim\sqrt[3]{\omega^2}$.*

**Proof**. We closely follow the proof of Theorem 1. We fix prime numbers $p_1$ such that $\frac{\sqrt[3]{\omega}}{2}<p_1<\sqrt[3]{\omega}$ and $p_2$ such that $\frac{\sqrt[3]{\omega}}{4}<p_2<\frac{\sqrt[3]{\omega}}{2}$. The general solution of the Diophantine equation $p_1^2\cdot x+p_2\cdot y=\omega$ has the form $x_t=x_0+t\cdot p_2$, $y_t=y_0-t\cdot p_1^2$, $t\in\mathbb{Z}$. We can now choose $t$ so that $\frac{\sqrt[3]{\omega}}{4}<x_t<\frac{3\sqrt[3]{\omega}}{4}$. For this $t$ we get $\frac{\omega}{16}<p_1^2\cdot x_t<\frac{3\omega}{4}$ and hence $\frac{\omega}{4}<p_2\cdot y_t=\omega-p_1^2\cdot x_t<\frac{15\omega}{16}$. It follows that $\frac{\sqrt[3]{\omega^2}}{2}<y_t<\frac{15\sqrt[3]{\omega^2}}{4}$. We let $\omega_1=p_1$, $\omega_3=x_t$, $\omega_4=p_2$, $\eta=y_t$. □

**Theorem 2.** *Every unlimited $\omega\in\mathbb{N}$ can be written in the form*

$$\omega=\omega_1\cdot\omega_2\cdot\omega_3+\omega_4\cdot\omega_5\cdot\omega_6+\omega_7\cdot\omega_8\cdot\omega_9,$$

*where $\omega_i>0$ and $\omega_i\sim\sqrt[3]{\omega}$ for $1\le i\le 9$.*

**Proof**. Use Theorem 1 to write $\eta=\omega_5\cdot\omega_6+\omega_8\cdot\omega_9$ where $\omega_5,\omega_6,\omega_8,\omega_9\sim\sqrt{\eta}\sim\sqrt[3]{\omega}$, then substitute into the expression from Lemma 3 and let $\omega_2=\omega_1$, $\omega_7=\omega_4$. □

**Corollary 2.** *Let $k\ge 3$ be a standard natural number. Every unlimited $\omega\in\mathbb{N}$ can be written in the form*

$$\omega=\sum_{i=1}^{k}\omega_{i,1}\cdot\omega_{i,2}\cdot\omega_{i,3}, \tag{5}$$

*where $\omega_{i,j}>0$ and $\omega_{i,j}\sim\sqrt[3]{\omega}$ for $1\le i\le k$, $1\le j\le 3$.*

**Proof**. By induction, starting with $k=3$ and using the observation that $\eta=\omega_1\cdot\omega_2\cdot\omega_3+\omega_4\cdot\omega_5\cdot\omega_6\sim\omega$ and hence, by Theorem 2, it can be expressed as $\eta=\omega'_1\cdot\omega'_2\cdot\omega'_3+\omega'_4\cdot\omega'_5\cdot\omega'_6+\omega'_7\cdot\omega'_8\cdot\omega'_9$, where $\omega'_i>0$ and $\omega'_i\sim\sqrt[3]{\omega}$ for $1\le i\le 9$. □

Lemma 3 generalizes as follows. Note that $r=2$ gives Theorem 1.

**Lemma 4.** *Let $r\ge 2$ be a standard natural number. Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\omega_1^{r-1}\cdot\omega_3+\omega_4\cdot\eta$ where $\omega_1,\omega_3,\omega_4\sim\sqrt[r]{\omega}$ and $\eta\sim\sqrt[r]{\omega^{r-1}}$.*

**Proof**. We fix prime numbers $p_1$ such that $\frac{\sqrt[r]{\omega}}{2}<p_1<\sqrt[r]{\omega}$ and $p_2$ such that $\frac{\sqrt[r]{\omega}}{4}<p_2<\frac{\sqrt[r]{\omega}}{2}$. The general solution of the Diophantine equation $p_1^{r-1}\cdot x+p_2\cdot y=\omega$ has the form $x_t=x_0+t\cdot p_2$, $y_t=y_0-t\cdot p_1^{r-1}$, $t\in\mathbb{Z}$. We can now choose $t$ so that $\frac{\sqrt[r]{\omega}}{4}<x_t<\frac{3\sqrt[r]{\omega}}{4}$ . For this $t$ we get $\frac{\omega}{2^{r+1}}<p_1^{r-1}\cdot x_t<\frac{3\omega}{4}$ and hence $\frac{\omega}{4}<p_2\cdot y_t=\omega-p_1^{r-1}\cdot x_t<\frac{(2^{r+1}-1)\omega}{2^{r+1}}$. It follows that $\frac{1}{2}\cdot\sqrt[r]{\omega^{r-1}}<y_t<\frac{2^{r+1}-1}{2^{r-1}}\cdot\sqrt[r]{\omega^{r-1}}$. We let $\omega_1=p_1$, $\omega_3=x_t$, $\omega_4=p_2$, $\eta=y_t$. □

**Theorem 3.** *Let $r\ge 2$ and $k\ge r$ be standard natural numbers. Every unlimited $\omega\in\mathbb{N}$ can be written in the form $\omega=\sum_{i=1}^{k}\prod_{j=1}^{r}\omega_{i,j}$, where $\omega_{i,j}>0$ and $\omega_{i,j}\sim\sqrt[r]{\omega}$ for $1\le i\le k$, $1\le j\le 3$.*

**Proof**. By induction on $r$. For $r=2$, this is Corollary 1. Assume the theorem is true for $r-1$. Then $k-1\geq r-1$ and we can write $\eta=\sum_{i=1}^{k-1}\prod_{j=1}^{r-1}\omega_{i,j}$ with all $\omega_{i,j}\sim\sqrt[r-1]{\eta}=\sqrt[r]{\omega}$ and substitute the result into the formula from Lemma 4. $\square$

Next, we present an explicit method to prove that all numbers that are similar in structure to $n!$ can be written in the form ($A_2$).

**Theorem 4.** *Let $(a_i)_{1\leq i\leq k}$ be a sequence of positive integers such that $a_1$ is limited, $k$ is unlimited and $a_{i+1}-a_i$ is limited positive for $i=1,2,\ldots,k-1$, and let $n=a_1a_2\cdots a_k$. There exist two unlimited positive integers $R_1$and $R_2$ such that $n=R_1\cdot R_2$ with $R_1\sim R_2$.*

**Proof**. Let $\lambda$ be a limited positive integer such that $0<a_{i+1}-a_i\leq\lambda$ for $1\leq i\leq k-1$. Indeed, such number exists since the set $\{a_{i+1}-a_i : i<k\}$ is internal, so it has a maximal element $a_{i^*+1}-a_{i^*}$ which is limited.

Now, we show that there exists a unique unlimited positive integer $t$ such that

$$\begin{cases} a_1a_2\cdots a_{t-1}a_t<a_{t+1}a_{t+2}\cdots a_{k-1}a_k, \\ a_1a_2\cdots a_ta_{t+1}\geq a_{t+2}\cdots a_{k-1}a_k. \end{cases} \tag{6}$$

Otherwise,

$$\begin{cases} a_1<a_2a_3\cdots a_{k-1}a_k \\ a_1a_2<a_3a_4\cdots a_{k-1}a_k \\ \qquad\vdots \\ a_1a_2\cdots a_{k-3}a_{k-2}<a_{k-1}a_k \\ a_1a_2\cdots a_{k-2}a_{k-1}<a_k. \end{cases} \tag{7}$$

But the last inequality of (7) leads to a contradiction because $a_{k-2}a_{k-1}>a_k$. Indeed, the numbers $a_{k-2}$, $a_{k-1}$ and $a_k$ are unlimited with $0<a_k-a_{k-1}<\lambda$ and $0<a_k-a_{k-2}<2\lambda$, which implies that $a_{k-1}=a_k-\lambda_1$ and $a_{k-2}=a_k-\lambda_2$ for some limited integers $\lambda_1$ and $\lambda_2$, since $\lambda$ is limited. Therefore,

$$a_{k-1}a_{k-2}=a_k^2\left(1-\frac{\lambda_1}{a_k}\right)\left(1-\frac{\lambda_2}{a_k}\right)=a_k^2\left(1-\phi\right)>a_k,$$

where $\phi\cong 0$. A contradiction. This proves (6).

Next, from (6) we also have

$$\frac{1}{a_{t+1}}\leq\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k}<a_{t+1}. \tag{8}$$

There are three cases to consider:

**Case 1.** $a_1a_2\cdots a_{t-1}a_t/a_{t+2}\cdots a_{k-1}a_k$ is appreciable. Since $a_{i+1}-a_i\leq\lambda$ with $\lambda$ limited, i.e., the elements $(a_i)_{1\leq i\leq k}$ are increasing by a limited quantity, there exists a positive integer $i_0$ with $i_0\leq t$ such that $a_{i_0}$ and $\sqrt{a_{t+1}}$ have the same order, that is, $a_{i_0}/\sqrt{a_{t+1}}$ is appreciable. We put $R_1=a_1a_2\cdots a_{t-1}a_ta_{t+1}/a_{i_0}$ and $R_2=a_{t+2}\cdots a_{k-1}a_ka_{i_0}$. It is clear that $n=R_1\cdot R_2$, where

$$\frac{R_1}{R_2}=\frac{a_1a_2\cdots a_{t-1}a_ta_{t+1}}{a_{i_0}^2a_{t+2}\cdots a_{k-1}a_k}=\frac{a_1a_2\cdots a_{t-1}a_t}{a_{t+2}\cdots a_{k-1}a_k}\cdot\frac{a_{t+1}}{a_{i_0}^2}$$

is appreciable since $a_{t+1} \sim a_{i_0}^2$.

**Case 2.** $a_1 a_2 \cdots a_{t-1} a_t / a_{t+2} \cdots a_{k-1} a_k \cong 0$. Here by (8), there exists an unlimited positive integer $l \leq a_{t+1}$ such that $\frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot l$ is appreciable. We have the following subcases:

**Case 2.1.** $a_{t+1}/l = A$ with $A$ appreciable. Here, we put $R_1 = a_1 a_2 \cdots a_{t-1} a_t a_{t+1}$ and $R_2 = a_{t+2} \cdots a_{k-1} a_k$, in which case $n = R_1 \cdot R_2$, where

$$\frac{R_1}{R_2} = \frac{a_1 a_2 \cdots a_{t-1} a_t a_{t+1}}{a_{t+2} \cdots a_{k-1} a_k} = \frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot lA,$$

which is appreciable.

**Case 2.2.** $a_{t+1}/l$ is unlimited. As above, let $i_0$ be a positive integer with $i_0 \leq t$ such that $a_{i_0}$ and $\sqrt{a_{t+1}/l}$ have the same order. We put $R_1 = a_1 a_2 \cdots a_{t-1} a_t a_{t+1} / a_{i_0}$ and $R_2 = a_{t+2} \cdots a_{k-1} a_k a_{i_0}$. It follows that $R_1/R_2 = \frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot \frac{a_{t+1}}{a_{i_0}^2 l}$ is appreciable since $a_{t+1}/l \sim a_{i_0}^2$.

**Case 3.** $a_1 a_2 \cdots a_{t-1} a_t / a_{t+2} \cdots a_{k-1} a_k \cong +\infty$. In this case, by (8), there exists an unlimited positive integer $m \leq a_{t+1}$ such that $\frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot \frac{1}{m}$ is appreciable. We also have the following subcases:

**Case 3.1.** $a_{t+1}/m = A$ with $A$ appreciable. Here we put $R_1 = a_1 a_2 \cdots a_{t-1} a_t$ and $R_2 = a_{t+2} \cdots a_{k-1} a_k a_{t+1}$, where $n = R_1 \cdot R_2$ and $R_1/R_2 = \frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k a_{t+1}} = \left(\frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot \frac{1}{m}\right) \cdot \frac{1}{A}$ which is appreciable.

**Case 3.2.** $a_{t+1}/m = \omega$ with $\omega$ unlimited. Let $i_0$, $j_0$ be two positive integers not exceeding $t$ with $i_0 \neq j_0$ such that $a_{i_0} \sim m$ and $a_{j_0} \sim \sqrt{\omega}$. Then we put $R_1 = a_1 a_2 \cdots a_{t-1} a_t a_{t+1} / a_{i_0} a_{j_0}$ and $R_2 = a_{t+2} \cdots a_{k-1} a_k a_{i_0} a_{j_0}$. We also observe that $n = R_1 \cdot R_2$, where

$$\frac{R_1}{R_2} = \frac{a_1 a_2 \cdots a_{t-1} a_t a_{t+1}}{a_{i_0}^2 a_{j_0}^2 a_{t+2} \cdots a_{k-1} a_k} = \left(\frac{a_1 a_2 \cdots a_{t-1} a_t}{a_{t+2} \cdots a_{k-1} a_k} \cdot \frac{1}{m}\right) \cdot \frac{m a_{t+1}}{a_{i_0}^2 a_{j_0}^2}$$

is appreciable since $m a_{t+1} = m^2 \omega \sim a_{i_0}^2 a_{j_0}^2$.

This completes the proof. □

Applying Theorem 4, we obtain the following corollaries.

**Corollary 3.** *Let $n$ be as in Theorem 4. Then $n$ is of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.*

**Proof**. Since $n = R_1 \cdot R_2$ with $R_1 \sim R_2$, we conclude that if one of these numbers is even, say $R_1$, then $n = (R_1/2) \cdot R_2 + (R_1/2) \cdot R_2$. If $R_1$ and $R_2$ are both odd, then $n = \left(\frac{R_1 - 1}{2}\right) R_2 + \left(\frac{R_1 - 1}{2} + 1\right) \cdot R_2$, as required. □

**Corollary 4.** *Let $n$ be unlimited. Then $n!$ is of the form $n! = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.*

**Proof**. By definition $n! = a_1 a_2 \cdots a_n$, where $a_i = i$ $(1 \leq i \leq n)$, that is, $(a_i)_{1\leq i\leq n}$ satisfy conditions of Theorem 4. Then the result follows by applying Corollary 3. $\square$

The proof of Theorem 4 can be adapted straightforwardly to obtain the following corollary.

**Corollary 5.** *Let $k$ be unlimited and let $(a_i)_{1\leq i\leq k}$ be a sequence of positive integers such that $a_1$ is limited and $a_{i+1} = s_i \cdot a_i$, where $s_i > 1$ is limited for $i = 1, 2, \ldots, k-1$, and let $n = a_1 a_2 \cdots a_k$. Then there exist two unlimited positive integers $R_1$ and $R_2$ such that $n = R_1 \cdot R_2$, where $R_1 \sim R_2$.*

## 3. Other similar representations

In this subsection, we provide some other representations of unlimited natural numbers. First, we need the following lemma:

**Lemma 5** (see [9]). *Let $n! = \prod_{p\leq n} p^{v_p(n!)}$ be the prime factorization of $n!$. If $v_p(n!) > v_q(n!)$, then $p^{v_p(n!)} > q^{v_q(n!)}$.*

**Remark 2.** *By Nathanson [16, Theorem 1.12, p. 29], for every positive integer $n$ and prime $p$, $v_p(n!) = \sum_{\alpha=1}^{+\infty} \left[\frac{n}{p^\alpha}\right] = \sum_{\alpha=1}^{\left[\frac{\log n}{\log p}\right]} \left[\frac{n}{p^\alpha}\right]$. It follows that for primes $p$ and $q$ with $p < q$ we have $v_p(n!) \geq v_q(n!)$. In particular, if $n \geq 4$, $p = 2$ and $q \geq 3$, then clearly $v_p(n!) = v_2(n!) > v_q(n!)$. Hence by Lemma 5, $2^{v_2(n!)} > q^{v_q(n!)}$.*

**Theorem 5.** *Let $n$ be unlimited. Then $n!$ can be written as $R_1 \cdot R_2$ where, $R_1, R_2$ are two unlimited positive integers with $R_1 \sim \sqrt[3]{n!} \sim \left[(n!)^{\frac{1}{3}}\right]$.*

**Proof**. By Stirling's formula we have $n! = n^n e^{-n}\sqrt{2\pi n}\,(1+\phi_1)$, $\phi_1 \cong 0$ (see [7, p. 49]). On the other hand, in 1808, Legendre determined the exact power $t$ of the prime $p$ that divides $n!$ (so $p^{t+1}$ does not divide $n!$) [18, p. 18], namely,

$$t = \sum_{\alpha=1}^{\infty} \left[\frac{n}{p^\alpha}\right] = \frac{n - (a_0 + a_1 + .. + a_r)}{p-1},$$

where the integers $a_0, a_1, \ldots, a_r$ are the digits of $n$ in base $p$, that is, $n = a_r p^r + a_{r-1}p^{r-1} + \cdots + a_1 p + a_0$ such that $0 \leq a_i \leq p-1$ for $i = 0, 1, \ldots, r$.

Now, assume that $n! = \prod_{i=1}^{m} p_i^{\alpha_i}$, where $2 = p_1 < p_2 < \cdots < p_m$ are primes and $\alpha_i \geq 1$ for all $i$. We have $\left[(n!)^{\frac{1}{3}}\right] = (n!)^{\frac{1}{3}}(1+\phi_2)$, $\phi_2 \cong 0$. By the formula above, the exponent $\alpha_2$ of 3 satisfies $\alpha_2 \leq n/2$. Since $\left[(n!)^{\frac{1}{3}}\right]/p_2^{\alpha_2} = \left[(n!)^{\frac{1}{3}}\right]/3^{\alpha_2} \geq \left[(n!)^{\frac{1}{3}}\right]/3^{n/2}$, it is easily seen that $\left[(n!)^{\frac{1}{3}}\right]/p_2^{\alpha_2} \cong +\infty$. Then there exists a positive integer $k$ such that

$$p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \leq \left[(n!)^{\frac{1}{3}}\right] < p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_{k+1}^{\alpha_{k+1}}.$$

Since in the prime factorization of $n!$ we have $\alpha_1 > \alpha_{k+1}$, it follows from Lemma 5 that $p_1^{\alpha_1} > p_{k+1}^{\alpha_{k+1}}$. Hence there exists an integer $s$ with $0 \le s < \alpha_1$ such that

$$p_1^s \cdot p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \le \left[ (n!)^{\frac{1}{3}} \right] < p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_1^{s+1}.$$

Therefore, $1 \le \left[ (n!)^{\frac{1}{3}} \right] / p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} < 2$, that is, $\left[ (n!)^{\frac{1}{3}} \right] \sim p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$. Hence, $n! = p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \cdot p_1^{\alpha_1 - s} p_{k+1}^{\alpha_{k+1}} \cdots p_m^{\alpha_m}$, which is of the form $R_1 \cdot R_2$, where $R_1 = p_1^s p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ and $R_2 = p_1^{\alpha_1 - s} p_{k+1}^{\alpha_{k+1}} \cdots p_m^{\alpha_m}$. This completes the proof. □

**Corollary 6.** *$n!$ is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \sqrt[3]{n!}$ for $1 \le i,\ j \le 6$.*

**Proof**. Since $n! = R_1 \cdot R_2$, *where* $R_1 \sim \sqrt[3]{n!}$, *we have* $R_2 \sim \sqrt[3]{(n!)^2}$. Use Theorem 1 to write $R_2 = \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5$ where $\omega_2, \omega_3, \omega_4, \omega_5 \sim \sqrt{R_2} = \sqrt[3]{n!}$. □

Consider the sequence of Fibonacci numbers $(F_n)$, where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $n \ge 2$. It is well-known that the generalized Fibonacci sequence is defined by $G_n = G_{n-1} + G_{n-2}$, where $G_1 = a$ and $G_2 = b$ ($a, b \in \mathbb{N}$ and $n \ge 3$), see Koshy [14, page 109].

**Theorem 6.** *Let $n$ be unlimited. If $a$ and $b$ are limited, then $G_{3n}^2 - G_n^2$ is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \omega_j$ for $1 \le i$, $j \le 6$.*

**Proof**. By [14, Theorem 7.1, p. 109], we have

$$G_n = aF_{n-2} + bF_{n-1}. \tag{9}$$

Moreover, the terms of this sequence verify the following equality: $G_{m+n}^2 - G_{m-n}^2 = G_{m+1} G_m F_{2n} + G_{m-1} G_m F_{2n}$ (see [14, Identity 3, p. 214]. In particular, for $m = 2n$ we get $G_{3n}^2 - G_n^2 = G_{2n+1} G_{2n} F_{2n} + G_{2n-1} G_{2n} F_{2n}$, which is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ are unlimited ($1 \le i \le 6$). Applying (9) we have $\omega_i \sim \omega_j$ for $1 \le i,\ j \le 6$. □

Note that Corollary 5 and Theorem 6 are interesting because it is not known whether every unlimited $\omega$ is of the form $\omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$ with $\omega_i \sim \omega_j$ for $1 \le i,\ j \le 6$.

**Proposition 2.** *There are infinitely many unlimited positive integers $n$ such that $F_n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{N}$ are unlimited, pairwise relatively prime with $\omega_i \sim \omega_j$ for $1 \le i, j \le 4$.*

**Proof**. Let $k$ be a positive integer with $3 \nmid (k+1)$ and let $n = 2k$. Applying Andrica [2, Equation (2), p. 194] ($F_{m+n} = F_{m+1} \cdot F_n + F_m \cdot F_{n-1}$), if $m = n+1$, then $F_{2n+1} = F_{n+2} \cdot F_n + F_{n+1} \cdot F_{n-1}$. Let $x, y \in \{n-1, n, n+1, n+2\}$. We can verify easily that $\gcd(x, y) = 1$ or 2, and by Koshy [14, Theorem 16.3, p. 198] we have $\gcd(F_x, F_y) = F_{\gcd(x,y)} = 1$ since $F_1 = F_2 = 1$. On the other hand, we see that $F_x / F_y$ is appreciable since $|x - y| \le 3$. □

**Theorem 7.** *Every unlimited positive integer $n$ can be written in the form* $(\mathrm{A}_2)$*, where $\omega_i \in \mathbb{Z}$ is unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$.*

The proof is based on the fact that a positive integer $n$ can be represented as the difference of two squares if and only if $n$ is not of the form $4k+2$ (see, e.g. Dujella [8]).

***Proof of Theorem** 7.* Let $n$ be an unlimited positive integer. If $n$ is not of the form $4k+2$, then $n = x^2 - y^2$ for some positive integers $x, y$ with $x$ unlimited, and if $n$ is of the form $4k+2$, then $n = 2m$ with $m$ odd, i.e., $m$ is not of the form $4k+2$. Thus, $n$ is of the form $2x^2 - 2y^2$. In both cases, $n$ is of the form $\lambda\left(x^2 - y^2\right)$, where $\lambda \in \{1, 2\}$. There are two cases to consider:

*Case 1.* $x$ and $y$ are of the same order. In this case we have nothing to prove and we can put $\omega_1 = \lambda x$, $\omega_2 = x$, $\omega_3 = -\lambda y$ and $\omega_4 = y$.

*Case 2.* $y/x \cong 0$. We distinguish two cases:

*Case 2.1.* Assume that $x+y$ is even. Then

$$n = \lambda(x-y)(x+y) = \lambda(x-y)\left(\frac{x+y}{2}\right) + \lambda(x-y)\left(\frac{x+y}{2}\right),$$

which is of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{Z}$ is unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$.

*Case 2.2.* Assume that $x+y$ is odd. Then

$$\begin{aligned} n &= \lambda(x-y)(x+y-1) + \lambda(x-y) \\ &= \lambda(x-y)\left(\frac{x+y-1}{2}\right) + \lambda(x-y)\left(\frac{x+y-1}{2}\right) + \lambda(x-y) \\ &= \lambda(x-y)\left(\frac{x+y-1}{2}\right) + \lambda(x-y)\left(\frac{x+y+1}{2}\right), \end{aligned}$$

which is also of the form $\omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$ with $\omega_i \in \mathbb{Z}$ unlimited and $|\omega_i/\omega_j| \in \{1/2, 1, 2\}$ for $1 \leq i, j \leq 4$. This completes the proof. $\square$

**Theorem 8.** *Every unlimited positive integer is either of the form $\omega_1^2 - \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited with $\omega_1/\omega_2 \cong 1$, or of the form $\omega_1^2/2 - \omega_2^2/2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are even and unlimited with $\omega_1/\omega_2 \cong 1$.*

**Proof**. We distinguish two cases:

*Case 1.* Assume that $n$ is not of the form $4k+2$. Then $n = a^2 - b^2$ for some positive integers $a, b$. This means that either $n$ is odd or it is of the form $4k$. If it is odd, then $n-1$ and $n+1$ are both even, in which case

$$n = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2. \tag{10}$$

On the other hand, if $n$ is divisible by 4, then $n = \left(\frac{n}{4}+1\right)^2 - \left(\frac{n}{4}-1\right)^2$. In both cases, $n$ is of the form $\omega_1^2 - \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited and $\omega_1/\omega_2 \cong 1$.

*Case 2.* Assume that $n = 4k+2$, then $n = 2m$ with $m$ odd. Since $m$ satisfies (10), we conclude that $n = (m+1)(m+1)/2 - (m-1)(m-1)/2$, which is of the form $\omega_1^2/2 - \omega_2^2/2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited and $\omega_1/\omega_2 \cong 1$. This completes the proof. $\square$

**Proposition 3.** *Let $p$ be a limited prime number such that $p \equiv 1\,(mod\; 4)$. There exist infinitely many positive integers $n$ such that $n$ is of the form* $(\mathrm{A}_2)$ *with $\omega_1/\omega_2 = \omega_3/\omega_4 = p$.*

**Proof**. Let $a$ and $b$ be two limited positive integers such that $p = a^2 + b^2$ and $\gcd(a, b) = 1$. Consider the Diophantine equation $a \cdot x + b \cdot y = 1$. Then there are limited integers $x_0$ and $y_0$ for which $a \cdot x_0 + b \cdot y_0 = 1$ and all solutions are given by $x_t = x_0 + bt$ and $y_t = y_0 - at$, where $t \in \mathbb{Z}$. For $t \cong \infty$ we see that $|x_t| \sim |y_t|$. For each such values of $t$ it follows from Lagrange's identity (Jarvis [13, Lemma 1.18, p. 9]) that $p\left(x_t^2 + y_t^2\right) = (ax_t + by_t)^2 + (ay_t - bx_t)^2 = 1 + k^2$, where $k = ay_t - bx_t$. Thus, $1 + k^2 = px_t^2 + py_t^2$. The proof is finished if we put $n = 1 + k^2$, $\omega_1 = p\,|x_t|$, $\omega_2 = |x_t|$, $\omega_3 = p\,|y_t|$ and $\omega_4 = |y_t|$. $\square$

**Proposition 4.** *Every unlimited positive integer $n$ can be written as one of the following four forms:*

(*1*) $n = \lambda\omega_1^2 + \omega_2^2 + \omega_2^2$, *where $\lambda \in \{1, 2\}$ and $\omega_i \sim \omega_j$ for $1 \le i, j \le 3$.*

(*2*) $n = (\lambda + 1)\,\omega_1^2 + \omega_2^2 - \omega_3 \cdot \omega_4$, *where $\lambda \in \{1, 2\}$ and $\omega_i \sim \omega_j$ for $1 \le i, j \le 4$.*

(*3*) $n = (\lambda + 2)\,\omega_1^2 - \omega_2 \cdot \omega_3 - \omega_4 \cdot \omega_5$, *where $\lambda \in \{1, 2\}$ and $\omega_i \sim \omega_j$ for $1 \le i, j \le 5$.*

(*4*) $n = 2\omega_1^2 + 2\omega_2^2 - \omega_3 \cdot \omega_4$, *where $\omega_i \sim \omega_j$ for $1 \le i, j \le 4$.*

**Proof**. Let $n$ be an unlimited positive integer. From [1, Theorem 8.25, p. 236], $n$ can be written in the form $x^2 + y^2 + \lambda z^2$, where $\lambda = 1$ or $\lambda = 2$.

First, assume that $z = \max\{x, y, z\}$. We distinguish the following cases:

*Case 1.* $x$ and $y$ are of the same order as $z$. In this case, we have nothing to prove and we can put $\omega_1 = z$, $\omega_2 = y$ and $\omega_3 = x$. Then $n$ is in form (*1*).

*Case 2.* $x/z \cong 0$ and $y/z$ is appreciable. Here, $n = (x + z)\,(x - z) + y^2 + (\lambda + 1)\,z^2$. Hence, $\omega_1 = z$, $\omega_2 = y$, $\omega_3 = x + z$ and $\omega_4 = z - x$. Thus, $n$ is in form (*2*).

*Case 3.* $y/z \cong 0$ and $x/z$ is appreciable. This case is very similar to that of Case 2 with $x$, $y$ exchanged. Thus, $n$ is in form (*2*).

*Case 4.* $x/z \cong 0$ and $y/z \cong 0$. Then, $n = (x + z)\,(x - z) + (y + z)\,(y - z) + (\lambda + 2)\,z^2$. Hence we can put $\omega_1 = z$, $\omega_2 = z + x$, $\omega_3 = z - x$, $\omega_4 = z + y$ and $\omega_5 = z - y$. Then $n$ is in form (*3*).

Now, assume that $\lambda = 2$ and $\max\{x, y, z\}$ is either $x$ or $y$, say $x$. We also have the following cases:

*Case 1.* $y$ and $z$ are of the same order as $x$. Here $n$ is in form (*1*).

*Case 2.* $y/x \cong 0$ and $z/x$ is appreciable. In this case, $n = 2x^2 + 2z^2 - (x + y)\,(x - y)$. Hence, $\omega_1 = x$, $\omega_2 = z$, $\omega_3 = x + y$ and $\omega_4 = x - y$. Then $n$ is in form (*4*).

*Case 3.* $z/x \cong 0$ and $y/x$ is appreciable. We can do the same reasoning as above, that is, $n$ is in form (*4*).

*Case 4.* $y/x \cong 0$ and $z/x \cong 0$. Then, $n = 4x^2 - 2(x+z)(x-z) - (x+y)(x-y)$. Hence, $\omega_1 = 2x$, $\omega_2 = 2(x+z)$, $\omega_3 = x - z$, $\omega_4 = x + y$ and $\omega_5 = x - y$. Then $n$ is in form *(3)*.

This completes the proof. □

## 4. Unlimited integers of the form $a \cdot \omega_1^2 + b \cdot \omega_2^2$, where $\omega_1 \sim \omega_2$

Let $n$ be an arbitrary unlimited number and let $a, b$ be limited. We want to represent $n$ in the form: $a \cdot \omega_1^2 + b \cdot \omega_2^2$, where $\omega_1 \sim \omega_2$.

Let $\omega$ be unlimited and let $F_\omega$ be the $\omega$-th Fibonacci number. Then $F_{2\omega+1}$ is of the form $\omega_1^2 + \omega_2^2$, where $\omega_1 \sim \omega_2$ and $\gcd(\omega_1, \omega_2) = 1$. In fact, from Koshy [14, Identity 30, p. 97] we have $F_{2\omega+1} = F_\omega^2 + F_{\omega+1}^2$, where $\gcd(F_\omega, F_{\omega+1}) = 1$ by [14, Theorem 16.3, p. 198].

Let us start with the following result:

**Proposition 5.** *There exist unlimited prime numbers $p$ such that $p = \omega_1^2 + \omega_2^2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited.*

**Proof**. From Dirichlet's theorem about primes in arithmetic progressions there exists an unlimited prime $q$ of the form $4k - 1$. Let $n$ be an unlimited positive integer with $n < q$. It is not difficult to see that the numbers $q$ and

$$4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2$$

are coprime. By Dirichlet's theorem once again, there exists a positive integer $k'$ such that the number $p = 4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2 \cdot k' - q$ is prime. Clearly, it is of the form $4t + 1$. By Nathanson [16, Theorem 13.3, p. 407], there exist two positive integers $\omega_1, \omega_2$ with $\omega_1 < \omega_2$ such that $p = \omega_1^2 + \omega_2^2$. Now, assume by way of contradiction that $\omega_1$ is limited, i.e., $\omega_1 < n$. It follows that

$$\begin{aligned}
\omega_2^2 &= p - \omega_1^2 = 4\left(q+1^2\right)^2\left(q+2^2\right)^2\cdots\left(q+n^2\right)^2 \cdot k' - \left(q+\omega_1^2\right) \\
&= \left(q+\omega_1^2\right)\left[\begin{array}{c} 4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\left(q+(\omega_1+1)^2\right)^2\cdots \\ \left(q+n^2\right)^2 \cdot k' - 1 \end{array}\right].
\end{aligned}$$

Note also that the above factors are relatively prime, i.e.,

$$\gcd\left(q+\omega_1^2, 4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\cdots\left(q+n^2\right)^2 \cdot k' - 1\right) = 1,$$

and so $4\left(q+1^2\right)^2\cdots\left(q+(\omega_1-1)^2\right)^2\left(q+\omega_1^2\right)\left(q+(\omega_1+1)^2\right)^2\cdots\left(q+n^2\right)^2 \cdot k' - 1$ must be square. This is impossible because it is of the form $4t - 1$. Thus, $\omega_2 > \omega_1 \geq n \cong \infty$. This completes the proof. □

**Proposition 6.** *Let $n \in \mathbb{N}$ be unlimited such that $n$ is representable as the sum of two squares. Then either $n = a^2 + b^2$ with $a \sim b$ or $2n = a^2 + b^2$ with $a \sim b$.*

**Proof**. Suppose that $n = a^2 + b^2$ with $b \leq a$. If $a \sim b$, the desired assertion holds in this case; otherwise, $b/a \cong 0$ and so $2n = (a-b)^2 + (a+b)^2$, where in this case $a - b \sim a + b$. This completes the proof. □

## 5. Representation of unlimited integers using quadratic forms

In this section, we aim to represent unlimited positive integers as in ($A_2$), where some of the factors $\omega_i$ $(1 \le i \le 4)$ are in $\mathbb{Z}$. In addition, we give the values of $\omega_i$ $(1 \le i \le 4)$.

Recall that a quadratic form is a homogeneous polynomial of degree two. The quadratic form $Q(x, y, \dots, z)$ represents the integer $n$ if there exist integers $a, b, \dots, c$ such that $n = Q(a, b, \dots, c)$. A binary quadratic form is a quadratic form in two variables. We consider the following definition:

**Definition 2.** *Let $f(x, y) = ax^2 + bxy + cy^2$. We say that $f$ represents an integer $n$ if $f(u, v) = n$ for some integers $u$ and $v$, and that $f$ properly represents $n$ if $f(u, v) = n$ with $\gcd(u, v) = 1$.*

In what follows, we give two results, where in the first we show that every unlimited integer $n$, which can be represented by a quadratic form $f(x, y) = ax^2+bxy+cy^2$ such that $a, b$ and $c$ are all nonzero limited integers with $b^2 - ac \neq 0$, can be written in the form ($A_2$), where $\omega_i \in \mathbb{Z}$ is unlimited for $1 \le i \le 4$. More precisely, we give the value of $\omega_i$ in terms of $n$ for $1 \le i, j \le 4$. In the second theorem, we present some types of quadratic forms for which any unlimited positive integer $n$ that can be represented by one of these forms is of the form:

$$\begin{cases} n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4 \\ \omega_i \sim \omega_j \ (1 \le i, j \le 4), \\ \gcd\left(\omega_1 \cdot \omega_2, \omega_3 \cdot \omega_4\right) \text{ is limited} \end{cases} \tag{$A_2'$}$$

where $\omega_i \in \mathbb{Z}$ is unlimited for $1 \le i \le 4$. Here we also give the value of $\omega_i$ in terms of $n$ for $1 \le i, j \le 4$.

**Theorem 9.** *Let $n$ be an unlimited positive integer. Assume that $n$ is represented by the quadratic form $f(x, y) = ax^2+bxy+cy^2$, where $a, b$ and $c$ are all nonzero limited integers with $b^2 - ac \neq 0$. Then by rewriting this quadratic form $n$ can always be represented explicitly in the form ($A_2$), where some of the $\omega_i$ may be negative integers.*

**Proof**. We suppose that $n$ is represented by $f$, i.e., $n = ax^2 + bxy + cy^2$. We have the following cases:

**I.** ($x = 0$ and $y \neq 0$) or ($x \neq 0$ and $y = 0$ ). In this case, $n = cy^2$ with $c \neq 0$ or $n = ax^2$ with $a \neq 0$. Let us take, for instance, $n = cy^2$. Then $n = c\,(y - t + t)^2$. Hence, $n = c\left((y - t)^2 + t^2 + 2t\,(y - t)\right) = c\,(y - t)^2 + ct\,(2y - t)$. We end this case if we take $t = [y/2]$ and put $\omega_1 = y - t$, $\omega_2 = c\,(y - t)$, $\omega_3 = ct$ and $\omega_4 = 2y - t$.

**II.** $x, y \neq 0$. We distinguish two subcases:

**II-1.** $a, b, c \neq 0$. Consider the following possibilities:

**II-1-1.** $y/x$ is appreciable. Clearly, we have $n = x\,(ax + by) + cy^2$. Since $ax + by$ is of the same order as $x$ and $y$, we put $\omega_1 = x$, $\omega_2 = ax + by$, $\omega_3 = cy$ and $\omega_4 = y$. Then $n$ can be represented in the form ($A_2$).

**II-1-2.** $y/x$ is unlimited. Here we see that

$$\begin{aligned} n &= ax^2 + bxy + cy^2 = ax^2 + y\,(bx + cy) = a\,(x - y + y)^2 + y\,(bx + cy) \\ &= a\,(x - y)\,(x + y) + y\,(ay + bx + cy) = a\,(x - y)\,(x + y) + y\,(y\,(a + c) + bx)\,. \end{aligned}$$

We end this case if $a + c \neq 0$ because we can put $\omega_1 = a\,(x - y)$, $\omega_2 = x + y$, $\omega_3 = y$ and $\omega_4 = y\,(a + c) + bx$. Otherwise, $c = -a$ and so $n = ax^2 + bxy - ay^2$. Since $x = x - y + y$, we conclude that $n = (x - y)\,(ax + (a + b)\,y) + by^2$. Similarly, when $a + b \neq 0$, we put $\omega_1 = x - y$, $\omega_2 = ax + (a + b)\,y$, $\omega_3 = by$ and $\omega_4 = y$. Otherwise, $b = -a$ and then $n = ax^2 - axy - ay^2$. Here, we can easily see that $n = a\,(x + 2y)^2 - 5ya\,(x + y)$. To finish the proof for this case, we only need to put $\omega_1 = a\,(x + 2y)$, $\omega_2 = x + 2y$, $\omega_3 = -5ya$ and $\omega_4 = x + y$. In addition, the proof of our claim for the case that $x/y$ is unlimited is similar to our previous discussion.

**II-2.** At least one of the coefficients $a$, $b$ and $c$ is zero.

**II-2-1.** Only one coefficient among the numbers $a$, $b$ and $c$ is zero. We have the following cases:

• $b = 0$. Then $n = ax^2 + cy^2$. Here we can assume that $x$ and $y$ are positive with $y \geq x$. If $y/x$ is appreciable, then the proof in this case is obviously met by taking appropriate values for $\omega_i$ $(1 \leq i \leq 4)$. Otherwise, $y/x$ is unlimited from which we get $n = ax^2 + cy^2 = a\,(x - y + y)^2 + cy^2 = a\,(x - y)\,(x + y) + (a + c)\,y^2$. Hence,

$$n = \begin{cases} a\,(x - y)\,(x + y) + (a + c)\,y^2, \text{ if } a + c \neq 0 \\ a\,(x - y)^2 + 2ay\,(x - y)\,, \text{ otherwise.} \end{cases}$$

The proof in this case is met by taking appropriate values for $\omega_i$ $(1 \leq i \leq 4)$. The case $x > y$ is treated in the same way.

• $a = 0$. Then $n = bxy + cy^2$. Suppose that $|y| \geq |x|$. If $y/x$ is appreciable, then the proof in this case is obviously met by taking appropriate values for $\omega_i$ $(1 \leq i \leq 4)$. Otherwise, $y/x$ is unlimited and then $n = b\,(x - y)\,y + (c + b)\,y^2$. If $c + b \neq 0$, then the proof is finished for this case by taking appropriate values for $\omega_i$ $(1 \leq i \leq 4)$. Otherwise, $c + b = 0$ and then

$$n = b\,(x - y)\,y = b\,(x - y)\,(y - t + t) = b\,(x - y)\,(y - t) + b\,(x - y)\,t$$

where $t = [y/2]$. Also the proof is finished for this case by taking appropriate values for $\omega_i$ $(1 \leq i \leq 4)$. The case $|x| > |y|$ is treated in the same way.

• $c = 0$. Then $n = ax^2 + bxy$. This case is treated in the same way as the case $(a = 0)$.

**II-2-2.** Exactly two coefficients among $a$, $b$ and $c$ are zero. We distinguish the following possibilities:

• $a = b = 0$. Then $n = cy^2$. This case is treated in the same way as the case (**I**).

• $a = c = 0$. Then $n = bxy$. Suppose that $|y| \geq |x|$. If $y/x$ is appreciable, then

$$n = bx\,(y - t + t) = bx\,(y - t) + bxt,$$

where $t = [y/2]$. This complete the proof for this case by taking $\omega_1 = bx$, $\omega_2 = y - t$, $\omega_3 = bx$ and $\omega_4 = t$. If $y/x$ is unlimited, then $n = by\,(x - y + y) = by\,(x - y) + by^2$. This completes the proof by taking $\omega_1 = by$, $\omega_2 = x - y$, $\omega_3 = by$ and $\omega_4 = y$. The case $|x| > |y|$ is treated in the same way.

• $b = c = 0$. Then $n = ax^2$. This case is treated in the same way as the case $(a = b = 0)$ of the previous case.

This completes the proof of Theorem 9. □

By a similar proof we obtain the following result:

**Theorem 10.** *Let $n$ be an unlimited positive integer represented by a quadratic form $f(x, y) = ax^2 + bxy + cy^2$, where $a$, $b$ and $c$ are limited integers with* $\gcd(x, y) = 1$. *Then $n$ is represented as in* $(\mathrm{A}_2')$ *whenever $f$ corresponds to one of the following cases:*

*(1) $f(x, y) = ax^2$.*

*(2) $f(x, y) = ax^2 + cy^2$ with $a \neq -c$.*

*(3) $f(x, y) = ax^2 + bxy + cy^2$ such that $a, b, c \neq 0$ and $y/x$ is appreciable.*

*(4) $f(x, y) = ax^2 + bxy + cy^2$ such that $a, b \neq 0, c = -a$ and $y/x$ is not appreciable.*

*(5) $f(x, y) = ax^2 + bxy + cy^2$ such that $b = c = -a$.*

**Proof**. (1) $n = ax^2$. Then $a, x \neq 0$. Put $n = a\,(x - t + t)^2$, where $t \geq 3x$ is prime with $t \sim 3x$. Therefore, $n = a\left((x - t)^2 + t^2 + 2t\,(x - t)\right) = a\,(x - t)^2 + at\,(2x - t)$. Let $\omega_1 = a\,(x - t)$, $\omega_2 = x - t$, $\omega_3 = at$ and $\omega_4 = 2x - t$. Clearly, $\omega_i$ is unlimited for $1 \leq i \leq 4$ and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$. Moreover, we can prove that $\gcd(\omega_1 \cdot \omega_2\,,\, \omega_3 \cdot \omega_4)$ is limited. Indeed, first we see that $\gcd(t, 2x - t) = 1$ since $t$ is prime and $t \geq 3x$. Suppose further that $\gcd(\omega_1\omega_2, \omega_3\omega_4) = ad_1$, where $d_1 \geq 2$. Then $d_1 \mid (x - t)^2$ and $d_1 \mid t\,(2x - t)$. Hence, $d_1 \mid (x - t)^2 + t\,(2x - t) = x^2$. There are two possibilities:

- $d_1 \mid x$. Then $d_1 \mid t$ since $d_1 \mid t\,(2x - t)$, which is impossible since $\gcd(x, t) = 1$.
- $d_1 \nmid x$. We put $x^2 = q_1^{2\alpha_1} q_2^{2\alpha_2} \cdots q_r^{2\alpha_r}$, where $q_1, q_2, \ldots, q_r$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers, and let $d_1 = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ with $0 \leq a_i \leq 2\alpha_i$ for $1 \leq i \leq r$. We prove that every prime factor of $d_1$ is limited; otherwise, if $p$ is an unlimited prime number with $p \mid d_1$, then $p \mid t$ and so $p = t$. A contradiction. Now, let $q_{i_0}^{a_0}$ be an unlimited prime power such that $q_{i_0}^{a_{i_0}} \mid d_1$, i.e., $q_{i_0}$ is limited and $a_{i_0}$ is unlimited. Since $q_{i_0}^{a_{i_0}} \mid x^2$, we conclude that $q_{i_0}^{\omega} \mid x$, where $\omega = a_{i_0}/2$ if $a_{i_0}$ is even or $\omega = (a_{i_0} - 1)/2$; otherwise. Since $q_{i_0}^{\omega} \mid 2x - t$, we deduce that $q_{i_0}^{\omega} = t$. This is a contradiction since $t$ is prime. Therefore, all the prime powers $q_1^{a_1}, q_2^{a_2}, \ldots, q_r^{a_r}$ are limited and so $d_1$ is also limited.

(2) Here we can assume that $x$ and $y$ are positive and $a$, $c$ are both non-zero; otherwise, if $a$ or $c$ is zero, then we are in case (1). Suppose that $y > x$. If $y/x$ is appreciable, then the proof is easy. In the case when $y/x$ is unlimited, we see that

$$n = ax^2 + cy^2 = a\,(x - y + y)^2 + cy^2 = a\,(x - y)\,(x + y) + (a + c)\,y^2.$$

Let $\omega_1 = a\,(x - y)$, $\omega_2 = (x + y)$, $\omega_3 = (a + c)\,y$ and $\omega_4 = y$. Clearly, $\omega_i$ is unlimited for $1 \le i \le 4$ and $\omega_i \sim \omega_j$ for $1 \le i, j \le 4$. Moreover, $\gcd(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4)$ is limited. Indeed, if $d = \left(a\,(x - y)\,(x + y)\,,\ (a + c)\,y^2\right) \cong +\infty$, then $d \mid a\,(x - y)\,(x + y)$ and $d \mid (a + c)\,y^2$. As in case (1), let $p^a$ be an unlimited prime power such that $p^a$ divides both $d$ and $y$, from which it follows that $p^a \mid a\,(x - y)\,(x + y)$. This contradicts the fact that $x$ and $y$ are relatively prime, i.e., $d$ is limited.

(3) Assume that $n = ax^2 + bxy + cy^2$, where $a, b, c \neq 0$ and $y/x$ is appreciable. In this case, $n = x\,(ax + by) + cy^2$. Now, if $ax + by = 0$, then $n = cy^2$ and this case can be treated as in case (1); otherwise, if $(ax + by)\,/x$ is appreciable, then we put $\omega_1 = x$, $\omega_2 = ax + by$, $\omega_3 = cy$ and $\omega_4 = y$. Then we can easily prove that $\gcd(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4)$ is limited since $\gcd(x, y) = 1$. But, if $(ax + by)\,/x \cong 0$, then we can write $n$ as $n = ax^2 + y\,(bx + cy)$, where $(bx + cy)\,/y$ must be appreciable and we end the proof as before. It remains to prove that $(ax + by)\,/x$ and $(bx + cy)\,/y$ cannot be simultaneously infinitesimal. Indeed, suppose we have $ax + by = \phi_1 x = w_1$ and $bx + cy = \phi_2 y = w_2$, where $\phi_1$ and $\phi_2$ are two infinitesimal numbers, that is, we have the following system:

$$\begin{cases} a \cdot x + b \cdot y = w_1 \\ b \cdot x + c \cdot y = w_2. \end{cases}$$

The solution of this system is $y = \dfrac{b \cdot w_1 - a \cdot w_2}{b^2 - ac}$ and $x = \dfrac{b \cdot w_2 - c \cdot w_1}{b^2 - ac}$. But this is a contradiction because this means that $y = \phi y$ and $x = \widetilde{\phi} x$, where $\phi$ and $\widetilde{\phi}$ are also infinitesimal.

(4) Consider the case when $n = ax^2 + bxy + cy^2$, where $a, b \neq 0, c = -a$ and $y/x$ is unlimited. Then $n = ax^2 + bxy - ay^2$. Put $x = x - y + y$ we get

$$n = (x - y)\,(ax + (a + b)\,y) + by^2.$$

If $a + b \neq 0$, then the proof is completed for this case by choosing $\omega_1 = x - y$, $\omega_2 = ax + (a + b)\,y$, $\omega_3 = by$ and $\omega_4 = y$. Otherwise, $b = -a$, and so $n = ax^2 - axy - ay^2$, in which case we get $n = a\,(x + 2y)^2 - 5ya\,(x + y)$. This ends the proof for this case by setting $\omega_1 = a\,(x + 2y)$, $\omega_2 = x + 2y$, $\omega_3 = -5ya$ and $\omega_4 = x + y$. As before, we can prove that $\gcd(\omega_1 \cdot \omega_2\ , \omega_3 \cdot \omega_4)$ is limited. Using the same way as above we can consider the case when $n = ax^2 + bxy + cy^2$, where $a, b \neq 0, c = -a$ and $x/y$ is unlimited.

(5) Here we can follow the same argument as in the proof of (4).

The proof of Theorem 10 is finished. □

## 5.1. Examples

Applying the above theorems we find the following examples:

**1)** Let $p$ be an unlimited prime number with $p \equiv 1 (\mathrm{mod}\ 4)$. By Niven [17, Lemma 2.13, p. 54], there exist positive integers $s, t$ for which $p = s^2 + t^2$. Hence by Theorem 9, $p$ can be written as in $(\mathrm{A}'_2)$.

**2)** Let $p$ be an unlimited prime number such that $(p/13) = (p/17) = 1$. By [17, Proposition 11.3.3, p. 324], either $p = x^2 + xy - 55y^2$ or $p = -x^2 + xy + 55y^2$, but not both represent $p$. Hence by Theorem 9, $p$ can be written as in $(\mathrm{A}'_2)$.

**3)** Let $p$ be an unlimited prime number such that $(-2/p) = (p/13) = 1$. Then at least one of the following statements is true: (a) both $p$ and $2p$ can be written as in $(\mathrm{A}'_2)$. (b) both $3p$ and $5p$ can be written as in $(\mathrm{A}'_2)$. Indeed, by Lehman [15, Proposition 7.3.2, p. 216], one and only one of the following is true: (a) The equations $x^2 + 26y^2 = p$ and $2x^2 + 13y^2 = 2p$ both have solutions in integers. (b) The equations $x^2 + 26y^2 = 3p$ and $2x^2 + 13y^2 = 5p$ both have solutions in integers. Hence, Theorem 9 gives us the response. Here, we remark that if we can write $p$ and $2p$ as in $(\mathrm{A}'_2)$, then we can do the same for $3p$ and $5p$, while the converse is not true.

**4)** Let $p$ be an unlimited prime number which is not congruent to 13, 17, 19, or 23 modulo 24. Since $p$ is not divisible by 4 and 9, we conclude from Lehman [15, Proposition 7.2.3, p. 207] that $p$ is either properly represented by $x^2 + 6y^2$ or by $2x^2 + 3y^2$. Hence, by Theorem 10, $p$ can be written as in $(\mathrm{A}'_2)$.

**5)** Let $p$ be an unlimited prime number which is not divisible by any prime congruent to $3, 5, 6$ (mod 7). Then $p$ is represented as in $(\mathrm{A}'_2)$. Indeed, in this case, $p$ is not divisible by 49. Then, by [15, Corollary 2.5.4, p. 84], $p$ is properly represented $x^2 + 7y^2$. Applying Theorem 10, $p$ can be written as in $(\mathrm{A}'_2)$.

## 6. Some equivalent internal statements

All variables range over positive integers. First, let us consider $(F_3)$: Every unlimited $v$ can be written in the form $v = a \cdot x^2 + b \cdot y^2$, where $a, b$ are limited. The external statement $(F_3)$ is equivalent to the following internal statement $(S_3)$: *There is a finite set* $\{\langle a_1, b_1\rangle, \dots, \langle a_k, b_k\rangle\}$ *and a number* $s$ *such that for every* $n \geq s$ *there exist* $i \leq k$ *and* $x, y$ *such that* $n = a_i \cdot x^2 + b_i \cdot y^2$.

**Proposition 7.** $(F_3) \Leftrightarrow (S_3)$.

**Proof**. First, assume that $(S_3)$ holds. By transfer, the set $\{\langle a_1, b_1\rangle, \dots, \langle a_k, b_k\rangle\}$ and the number $s$ can be taken to be standard. If $v$ is unlimited, then $v > s$, so $a_i \cdot x^2 + b_i \cdot y^2$ for some standard $i$, $a_i$ and $b_i$. This proves $(F_3)$. Conversely, assume that $(S_3)$ holds. Then for every standard finite set $\{\langle a_1, b_1\rangle, \dots, \langle a_k, b_k\rangle\}$ and every standard number $s$ there exists $n$ such that for every $i \leq k$ we have $n \geq s \wedge \forall x, y$ $(n \neq a_i \cdot x^2 + b_i \cdot y^2$. By idealization[‡], there is $v$ such that for every standard $\langle a, b\rangle$

[‡]Idealization (see F. Diener [6, pp.9, 21]): $\forall^{\mathrm{stfin}} z \exists y \forall x \in z \ \ B(x, y, t) \Leftrightarrow \exists y \forall^{st} x\ B(x, y, t)$. The only nonlogical symbol of $B$ must be $\in$ (that is, $B$ must be internal). The parameter $t$ may take

and every standard $s$ we have $v \geq s \wedge \forall x, y\ (v \neq a \cdot x^2 + b \cdot y^2)$. So $v$ is unlimited and it cannot be written in the desired form. $\square$

Next, let us consider $(F_3^*)$, which is obtained from $(F_3)$ by adding the requirement that $x/y$ be appreciable. Note that $(F_3^*)$ is equivalent to the following internal statement $(S_3^*)$: *There is a finite set* $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ *and numbers m, s such that for every* $n \geq s$ *there exist* $i \leq k$ *and* $x, y \geq \sqrt{n}/m$ *such that* $n = a_i \cdot x^2 + b_i \cdot y^2$.

**Proposition 8.** $(F_3^*) \Leftrightarrow (S_3^*)$.

**Proof**. Assume $(S_3^*)$ holds. By transfer, the set $\{\langle a_1, b_1\rangle, \ldots, \langle a_k, b_k\rangle\}$ and the numbers $m, s$ can be taken to be standard. If $v$ is unlimited, then $v > s$, so $v = a_i \cdot x^2 + b_i \cdot y^2$ for some standard $a, b$ and $x, y \geq \frac{\sqrt{v}}{m}$. Of course, also $x, y \leq \sqrt{v}$, hence $1/m \leq x/y \leq m$.

Assume the negation of $(S_3^*)$ holds. As in the proof of "$(F_3)$ implies $(S_3)$", we obtain $v$ such that for every standard $\langle a, b\rangle$ and every standard $m, s$ we have $v \geq s \wedge \forall x, y \geq \frac{\sqrt{v}}{m}\ (v \neq a \cdot x^2 + b \cdot y^2)$.

Suppose that for some standard $a, b$ we have $v = a \cdot x^2 + b \cdot y^2$, where $x/y$ is appreciable. Then $1/\ell \leq x/y \leq 1/\ell$ holds for some standard $\ell$. It follows that $y \leq x \cdot \ell$ and $x \leq y \cdot \ell$, hence $v \leq (a + b \cdot \ell^2) \cdot x^2$ and $v \leq (a \cdot \ell^2 + b) \cdot y^2$. Fix a standard $m \geq \max(\sqrt{a + b \cdot \ell^2}, \sqrt{a \cdot \ell^2 + b})$. Then $x, y \geq \frac{\sqrt{v}}{m}$, a contradiction. $\square$

If $(F_3^*)$ is true, then $(F_2)$: *Every unlimited* $v$ *can be written in the form* $v = x_1 \cdot x_2 + x_3 \cdot x_4$, *where all* $x_i$ *are unlimited and* $x_i/x_j$ *is always appreciable is true.* Statement $(F_2)$ is equivalent to the internal statement

$(S_2)$: *There are numbers* $m, s$ *such that for every* $n \geq s$ *there exist* $x_1, x_2, x_3, x_4$ *such that* $n = x_1 \cdot x_2 + x_3 \cdot x_4$ *and* $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ *holds for* $1 \leq i \leq 4$.

**Proposition 9.** $(F_2) \Leftrightarrow (S_2)$.

**Proof**. Similar to the preceding proof. On the one hand, note that the condition $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ implies that $1/m^2 \leq x_i/x_j \leq m^2$, so all the ratios $x_i/x_j$ are appreciable. On the other hand, if $1/k \leq x_i/x_j \leq k$ holds for all $i, j$ (where $k$ is standard), we have $(1/k)x_j \leq x_i \leq k \cdot x_j$ for all $i, j$. From this one gets $(1/k + 1/k^2) \cdot x_i^2 \leq x_1 \cdot x_2 + x_3 \cdot x_4 = v \leq (k + k^2) \cdot x_i^2$. Let $m \geq \max\left(\sqrt{k + k^2}, k/\sqrt{1 + k}\right)$ be standard. The above inequality gives $(1/m^2) \cdot x_i^2 \leq v \leq m^2 \cdot x_i^2$ and $\sqrt{v}/m \leq x_i \leq m \cdot \sqrt{v}$ for all $i$. $\square$

In addition, Theorem 1 is equivalent to the following internal statement:

**Theorem 11.** *There exists* $(i, j) \in \mathbb{N}^2$ *such that every* $\omega \geq i$ *can be written as* $\omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, *where* $\omega_l$ *is a positive integer with* $\omega_l/\sqrt{\omega} \in [1/j, j]$ *for* $1 \leq l \leq 4$.

**Proof**. We write Theorem 1 as follows:

$$\begin{aligned} &\forall\ \omega\ \left[\ \forall^{st}\ i\ \ (\ \omega > i) \Rightarrow \exists\ (\omega_1, \omega_2, \omega_3, \omega_4) \right. \\ &\left. \exists^{st}\ j\ \forall\ l \in \{1, \ldots, 4\}\ \left(\omega_l/\sqrt{\omega} \in [1/j, j]\right)\ \&\ \omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4\ \right], \end{aligned}$$

---

any value.

where all variables range over positive integers. This is equivalent to

$$\begin{aligned}&\forall\ \omega\ \exists^{st}\ j\ \exists^{st}\ i\quad [\ (\ \omega > i)\\ \Rightarrow\ &\exists\ (\omega_1, \omega_2, \omega_3, \omega_4)\ \ \forall\ l \in \{1, \dots, 4\}\,, \big(\omega_l/\sqrt{\omega} \in [1/j, j]\big)\ \ \&\ \omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4\ ]\,.\end{aligned}$$

By idealization, we obtain

$$\begin{aligned}&\exists^{st}\ i\ \exists^{st}\ j\ \forall\ \omega\ \ [\ (\ \omega > i)\\ \Rightarrow\ &\exists\ (\omega_1, \omega_2, \omega_3, \omega_4)\ \ \forall\ l \in \{1, \dots, 4\}\,, \big(\ \omega_l/\sqrt{\omega} \in [1/j, j]\big)\ \ \&\ \omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4\ ]\,.\end{aligned}$$

Now, by transfer, the last formula is equivalent to

$$\begin{aligned}&\exists\ i\ \exists\ j\ \forall\ \omega\ \ [\ (\ \omega > i)\\ \Rightarrow\ &\exists\ (\omega_1, \omega_2, \omega_3, \omega_4)\ \ \forall\ l \in \{1, \dots, 4\}\,, \big(\omega_l/\sqrt{\omega} \in [1/j, j]\big)\ \ \&\ \omega = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4\ ]\,.\end{aligned}$$

This completes the proof. □

Finally, we obtain a generalization of the above theorem as follows:

**Corollary 7.** *Let $k \geq 2$ be a fixed standard integer. Then there exists $(i, j) \in \mathbb{N}^2$ such that every $\omega \geq i$ can be written as $\omega = \omega_1 \cdot \omega_2 + \cdots + \omega_{2k-1} \cdot \omega_{2k}$, where $\omega_l$ is a positive integer with $\omega_l/\sqrt{\omega} \in [1/j, j]$ for $l = 1, 2, \dots, 2k$.*

**Proof**. Corollary 1 is equivalent to the following internal statement:

$$\begin{aligned}\forall\ \omega\ \ \big[\ \forall^{st}\ i\quad (\ \omega > i)\ \Rightarrow \exists\,\{\omega_1, \dots, \omega_{2k}\}\,\exists^{st} j\ \forall l \in \{1, \dots, 2k\}\,,&\\ \big(\omega_l/\sqrt{\omega} \in [1/j, j]\big)\ \&\ \omega = \omega_1\omega_2 + \cdots + \omega_{2k-1}\omega_{2k}\ ]\,,&\end{aligned}$$

where $k$ is a standard positive integer. The unique free variable is $k$ and it is standard, so we can apply the same method as before to show that the last formula is equivalent to

$$\begin{aligned}\exists\ i\ \exists\ j\ \forall\ \omega\ \ [\ (\ \omega > i) \Rightarrow \exists\ (\omega_1, \dots, \omega_{2k})\ \ \forall\ l \in \{1, \dots, 2k\}\,,&\\ \big(\omega_l/\sqrt{\omega} \in [1/j, j]\big)\ \&\ \omega = \omega_1\omega_2 + \cdots + \omega_{2k-1}\omega_{2k}\ ]\,,&\end{aligned}$$

as required. □

## 7. Open questions

For further research, we propose the following questions on the representation of unlimited integers as in ($A_2$).

1. We ask if every unlimited positive integer $n$ is of the form $n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \in \mathbb{N}$ is unlimited and $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$ with $\gcd(\omega_i, \omega_j) = 1$ for $i \neq j$.

2. Let $\omega$ be unlimited. Consider the numbers $n = a_1 a_2 \cdots a_\omega$, where $a_i$ is standard for every $i$ standard and $a_{i+1}/a_i \cong \infty$ for $i \cong \infty$. For example, $n$ is the product of Fermat numbers, i.e., $n = f_0 f_1 \cdots f_\omega$ with $\omega \cong \infty$, where $f_n = 2^{2^n} + 1$ ($n \geq 0$). As in the proof of Theorem 4, we ask if we can determine effective values $\omega_1, \omega_2, \omega_3, \omega_4$ such that $n = \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_4$, where $\omega_i \sim \omega_j$ for $1 \leq i, j \leq 4$.

3. Does result (5) in Corollary 2 in Section 2 hold for $k = 2$? In other words, we ask whether every unlimited positive integer $n$ is of the form $n = \omega_1 \cdot \omega_2 \cdot \omega_3 + \omega_4 \cdot \omega_5 \cdot \omega_6$, where $\omega_i \in \mathbb{N}$ is unlimited with $\omega_i \sim \omega_j$ for $1 \leq i,\ j \leq 6$.

## Acknowledgement

We thank the referee for very helpful and detailed comments which improved the quality of this paper. We would like to thank professor Karel Hrbáček for valuable talks with him. In particular, some proofs in sections 2 and 6 came from him.

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