# The New ADE-TLM Algorithm for Modeling Debye Medium

Original Scientific Paper

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***Abstract*** *– In this paper, we present a novel approach to simulating the interaction between electromagnetic waves and a Debye medium utilizing a Transmission Line Matrix (TLM) algorithm with the symmetrical condensed node (SCN-TLM) technique. The proposed method utilizes the polarization current within the media and incorporates the auxiliary differential equation (ADE) technique to handle scattering following the conventional discretization process. The averaged approximation is employed to utilize the polarization current density J and the electric voltage. By reducing the number of operations required per iteration, the New ADE-TLM method has successfully decreased the computational time compared to time convolution techniques. Despite this reduction in computational time, the New ADE-TLM method maintains a numerical accuracy that is comparable to that of time convolution techniques. The efficiency and precision of this approach are confirmed by the agreement between the results obtained and those predicted by the analytic model.*

***Keywords****: Auxiliary Differential Equation (ADE), Symmetrical Condensed Node (SCN), Transmission line matrix (TLM) approach, Debye medium*

## 1. INTRODUCTION

The Transmission Line Matrix (TLM) approach was designed as a simple analogy between the principles of electromagnetic wave propagation and those of electrical currents and voltages in a network of continuous transmission lines. This network, representing the discretization of the physical environment to be studied, is formed by interconnected transmission lines, which constitute the basic elements called nodes.

The TLM approach is immediately derived from Maxwell's equations using the centered derivative approximation [1], shares similarities with the finite difference model. This suggests that the TLM approach offers a similar level of accuracy and efficiency in representing and solving electromagnetic problems, making it a viable choice for modeling such systems.

For example, it has been shown that the equations of the extended TLM node model can be transformed into the finite difference equations used by Brodwin and Taflove [2, 3].

The TLM technique can be considered either as a mathematical representation obtained from Maxwell's equations using the finite difference approach or a physical representation based on the Huygens principle for modeling transmission line networks.

In recent years, several approaches based on the FDTD model [4, 5] have been presented to analyze problems related to the propagation of electromagnetic wave fields in linear media [6, 7] and to predict the gain, absorption, and scattering in a nonlinear medium [8].

The technique, which depends on the auxiliary differential equation (ADE-FDTD), has been effectively used in modeling nonlinear media [9, 10].

The TLM approach effectively simulated the performance of electromagnetic waves in nonlinear and linear media [11, 12], allowing the resolution of three-dimensional problems as well as the treatment of inhomogeneous, anisotropic and dispersive waves [13].

In recent years, we report a many TLM-based approaches for the analysis of a dispersive medium. We find that : the use of equations characterizing the conductive medium in terms of equivalent node sources [14], the use of Constant Recursive Convolution (CRC) [15], which was later improved to the Current Density Recursive Convolution (CDRC-TLM) technique [16], and the Piecewise Linear Current Density Recursive Convolution (PLCDRC-TLM) technique [17] ,which are applied to magnetized plasma, the application of the Z-transform approach [18] to characterize the electric properties of the dispersive media, and the other technique includes the JE-TLM with voltage sources [19] which is applied to model isotropic plasma.

The auxiliary differential equation (ADE-TLM) approach is employed to simulate the chiral media [20], this is achieved by utilizing the ADE approximation, and employed to model Cole-Cole media [21].

In this paper, we propose the new ADE-TLM algorithm in terms of polarization current density and the constitutive relations between the voltage V and the polarization current density J, which are presented in [22], to simulate the propagation of EM wave in the Debye medium. Therefore, we will discuss the formulation of this technique in detail.

The New ADE-TLM algorithm is based on the SCN-TLM by using 12 principal ports to model the free space and modeling the properties of the Debye medium by using 3 additional ports as voltage sources.

In contrast, the derivation of our new ADE-TLM algorithm, which utilizes the centered derivative approximation, is simpler in comparison to the derivation of the ADE-TLM algorithm discussed in [23].

When comparing the ADE schemes to the RC and PLRC schemes [16, 17], it is found that the ADE schemes require a smaller number of variables to be stored and involve fewer arithmetic operations. As a result, the ADE schemes significantly reduce the computational complexity involved in the calculations, making them a more efficient option. Also, the derivation of our proposed method is simpler compared to the derivation of the JE-TLM method [19]. However, it typically leads to enhanced accuracy.

This new ADE-TLM method is examined and validated by calculations of the reflection coefficient of a Gaussian pulse incident at the air-water interface. An excellent agreement has been found between the new proposed ADE-TLM calculation and the analytical solutions for the reflection coefficient.

The present paper is organized in the following sections: In Section 2, we present the theoretical formulations and equations related to the Debye model. In Section 3, we introduce the derivation of the New ADE-TLM algorithm, which is used for calculating sources. The efficiency and validity of this algorithm are demonstrated in the frequency domain, and the results are detailed in Section 4. In conclusion, section 5 presents our final remarks.

## 2. FORMULATIONS AND EQUATIONS

In the Debye medium, the polarization current density is described by Maxwell equations as follows:

$$\nabla \times H = \varepsilon_0 \varepsilon_\infty \frac{\partial E}{\partial t} + J \tag{1}$$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \tag{2}$$

Where $H$ represents the magnetic field strength, $E$ defines the electric field strength, $\varepsilon_0$ is the permittivity of vacuum, $\varepsilon_\infty$ is the relative permittivity at infinite frequency, $\mu_0$ is the magnetic permeability of vacuum, and $J = \sum_{p=1}^{p} J_p$ defines the polarization current density with multiple poles, where $J$ represents the polarization current density and $p$ is the number of poles.

In the frequency domain, we express the relative permittivity of the Debye media as [24]:

$$\varepsilon(\omega) = \varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty) \sum_{p=1}^{p} \frac{G_p}{1 + j\omega\tau_p} \tag{3}$$

$\tau_p$ is the relaxation time for the $p_{th}$ pole, the amplitude for each pole is: $\Delta\varepsilon_p = \varepsilon_s - \varepsilon_\infty$. Where $\varepsilon_s$ is the static permittivity. With [24]: $\sum_{p=1}^{p} G_p = 1$.

In the frequency domain, the polarization current density caused by one pole is given as [25, 26]:

$$\tilde{J}_p = (\varepsilon_s - \varepsilon_\infty) \times G_p \times \varepsilon_0 \left( \frac{j\omega}{1 + j\omega\tau_p} \right) \times \tilde{E} \tag{4}$$

An efficient means to obtain $\tilde{J}_p$ from Eq. (4) is to first multiply both sides of this equation by $(1+j\omega\tau_p)$. This gives:

$$\tilde{J}_p + jw\tau_p \tilde{J}_p = (\varepsilon_s - \varepsilon_\infty) \times \varepsilon_0 \times G_p jw\tilde{E} \tag{5}$$

We obtain by applying the inverse Fourier transform to each term in equation (5):

$$J_p + \tau_p \frac{\partial J_p}{\partial t} = (\varepsilon_s - \varepsilon_\infty) \times \varepsilon_0 \times G_p \frac{\partial E}{\partial t} \tag{6}$$

## 3. ADE-TLM ALGORITHM

We obtain by using the finite difference time in Eq. (6) at time step $n+1$:

$$\left( \frac{J_p^{n+1} + J_p^n}{2} \right) + \tau_p \left( \frac{J_p^{n+1} - J_p^n}{\Delta t} \right) = (\varepsilon_s - \varepsilon_\infty) \times \varepsilon_0 \times G_p \left( \frac{E^{n+1} - E^n}{\Delta t} \right) \tag{7}$$

The updated equation is:

$$J_p^{n+1} = a_d J_p^n + b_d \left( \frac{E^{n+1} - E^n}{\Delta t} \right) \tag{8}$$

With the coefficients:

$$a_d = \frac{2\tau_p - \Delta t}{2\tau_p + \Delta t}$$
$$b_d = \frac{2(\varepsilon_s - \varepsilon_\infty) \times \varepsilon_0 \times G_p \times \Delta t}{2\tau_p + \Delta t} \tag{9}$$

The time discretization centered at step $n + 1/2$ of Eq. (1) gives:

$$E^{n+1} = E^n + \frac{\Delta t}{\varepsilon_0 \varepsilon_\infty} \left( \nabla \wedge H^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \right) \tag{10}$$

From Eq. (8), we obtain the polarization current centered at step $n + 1/2$ by:

$$J_p^{n+\frac{1}{2}} = \frac{1}{2}\left( J^{n+1} + J^n \right) \tag{11}$$

We convert the electric field to a voltage utilizing the following equation:

$$E^n = \frac{V^n}{\Delta l} \tag{12}$$

Where $\Delta l$ represents the space step as well as $V$ defines the electric voltage.

By substituting Eq. (12) in Eq. (8) and Eq. (10) respectively, we find:

$$J_p^{n+1} = a_d J_p^n + b_d \left( \frac{V^{n+1} - V^n}{\Delta t \Delta l} \right) \tag{13}$$

$$V^{n+1} = V^n + \frac{\Delta t \Delta l}{\varepsilon_0 \varepsilon_\infty} \left( \nabla \wedge H^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \right) \tag{14}$$

We first update Eq. (13) to improve the efficiency of ADE-TLM, and we find by utilizing the average approximation defined in [27]:

$$V^n = \frac{V^{n+1} + V^{n-1}}{2} \tag{15}$$

We obtain by substituting Eq. (15) into Eq. (13):

$$J_p^{n+1} = a_d J_p^n + b_d \left( \frac{V^n - V^{n-1}}{\Delta t \Delta l} \right) \tag{16}$$

We find by utilizing the SCN-TLM method [28] to Eq. (2):

$$\begin{pmatrix} \nabla \wedge H_x{}^{n+\frac{1}{2}} \\ \nabla \wedge H_y{}^{n+\frac{1}{2}} \\ \nabla \wedge H_z{}^{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_0}{2\Delta t \Delta l}\left[ \left( V_1^i + V_2^i + V_9^i + V_{12}^i \right)^{n+1} - \left( V_1^r + V_2^r + V_9^r + V_{12}^r \right)^n \right] \\ \frac{\varepsilon_0}{2\Delta t \Delta l}\left[ \left( V_3^i + V_4^i + V_8^i + V_{11}^i \right)^{n+1} - \left( V_3^r + V_4^r + V_8^r + V_{11}^r \right)^n \right] \\ \frac{\varepsilon_0}{2\Delta t \Delta l}\left[ \left( V_5^i + V_6^i + V_7^i + V_{10}^i \right)^{n+1} - \left( V_5^r + V_6^r + V_7^r + V_{10}^r \right)^n \right] \end{pmatrix} \tag{17}$$

Applying the charge conservation's laws:

$$\begin{pmatrix} V_1^r + V_2^r + V_9^r + V_{12}^r \\ V_3^r + V_4^r + V_8^r + V_{11}^r \\ V_5^r + V_6^r + V_7^r + V_{10}^r \end{pmatrix}^n = \begin{pmatrix} V_1^i + V_2^i + V_9^i + V_{12}^i \\ V_3^i + V_4^i + V_8^i + V_{11}^i \\ V_5^i + V_6^i + V_7^i + V_{10}^i \end{pmatrix}^n + \begin{pmatrix} V_{sx} \\ V_{sy} \\ V_{sz} \end{pmatrix}^n \tag{18}$$

Replacing Eq. (17) in Eq. (14), utilizing Eq. (18), and employing the Symmetrical Condensed Node (SCN), the total voltage is written as follows:

$$\begin{pmatrix} V_x^{n+1} \\ V_y^{n+1} \\ V_z^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{2}{4+Y_{ox}} \\ \frac{2}{4+Y_{oy}} \\ \frac{2}{4+Y_{oz}} \end{pmatrix} \begin{pmatrix} \left[ V_1^i + V_2^i + V_9^i + V_{12}^i + \frac{1}{2}V_{sx} \right]^{n+1} \\ \left[ V_3^i + V_4^i + V_8^i + V_{11}^i + \frac{1}{2}V_{sy} \right]^{n+1} \\ \left[ V_5^i + V_6^i + V_7^i + V_{10}^i + \frac{1}{2}V_{sz} \right]^{n+1} \end{pmatrix} \tag{19}$$

$Y_{ox}$, $Y_{oy}$, and $Y_{oz}$ indicate the normalized admittances.

The normalized admittances $Y_{ox}$, $Y_{oy}$, and $Y_{oz}$ can be written after comparing between Eq. (14) and Eq. (19) as follows:

$$Y_{ou} = 4\left( \varepsilon_\infty - 1 \right) \tag{20}$$

And the voltage sources $V_{ox}$, $V_{oy}$, and $V_{oz}$ are :

$$V_{su}^{n+1} = -V_{su} - 4\left[ \frac{\Delta t \Delta l}{2\varepsilon_0} \sum_{p=1}^{p} \left( J_{pu}^{n+1} + J_{pu}^n \right) \right] \tag{21}$$

With: $u \in \{x, y, z\}$.

The ADE-TLM algorithm procedure in the Debye medium is as follows:

1. We start by conserving values from $V^n$, $V^{n-1}$ and $J_p^n$, and $J_p^{n-1}$.
2. We update $J_p^{n-1}$ made by Eq. (16), and we introduce the obtained values of $J_p^{n-1}$ in the voltage Sources employing Eq. (21).
3. According to Eqs. (20) and (21), we update the $V^{n+1}$ characterized in Eq. (19) using normalized admittance and voltage sources.
4. The reflected pulses are simulated by using the conserved values of $V^{n+1}$, employing the scattering matrix.
5. At the end, the connection matrix is used to simulate the propagation to neighboring nodes.

The flow chart of the New ADE-TLM algorithm for Debye dispersive media is proposed in Fig. 1.

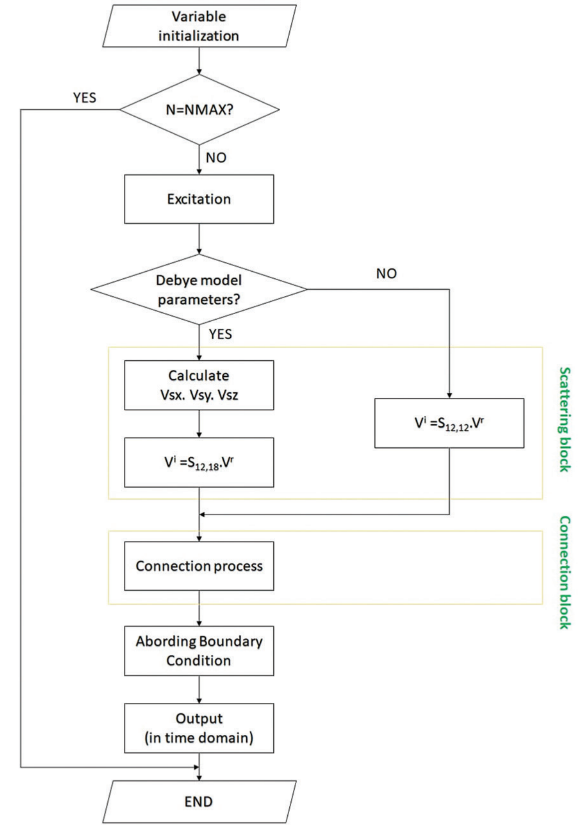


**Fig. 1.** Flowchart of New ADE-TLM process for Debye Dispersive Media

## 4. RESULTS AND DISCUSSION

To evaluate the efficiency and validity of the newly presented ADE-TLM algorithm, which includes voltage sources for the Debye medium. A simulation is performed to study the interaction between an electromagnetic wave and a Debye slab. We simulate the reflection coefficient of an interface between air and water (Debye medium) employing a one-dimensional, the results obtained are compared to the analytical solutions to validate and make comparisons.

A Gaussian plane wave is incident from air to a Debye media. First the TLM network is partitioned in (1,1,1000)$\Delta l$ with the space step: $\Delta l$=37.5µm and the time step: $\Delta t$=0.0625 ps.

We partitioned the one-dimensional computational space into 1000 cells, with 500 cells represented to simulating air and the remaining 500 cells representing water.Similar to [29], the following parameters describe the simulated Debye media in the structure:

$$\varepsilon_s = 81,\ \varepsilon_\infty = 1.8 \text{ and } \tau_0 = 9.4\times10^{-12}\,s\,.$$

The wave source was defined in the form of :

$$E(t)_s = 1000e^{-\frac{(t-t_0)^2}{T^2}} \text{ with } T = 152\Delta t$$

and $t_0$=400$\Delta t$. The simulations are programmed to run 5000 iterations.

After being reflected from the air-water interface (cell 499), the Gaussian pulse propagates across it.

Figs. 2-6 show the total electric field along the x-axis after different time steps: 750, 1300, 1500, 1800, and 3000. These figures clearly indicate the presence of dispersion and attenuation effects.

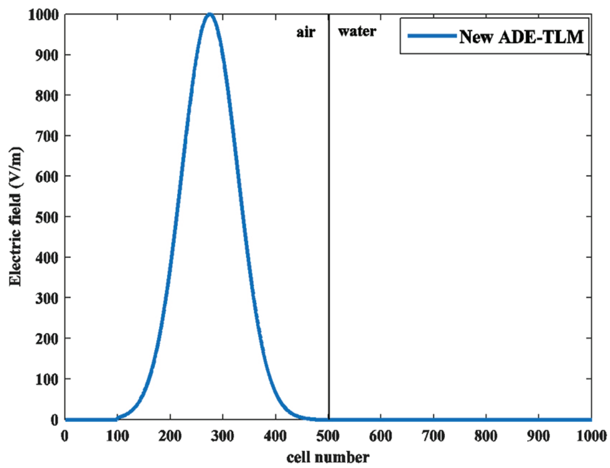


**Fig. 2.** The incidence Gaussian signal after 750 time steps

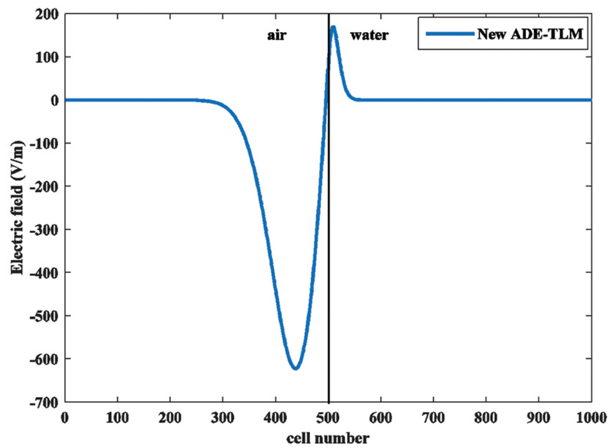


**Fig. 3.** The total electric field versus position after 1300 time steps

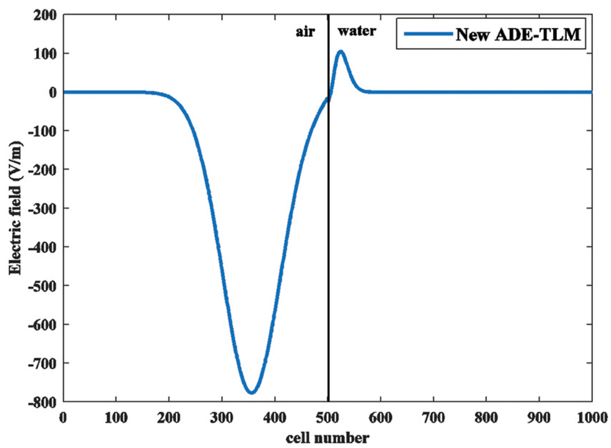


**Fig. 4.** The total electric field versus position after 1500 time steps

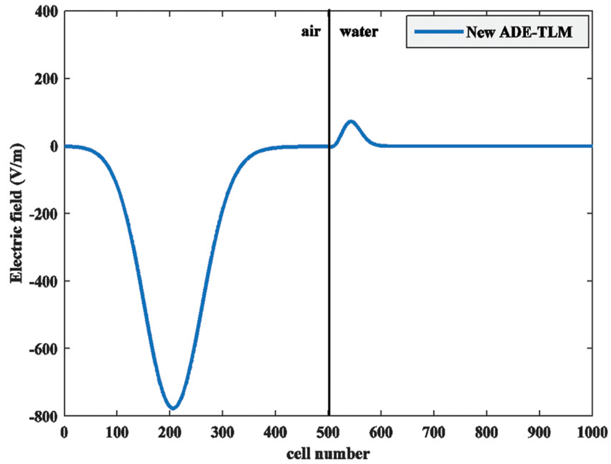


**Fig. 5.** The total electric field versus position after 1800 time steps

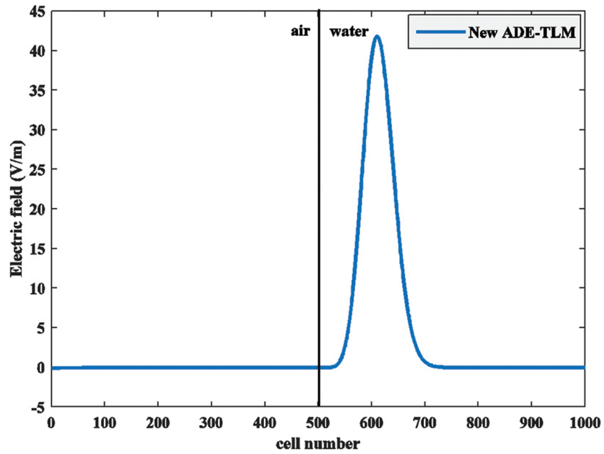


**Fig. 6.** The total electric field versus position after 3000 time steps

For the ADE-TLM approach, we simulate the air-water interface reflection coefficient, which is calculated by subtracting the Fourier transform of the incident field from the Fourier transform of the reflected field.

We calculate the analytical solution for the reflection coefficient, utilizing the next formulation [30]:

$$|\Gamma(\omega)| = \left| \frac{\sqrt{\varepsilon_0} - \sqrt{\varepsilon * (\omega)}}{\sqrt{\varepsilon_0} + \sqrt{\varepsilon * (\omega)}} \right| \tag{22}$$

Where $\varepsilon^*(\omega)$ is the complex permittivity.

Fig. 7. shows the results of the reflection coefficient simulation at the air-water interface.

An efficient agreement was noted between the simulation results obtained employing the new ADE-TLM method, the ADE-TLM method described in [23] and the analytical results for the reflection coefficient at different frequencies as defined by Eq. (22).

This agreement is excellent and is consistent with the findings presented in [29].

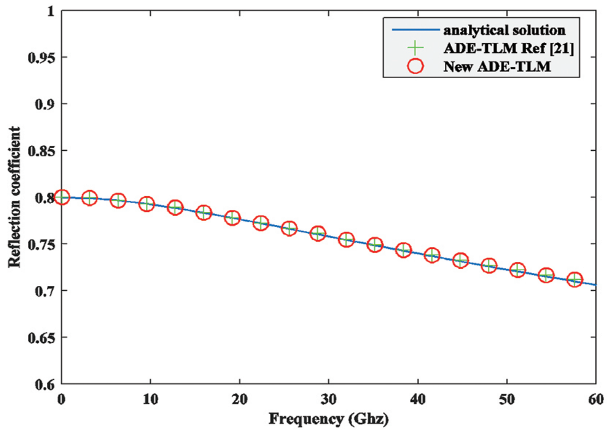


**Fig. 7.** Comparison of the new ADE-TLM calculated with the analytical solution (eq.22) for the magnitude of the reflection and the ADE-TLM method described in [23]

## 5. CONCLUSION

In this work, the New ADE-TLM algorithm was effectively used to analyze various polarization formulations and Maxwell's equations, allowing the simulation of reflected optical pulses propagation in a Debye media. This method utilizes the connection between centered derivatives approximations, electrical voltage, and polarization current density *J*.

In our approach, we included voltage sources that model linear characteristics and utilized the principle of variable admittance. In general, the proposed model offers simplicity as an advantage, as it avoids the need to handle convolution products, which are required in recursive convolution methods. This simplicity results in notable performance improvements when compared to other methods like RC, PLRC, and JE.

The results obtained using our new ADE-TLM approach are in excellent agreement with both the analytical values of the reflection coefficient and the ADE-TLM method described in [23], demonstrating the efficiency and validity of the suggested method.

Future work will integrate the new ADE-TLM algorithm into Kerr and Raman nonlinear dispersive media.

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