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Circles Related to a Complete Quadrangle

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ABSTRACT

This paper presents an overview of some properties of a complete quadrangle $ABCD$ in the Euclidean plane. We study the circles with diameters AB, AC, AD, BC, BD, and CD, as well as the pedal triangles and the pedal circles of the points A, B, C, D with respect to the triangles BCD, ACD, ABD and ABC, respectively. The presented results are known in literature, but here we prove them using a single method.

Key words: complete quadrangle, pedal triangles, pedal circles

MSC2020: 51N20

Kružnice pridružene potpunom četverovrhu

SAŽETAK

U radu dajemo pregled nekih svojstava potpunog četverovrha $ABCD$ u euklidskoj ravnini. Proučavamo kružnice s promjerima AB, AC, AD, BC, BD, CD, kao i nožišne trokute i nožišne kružnice točaka A, B, C, D s obzirom na trokute BCD, ACD, ABD, ABC redom navedene. Svi prikazani rezultati su poznati iz literature, ali ih ovdje dokazujemo koristeći istu metodu.

Ključne riječi: potpuni četverovrh, nožišni trokuti, nožišne kružnice

1 Introduction

Studying the geometry of the complete quadrangle in the Euclidean plane, we came across a large number of papers in which the properties of the quadrangle are proven in different ways. Our aim was to prove these claims using one method and, if possible, to prove some original claim. This paper is the third in a series of such works. In [12] we introduced the choice of the suitable coordinate system that enables us to prove all the properties in the same way, while in [13] we focused on the center, anticenter and a diagonal triangle of the quadrangle, as well as on the isogonality with respect to the four triangles formed by the sides of the quadrangle. In this paper we give an overview of some properties of the quadrangle regarding the circles related to it. Let us start by recalling some basic definitions and statements proved in [12] and [13].

The complete quadrangle $ABCD$ is formed by four points A,B,C,D and six lines AB, AC, AD, BC, BD, CD. There we distinguish the opposite sides, ones that have no common vertex. We use rectangular coordinates working with four parameters $a,b,c,d \neq 0$. For such a quadrangle we have proved: each quadrangle with no perpendicular opposite sides has a circumscribed rectangular hyperbola.
Choosing suitable coordinate system we get for the circumscribed hyperbola $\mathcal{H}$

$$xy = 1. \tag{1}$$

The center of this hyperbola is the point O and we will call it the center of the quadrangle $ABCD$. Asymptotes of $\mathcal{H}$ are the axes of the quadrangle $ABCD$.
Vertices of the quadrangle $ABCD$ are

$$A = \left(a, \frac{1}{a}\right), B = \left(b, \frac{1}{b}\right), C = \left(c, \frac{1}{c}\right), D = \left(d, \frac{1}{d}\right), \tag{2}$$

and the sides are

$$\begin{aligned} &AB \ldots x + aby = a + b, \quad AC \ldots x + acy = a + c, \\ &AD \ldots x + ady = a + d, \quad BC \ldots x + bcy = b + c, \\ &BD \ldots x + bdy = b + d, \quad CD \ldots x + cdy = c + d. \end{aligned} \tag{3}$$

Very often we will use elementary symmetric function in four variables a,b,c,d:

$$\begin{aligned} &s = a + b + c + d, \quad q = ab + ac + ad + bc + bd + cd, \\ &r = abc + abd + acd + bcd, \quad p = abcd. \end{aligned} \tag{4}$$

The Euler's circles of the triangles BCD, ACD, ABD, and ABC are given in the next equation on the example of the circle $\mathcal{N}_d$ of the triangle ABC

$$\mathcal{N}_d \quad \ldots \quad 2abc\left(x^2+y^2\right)+[1-abc\,(a+b+c)]\,x - \left(a^2b^2c^2-ab-ac-bc\right)y=0 \tag{5}$$

with the center

$$N_d=\left(\frac{1}{4}\left(a+b+c-\frac{1}{abc}\right),\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-abc\right)\right). \tag{6}$$

By H_a,H_b,H_c,H_d we denote the orthocenters of the triangles BCD, ACD, ABD, and ABC, respectively. Their forms are

$$H_a=\left(-\frac{1}{bcd},-bcd\right),\qquad H_b=\left(-\frac{1}{acd},-acd\right),$$
$$H_c=\left(-\frac{1}{abd},-abd\right),\qquad H_d=\left(-\frac{1}{abc},-abc\right). \tag{7}$$

The diagonal triangle UVW of the quadrangle $ABCD$ is given by the vertices

$$U=AB\cap CD=\left(\frac{ab(c+d)-cd(a+b)}{ab-cd},\frac{a+b-c-d}{ab-cd}\right),$$
$$V=AC\cap BD=\left(\frac{ac(b+d)-bd(a+c)}{ac-bd},\frac{a+c-b-d}{ac-bd}\right), \tag{8}$$
$$W=AD\cap BC=\left(\frac{ad(b+c)-bc(a+d)}{ad-bc},\frac{a+d-b-c}{ad-bc}\right),$$

and the sides are

$$\mathcal{U}=VW\ldots \tag{9}$$
$$(a+b-c-d)x+[ab(c+d)-cd(a+b)]y=2(ab-cd),$$
$$\mathcal{V}=UW\ldots$$
$$(a+c-b-d)x+[ac(b+d)-bd(a+c)]y=2(ac-bd),$$
$$\mathcal{W}=UV\ldots$$
$$(a+d-b-c)x+[ad(b+c)-bc(a+d)]y=2(ad-bc).$$

By A',B',C',D' we consider the points isogonal to the points A,B,C,D with respect to the triangles BCD, ACD, ABD, ABC, respectively. E. g.

$$D'=\left(\frac{2d-s}{p-1},\frac{r-2abc}{p-1}\right). \tag{10}$$

And, the following relations are also valid

$$AB\cdot CD=\left|\frac{(a-b)(c-d)}{p}\right|\sqrt{\lambda\lambda'},$$
$$AC\cdot BD=\left|\frac{(a-c)(b-d)}{p}\right|\sqrt{\mu\mu'}, \tag{11}$$
$$AD\cdot BC=\left|\frac{(a-d)(b-c)}{p}\right|\sqrt{\nu\nu'}.$$

where the next notations are used

$$\lambda=a^2b^2+1,\quad \mu=a^2c^2+1,\quad \nu=a^2d^2+1,$$
$$\lambda'=c^2d^2+1,\quad \mu'=b^2d^2+1,\quad \nu'=b^2c^2+1. \tag{12}$$

The circumscribed circles of the triangles BCD, ACD, ABD, ABC are given by

$$\mathcal{K}_a \quad \ldots \quad bcd\left(x^2+y^2\right)-[1+bcd\,(b+c+d)]\,x - \left(b^2c^2d^2+bc+bd+cd\right)y + b+c+d+bcd\,(bc+bd+cd)=0,$$
$$\mathcal{K}_b \quad \ldots \quad acd\left(x^2+y^2\right)-[1+acd\,(a+c+d)]\,x - \left(a^2c^2d^2+ac+ad+cd\right)y + a+c+d+acd\,(ac+ad+cd)=0,$$
$$\mathcal{K}_c \quad \ldots \quad abd\left(x^2+y^2\right)-[1+abd\,(a+b+d)]\,x - \left(a^2b^2d^2+ab+ad+bd\right)y + a+b+d+abd\,(ab+ad+bd)=0,$$
$$\mathcal{K}_d \quad \ldots \quad abc\left(x^2+y^2\right)-[1+abc\,(a+b+c)]\,x - \left(a^2b^2c^2+ab+ac+bc\right)y + a+b+c+abc\,(ab+ac+bc)=0$$

with the centers

$$O_a=\left(\frac{1}{2}\left(b+c+d+\frac{1}{bcd}\right),\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+bcd\right)\right),$$
$$O_b=\left(\frac{1}{2}\left(a+c+d+\frac{1}{acd}\right),\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}+acd\right)\right),$$
$$O_c=\left(\frac{1}{2}\left(a+b+d+\frac{1}{abd}\right),\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}+abd\right)\right),$$
$$O_d=\left(\frac{1}{2}\left(a+b+c+\frac{1}{abc}\right),\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+abc\right)\right)$$

and the radii

$$\rho_a=\frac{1}{2}\left|\frac{a}{p}\right|\sqrt{\lambda'\mu'\nu'},\quad \rho_b=\frac{1}{2}\left|\frac{b}{p}\right|\sqrt{\lambda'\mu\nu},$$
$$\rho_c=\frac{1}{2}\left|\frac{c}{p}\right|\sqrt{\lambda\mu'\nu},\quad \rho_d=\frac{1}{2}\left|\frac{d}{p}\right|\sqrt{\lambda\mu\nu'}, \tag{13}$$

respectively.

It would be important the following formula for two lines $\mathcal{L}$ and $\mathcal{L}'$ with slopes $\frac{m}{n}$ and $\frac{m'}{n'}$ and their oriented angle $\angle(\mathcal{L},\mathcal{L}')$

$$tg\angle(\mathcal{L},\mathcal{L}')=\frac{m'n-mn'}{mm'+nn'}. \tag{14}$$

2 Circles with diameters AB, AC, AD, BC, BD, CD and few more circles

The points $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ are incident to the circle with the equation

$$x^2+y^2-(x_1+x_2)x-(y_1+y_2)y+x_1x_2+y_1y_2=0 \qquad (15)$$

with the center in the midpoint $(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2))$ of these points, so (15) is the equation of the circle with the diameter P_1P_2. Using this formula, for the circle with diameter AB we get the equation

$$x^2+y^2-(a+b)x-\frac{a+b}{ab}y+ab+\frac{1}{ab}=0$$

so the power p_{AB} of the point $P=(x,y)$ with respect to that circle is

$$p_{AB}=x^2+y^2-(a+b)x-\frac{a+b}{ab}y+ab+\frac{1}{ab}.$$

Analogously, the power p_{CD} of the point P with respect to the circle with the diameter CD equals

$$p_{CD}=x^2+y^2-(c+d)x-\frac{c+d}{cd}y+cd+\frac{1}{cd},$$

so it follows

$$p_{AB}+p_{CD}=2x^2+2y^2-sx-\frac{r}{p}y+ab+cd+\frac{ab+cd}{p}.$$

For the power of the point P with respect to the circles with diameters AC,BD and AD,BC the following equalities are valid

$$p_{AC}+p_{BD}=2x^2+2y^2-sx-\frac{r}{p}y+ac+bd+\frac{ac+bd}{p},$$
$$p_{AD}+p_{BC}=2x^2+2y^2-sx-\frac{r}{p}y+ad+bc+\frac{ad+bc}{p}.$$

The midpoints of the sides AB and CD are points $(\frac{a+b}{2},\frac{a+b}{2ab})$, $(\frac{c+d}{2},\frac{c+d}{2cd})$, and a power p_u of the point P with respect to the circle whose the diameter is connecting line of these two midpoints, is equal to

$$p_u=x^2+y^2-\frac{s}{2}x-\frac{r}{2p}y+\frac{1}{4}(a+b)(c+d)+\frac{1}{4p}(a+b)(c+d).$$

Two more equalities are valid

$$p_v=x^2+y^2-\frac{s}{2}x-\frac{r}{2p}y+\frac{1}{4}(a+c)(b+d)+\frac{1}{4p}(a+c)(b+d),$$
$$p_w=x^2+y^2-\frac{s}{2}x-\frac{r}{2p}y+\frac{1}{4}(a+d)(b+c)+\frac{1}{4p}(a+d)(b+c)$$

for the powers of the point P with respect to the circles, for which the diameters are connecting lines of the midpoints of the sides AC, BD and AD, BC. Out of these equalities the following statement is valid

Theorem 1 *The powers of the point P with respect to the circles, for which the diameters are connecting lines of the midpoints of the sides AB, CD; AC, BD and AD, BC fulfil*

$$p_{AB}+p_{CD}+p_{AC}+p_{BD}=4p_w,$$
$$p_{AB}+p_{CD}+p_{AD}+p_{BC}=4p_v,$$
$$p_{AC}+p_{BD}+p_{AD}+p_{BC}=4p_u$$

and

$$p_{AB}+p_{CD}+p_{AC}+p_{BD}+p_{AD}+p_{BC}=2(p_u+p_v+p_w),$$

where p_u,p_v,p_w are powers of the point P with respect to the circle whose the diameter is connecting line of the midpoints of AB,CD; AC,BD and AD,BC.

The first three equalities can be found in [4], and the last equality is in [11].
Let $\mathcal{L}$ be the line with the equation $fx+gy+h=0$. Its intersection points with lines AB and CD from (3) are points $P_{AB}=(u_1,v_1)$ and $P_{CD}=(u_2,v_2)$, where

$$u_1=-\frac{ag+bg+abh}{abf-g},v_1=\frac{af+bf+h}{abf-g},$$
$$u_2=-\frac{cg+dg+cdh}{cdf-g},v_2=\frac{cf+df+h}{cdf-g}.$$

As $(abf-g)(cdf-g)=pf^2-(ab+cd)fg+g^2$, and

$$(abf-g)(cdf-g)(u_1+u_2)=$$
$$=(ab+cd)gh+sg^2-rfg-2pfh,$$
$$(abf-g)(cdf-g)(v_1+v_2)=$$
$$=(ab+cd)fh+rf^2-sfg-2gh,$$
$$(abf-g)(cdf-g)(uu'+vv')=$$
$$=ph^2+rgh+(q-ab-cd)(f^2+g^2)+sfh+h^2,$$

then the circle $\mathcal{K}_{AB,CD}$ with the diameter $P_{AB}P_{CD}$ has the equation

$$[pf^2-(ab+cd)fg+g^2](x^2+y^2)$$
$$-[(ab+cd)gh+sg^2-rfg-2pfh]x$$
$$-[(ab+cd)fh+rf^2-sfg-2gh]y+ph^2+rgh$$
$$+(q-ab-cd)(f^2+g^2)+sfh+h^2=0.$$

Analogously, the circle $\mathcal{K}_{AC,BD}$ with the diameter $P_{AC}P_{BD}$ has the equation

$$[pf^2-(ac+bd)fg+g^2](x^2+y^2)$$
$$-[(ac+bd)gh+sg^2-rfg-2pfh]x$$
$$-[(ac+bd)fh+rf^2-sfg-2gh]y$$
$$+ph^2+rgh+(q-ac-bd)(f^2+g^2)+sfh+h^2=0.$$

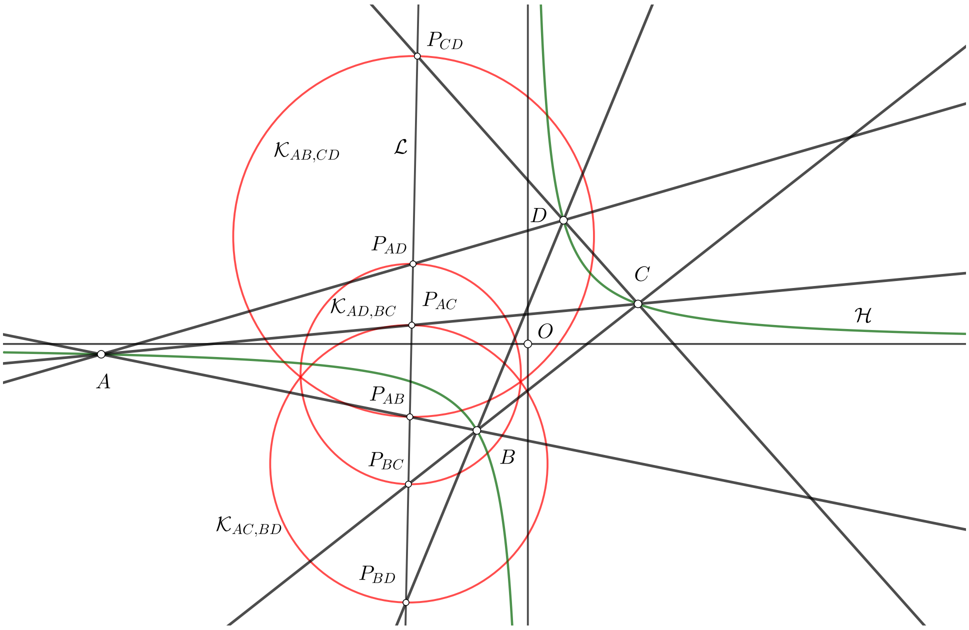

Figure 1: *Visualization of Theorem 2.*

Subtracting these two equations and dividing the obtained result by the common factor $(a-d)(b-c)$ we get the equation of a circle $\mathcal{K}$ in the form

$$fg(x^2+y^2)+ghx+fhy+f^2+g^2=0.$$

Hence, the circles $\mathcal{K}_{AB,CD}$, $\mathcal{K}_{AC,BD}$, $\mathcal{K}$ belong to the same pencil of circles. However, out of symmetry of the circle $\mathcal{K}$ on a,b,c,d we conclude that $\mathcal{K}_{AB,CD}$, $\mathcal{K}_{AD,BC}$, $\mathcal{K}$ belong to one pencil of circles. Hence,

Theorem 2 *Let $\mathcal{L}$ be a line. Three circles with diameters $P_{AB}P_{CD}$, $P_{AC}P_{BD}$, $P_{AD}P_{BC}$ belong to one pencil of circles, where $P_{AB},P_{CD},P_{AC},P_{BD},P_{AD},P_{BC}$ are intersection points of the line $\mathcal{L}$ with lines AB,CD,AC,BD,AD,BC.*

This result can be found in [7], [9] and [10]. See Figure 1.

3 Pedal triangles and pedal circles of the points A,B,C,D with respect to the triangles BCD, ACD, ABD, ABC

A normal from the point $A=(a,\frac{1}{a})$ to the line BC with equation $x+bcy=b+c$ has the equation $bcx-y=abc-\frac{1}{a}$, and these two lines are intersected in the point

$$A_d=\left(\frac{1}{a\nu'}(a^2b^2c^2+ab+ac-bc),\frac{1}{a\nu'}(ab^2c+abc^2-a^2bc+1)\right), \tag{16}$$

and, analogously, the pedal of the normal from A to the line BD is the point

$$A_c=\left(\frac{1}{a\mu'}(a^2b^2d^2+ab+ad-bd),\frac{1}{a\mu'}(ab^2d+abd^2-a^2bd+1)\right). \tag{17}$$

Because of that,

$$a^2\mu'^2\nu'^2A_cA_d{}^2=$$
$$=[\mu'(a^2b^2c^2+ab+ac-bc)-\nu'(a^2b^2d^2+ab+ad-bd)]^2$$
$$+[\mu'(ab^2c+abc^2-a^2bc+1)-\nu'(ab^2d+abd^2-a^2bd+1)]^2.$$

It is easy to see

$$(b^2d^2+1)(a^2b^2c^2+ab+ac-bc)$$
$$-(b^2c^2+1)(a^2b^2d^2+ab+ad-bd)]^2=$$
$$=(a-b)(c-d)(ab^2c+ab^2d-b^2cd+1),$$
$$(b^2d^2+1)(ab^2c+abc^2-a^2bc+1)$$
$$-(b^2c^2+1)(ab^2d+abd^2-a^2bd+1)=$$
$$=(a-b)(c-d)(ab^3cd-ab+bc+bd),$$
$$(ab^2c+ab^2d-b^2cd+1)^2+(ab^3cd-ab+bc+bd)^2=$$
$$=(a^2b^2+1)(b^2c^2+1)(b^2d^2+1)=\lambda\mu'\nu',$$

so $a^2\mu'^2\nu'^2A_cA_d{}^2=(a-b)^2(c-d)^2\lambda\mu'\nu'$ or, finally, $a^2\mu'\nu'A_cA_d{}^2=(a-b)^2(c-d)^2\lambda$. We proved the first of three analogous formulae

$$A_cA_d = \left|\frac{(a-b)(c-d)}{a}\right| \sqrt{\frac{\lambda}{\mu'\nu'}},$$
$$A_bA_d = \left|\frac{(a-c)(b-d)}{a}\right| \sqrt{\frac{\mu}{\lambda'\nu'}}, \quad (18)$$
$$A_bA_c = \left|\frac{(a-d)(b-c)}{a}\right| \sqrt{\frac{\nu}{\lambda'\mu'}}$$

for the lengths of the sides of the pedal triangle $A_bA_cA_d$ of the point A with respect to the triangle BCD. Analogous formulae for the lengths of the pedal triangle $B_aB_cB_d$ of the point B with respect to the triangle ACD are

$$B_cB_d = \left|\frac{(a-b)(c-d)}{b}\right| \sqrt{\frac{\lambda}{\mu\nu}},$$
$$B_aB_c = \left|\frac{(a-c)(b-d)}{b}\right| \sqrt{\frac{\mu'}{\lambda'\nu}},$$
$$B_aB_d = \left|\frac{(a-d)(b-c)}{b}\right| \sqrt{\frac{\nu'}{\lambda'\mu}}$$

Formulae for the lengths of the sides of the pedal triangles $C_aC_bC_d$ and $D_aD_bD_c$ of the points C and D with respect to the triangles ABD and ABC look similarly. Out of previously mentioned formulae

$$A_cA_d : B_cB_d = A_bA_d : B_aB_c = A_bA_c : B_aB_d = \left|\frac{b}{a}\right| \sqrt{\frac{\mu\nu}{\mu'\nu'}}$$

follow, meaning that triangles $A_bA_cA_d$ and $B_aB_dB_c$ are similar. Due to analogy, the triangles $C_dC_aC_b$ and $D_cD_bD_a$ are also similar to these triangles. So, we proved the result that can be found in [2], [3] and [6].

Theorem 3 *The pedal triangles of the points A,B,C,D with respect to the triangles BCD,ACD,ABD,ABC, respectively, are similar.*

Out of the corresponding equalities (11) and (18) we get the ratios

$$AB\cdot CD : A_cA_d = AC\cdot BD : A_bA_d = AD\cdot BC : A_bA_c =$$
$$= \sqrt{\lambda'\mu'\nu'} : |bcd|$$

i.e.

Theorem 4 *The lengths of sides of the pedal triangles of $A_bA_cA_d$, $B_aB_cB_d$, $C_aC_bC_d$, $D_aD_bD_c$ are related as the products of the lengths of pairs of opposite sides of the quadrangle ABCD.*

The last ratio equals to $2\rho_a$ because of (13). These statements can be found in [6].

The point A_d from (16) is incident to the circle $\mathcal{P}_a$ with the equation

$$a(p-1)(x^2+y^2) - a[a(p+1)-s]x + (p+1-ar)y = 0$$

i.e.

$$(p-1)(x^2+y^2) - [a(p+1)-s]x + \left(\frac{p+1}{a} - r\right)y = 0 \quad (19)$$

because of

$$(p-1)[(a^2b^2c^2+ab+ac-bc)^2+(ab^2c+abc^2-a^2bc+1)^2]-$$
$$-a(b^2c^2+1)(a^2b^2c^2+ab+ac-bc)(a(p+1)-s)+$$
$$+(b^2c^2+1)(ab^2c+abc^2-a^2bc+1)(p+1-ar) = 0.$$

Because of symmetry on b,c,d, of the equation (19) the circle $\mathcal{P}_a$ is a pedal circle of A with respect to the triangle BCD. Obviously, it is incident to the center O. Hence,

Theorem 5 *The pedal circles $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c, \mathcal{P}_d$ of the points A, B, C, D with respect to the triangles BCD, ACD, ABD, ABC, respectively are incident to the center O of the quadrangle ABCD.*

This result can be found in [1], [2], [5], [6].

The circle (19) has the center

$$P_a = \Big(\frac{1}{2(p-1)}(a^2bcd-b-c-d),$$
$$\frac{1}{2(p-1)}(abc+abd+acd-\frac{1}{a})\Big) \quad (20)$$

and the length OP_a is the radius r_a of that circle and easily we get

$$r_a = \frac{1}{2|a(p-1)|}\sqrt{(a^2b^2+1)(a^2c^2+1)(a^2d^2+1)} =$$
$$= \frac{1}{2|a(p-1)|}\sqrt{\lambda\mu\nu},$$

together with the first equality from (13) it proves the equality $\rho_a r_a = \frac{1}{4|p(p-1)|}\sqrt{\lambda\mu\nu\lambda'\mu'\nu'}$. This equality together with three analogous equalities prove that $\rho_a r_a = \rho_b r_b = \rho_c r_c = \rho_d r_d$, i.e.

Theorem 6 *The radii of the pedal circles $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c, \mathcal{P}_d$ of the points A, B, C, D with respect to the triangles BCD,ACD,ABD,ABC respectively, are inversely proportional to the radii of the circles BCD,ACD,ABD,ABC.*

This result can be reached in [6] and [8].

The point P_a from (20) is the midpoint of the point A and the point A' analogous to the point D' from (10), that is in accordance with the fact that the pedal circle of the point with respect to the triangle has the center in the midpoint of that point and its isogonal point with respect to this triangle. The ratio of the radii $r_a = \frac{1}{2|a(p-1)|}\sqrt{\lambda\mu\nu}$ and $r_b = \frac{1}{2|b(p-1)|}\sqrt{\lambda\mu'\nu'}$ is equal to the coefficient $|\frac{b}{a}|\sqrt{\frac{\mu\nu}{\mu'\nu'}}$ of the similarity of the triangles $A_bA_cA_d$ and $B_aB_dB_c$.

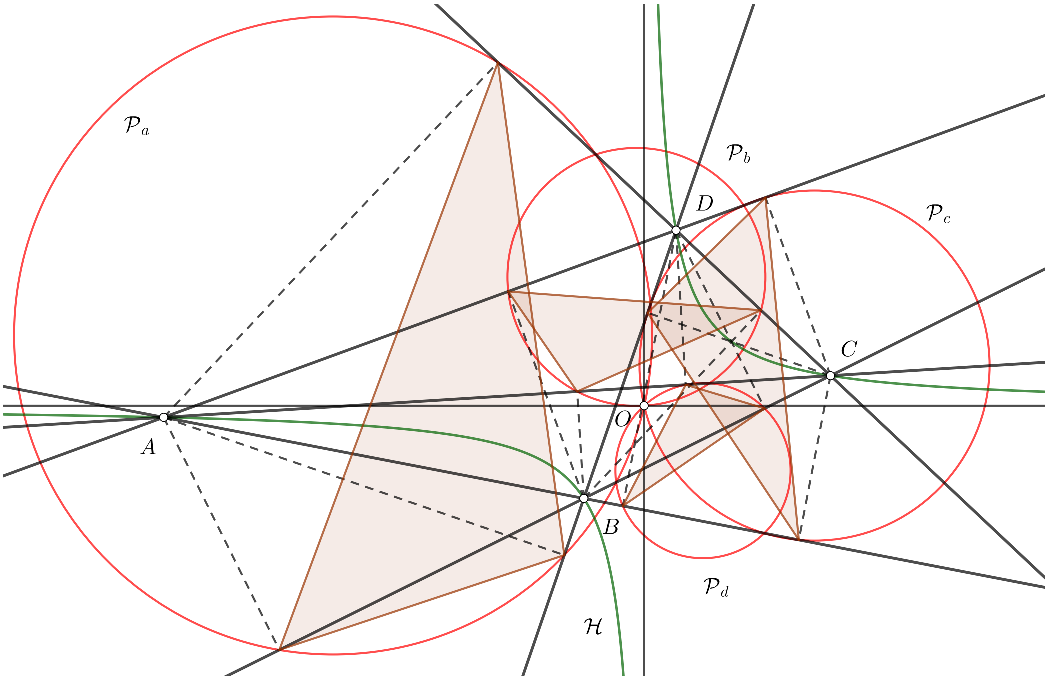

Figure 2: *Visualization of Theorem 5.*

The points A' and B' analogous to D' from (10) have the midpoint

$$M_{ab}=\left(-\frac{c+d}{p-1},ab\frac{c+d}{p-1}\right), \tag{21}$$

that is incident to the circle $\mathcal{P}_a$ with the equation (19). Taking the analogous results in consideration, we proved

Theorem 7 *The midpoints of the triples of segments $A'B', A'C', A'D'$; $A'B', B'C', B'D'$; $A'C', B'C', C'D'$; $A'D', B'D', C'D'$ are incident to the pedal circles $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c, \mathcal{P}_d$ of points A, B, C, D with respect to the triangles BCD, ACD, ABD, ABC, respectively.*

This result can be reached in [1].

The point A_c from (17) is incident to the line with equation

$$(a^2bd+abd^2-ab^2d+1)x+(a^2b^2d^2+ab+bd-ad)y= \\ =2b(a^2d^2+1),$$

and the point D_c is also incident to this line because of symmetry of this equation on a and d. We conclude that this is the line A_cD_c. It is incident to the point

$$\left(-\frac{2b}{(p-1)\lambda}(a^2bc+a^2bd-a^2cd+1),\right. \\ \left.\frac{2b}{(p-1)\lambda}(a^3bcd+ac+ad-ab)\right) \tag{22}$$

as well. Because the symmetry on c and d in the form of this point, obviously it lies on the line A_dC_d, hence this point is $A_cD_c\cap A_dC_d$.

The point

$$C_d=\left(\frac{1}{c\lambda}(a^2b^2c^2+ac+bc-ab),\frac{1}{c\lambda}(a^2bc+ab^2c-abc^2+1)\right)$$

is analogous to A_d from (16). It is incident to the line

$$c(a^3bcd+ab+ad-ac)x-c(a^2bc+a^2cd-a^2bd+1)y= \\ =(p-1)(a^2c^2+1),$$

and again because of symmetry on b and d, C_b is incident to it as well, so it is the line C_dC_b. This line is incident to the point

$$\left(\frac{p-1}{2acd\lambda}(a^2bc+a^2bd-a^2cd+1),\right. \\ \left.-\frac{p-1}{2acd\lambda}(a^3bcd+ac+ad-ab)\right),$$

and that point lies on the line D_cD_b because of the symmetry on c and d in the form of this point. Hence, this point is $C_dC_b\cap D_bD_c$. The obtained points $A_cD_c\cap A_dC_d$ and $C_dC_b\cap D_bD_c$ have the proportional coordinates. Homothety with the center O and coefficient $-\frac{1}{4p}(p-1)^2$ associates one point to another. As this coefficient is symmetric on parameters a,b,c,d then by cyclic permutation of b,c,d and B,C,D it follows that the same homothety associates the point $A_dB_d\cap A_bD_b$ to the point $D_bD_c\cap B_cB_d$, and the point $A_bC_b\cap A_cB_c$ to the point $B_cB_d\cap C_dC_b$, i.e. the mentioned homothety associates the triangle with vertices $A_cD_c\cap A_dC_d$, $A_dB_d\cap A_bD_b$, $A_bC_b\cap A_cB_c$ to the triangle formed by lines B_cB_d, C_dC_b, D_bD_c. It can be checked the following theorem and also three more analogous statements

Theorem 8 *Let A_b, A_c, A_d be pedal points and $\mathcal{P}_a$ pedal circle of A with respect to the triangle BCD. Let B_a, B_c, B_d, C_a, C_b, C_d, D_a, D_b, D_c be pedal points of B, C, D with respect to the triangle ACD, ABD, ABC, respectively. The points $A_cD_c \cap A_dC_d$, $A_dB_d \cap A_bD_b$ and $A_bC_b \cap A_cB_c$ are incident to $\mathcal{P}_a$.*

Because of the mentioned homothety, there is and the next result

Theorem 9 *The triangle formed by lines B_cB_d, C_dC_b, D_bD_c is inscribed to the circle that passes through the center O and at that point touches the circle $\mathcal{P}_a$.*

All of these results can be found in [1] and they are associated to Q.T. Bui.
In [1] the center of the quadrangle $ABCD'$ is studied as well. From Theorem 1 from [13] and Theorem 5 we know that the center O of the quadrangle $ABCD$ is incident to the Euler's circle $\mathcal{N}_d$ of the triangle ABC and to the pedal circle $\mathcal{P}_d$ of the point D with respect to that same triangle. So the center of the quadrangle $ABCD'$ is incident to the Euler's circle $\mathcal{N}_d$ of the triangle ABC and to the pedal circle of the point D' with respect to that triangle. The latter circle is the circle $\mathcal{P}_d$ because the isogonal points in the triangle have the same pedal circle. There is a question appearing: Is this center the center of the quadrangle O or the other intersection point of the circles $\mathcal{N}_d$ and $\mathcal{P}_d$? In the first case the point D' would lie on the hyperbola $\mathcal{H}$ and that is possible, but if it would be always like that then the same it should be valid for the points B', C' and D'. The point D' is incident to the hyperbola $\mathcal{H}$ under the condition that the equality $(d-a-b-c)(abd+acd+bcd-abc) = (p-1)^2$ is valid. The conditions for the points B', C' and D' look similarly. However, adding up these four conditions we get the equality $-16p = 4(p-1)^2 = 0$ i. e. $p = -1$ and the quadrangle $ABCD$ is the orthocentric. If we exclude this case, then we get the following statement.

Theorem 10 *The center of the quadrangle $ABCD'$ is the second intersection point of the circles $\mathcal{N}_d$ and $\mathcal{P}_d$ next to the center O.*

Three more analogous statements follow up.
The circle $\mathcal{P}_a$ with the equation (19) and the circle $\mathcal{P}_b$ with analogous equation

$$(p-1)(x^2+y^2) - [b(p+1)-s]x + \left(\frac{p+1}{b} - r\right)y = 0$$

have the radical axis with the equation $abx+y=0$. The midpoint of the point C and the point H_d from (7) is the point $(\frac{1}{2}(c - \frac{1}{abc}), \frac{1}{2}(\frac{1}{c} - abc))$ and it is incident to the radical axis. The same is valid and for the midpoint of points D and H_c. Points C and H_c are incident to the line $abdx - cy = p-1$ that passes through the point $(\frac{p-1}{ab(c+d)}, -\frac{p-1}{c+d})$. Because of symmetry on c and d, this point is also incident to DH_d. However, the intersection point $CH_c \cap DH_d$ is lying on the mentioned radical axis, see Figure 3. This result can be reached in [6] and [8]. The point M_{ab} from (21) is also incident to the mentioned radical axis with the equation $abx + y = 0$. The statement on the collinearity of these four points as well as five more such collinearities is given in [1]. Hence, the radical axis of the circles $\mathcal{P}_a$ and $\mathcal{P}_b$ bisects the segments CH_d, DH_c and $A'B'$. That radical axis is antiparallel to the line AB with respect the axes of the hyperbola $\mathcal{H}$, and the similar is valid for five more analogous radical axes. We have just proved the following theorem and five more analogous statements

Theorem 11 *Let H_c, H_d be orthocenters of ABD, ABC, respectively, and let A', B' be isogonal points of A, B and with respect to BCD, ACD respectively, and $\mathcal{P}_a, \mathcal{P}_b$ pedal circles of the points A, B with respect to the triangles BCD, ACD. Then the following four points lie on the radical axis of $\mathcal{P}_a$ and $\mathcal{P}_b$: midpoints of three segments $A'B'$, DH_c, CH_d and the intersection point $CH_c \cap DH_d$.*

The point M_{ab} obviously lies on the line CD as well as the points A_b and B_a. It is easy to check that the point M_{ab} is incident to the line

$$(a^2bc + ab^2d - abcd + 1)x + (a^2b^2cd + ac + bd - ab)y = \\ = (a^2b^2+1)(c+d),$$

as well as the point A_c from (17). By substituting $a \leftrightarrow b$ and $c \leftrightarrow d$ in the previous equation one obtains the line incident to the point B_d. Hence, the point M_{ab} is incident to the line A_cB_d, and analogously to the line A_dB_c. It means that the point M_{ab} is the center of the perspectivity for triangles $A_bA_cA_d$ and $B_aB_cB_d$. Out of (17) it follows that the line OA_c has the slope

$$\frac{m'}{n'} = \frac{ab^2d + abd^2 - a^2bd + 1}{a^2b^2d^2 + ab + ad - bd},$$

and, analogously, the line OA_b has the slope

$$\frac{m}{n} = \frac{ac^2d + acd^2 - a^2cd + 1}{a^2c^2d^2 + ac + ad - cd}.$$

After some calculation we get

$$m'n - mn' = (a^2d^2+1)(a-d)(b-c)(p-1),$$
$$mm' + nn' = (a^2d^2+1)[(p-1)^2 + ad(b^2+c^2) + bc(a^2+d^2)],$$

so due to (14) it follows

$$tg\angle A_bOA_c = \frac{(a-d)(b-c)(p-1)}{(p-1)^2 + ad(b^2+c^2) + bc(a^2+d^2)}.$$

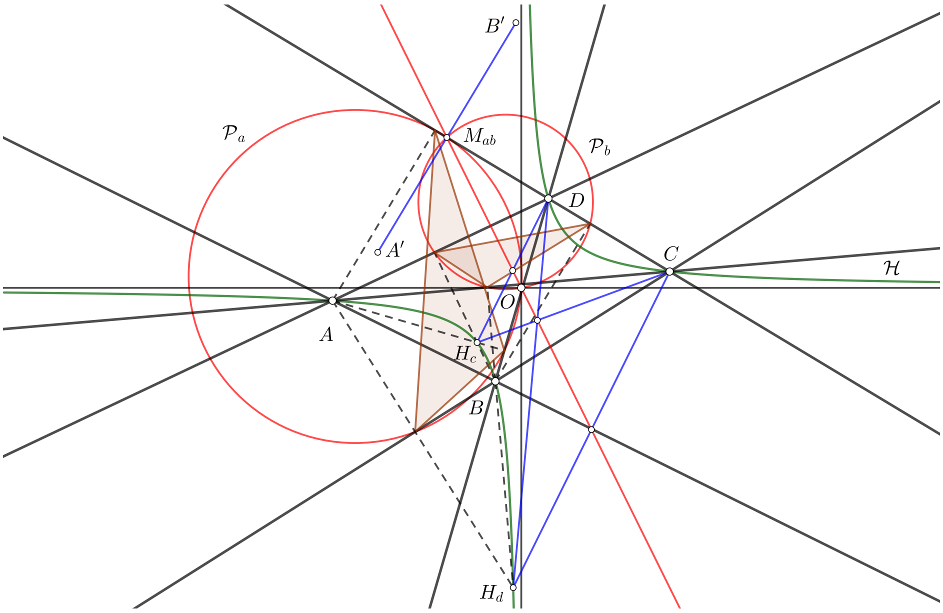

Figure 3: *Visualization of Theorem 11*

Substituting $a \leftrightarrow b$ and $c \leftrightarrow d$ the equality

$$tg\angle B_aOB_d = \frac{(a-d)(b-c)(p-1)}{(p-1)^2+ad(b^2+c^2)+bc(a^2+d^2)}$$

follows up. By this we achieved the equality of the oriented angles $\angle A_bOA_c = \angle B_aOB_d$, as well as $\angle A_bOA_d = \angle B_aOB_c$. However, out of these equalities the equality $\angle A_bOB_a = \angle A_cOB_d = \angle A_dOB_c$ is valid meaning that the center O is the center of the similarity of triangles $A_bA_cA_d$ and $B_aB_dB_c$. So, we have just proved the following result and five more analogous results that can be found in [6]:

Theorem 12 *The triangles $A_bA_cA_d$ and $B_aB_dB_c$ are similar and perspective where the center of the similarity is the center O, one intersection point of the circles $A_bA_cA_d$ and $B_aB_dB_c$, and the center of the perspectivity is their other intersection point M_{ab}.*

For the oriented segments $\overrightarrow{AB}$ and $\overrightarrow{P_aP_b}$ the following equalities are valid

$$\overrightarrow{AB} = \left(b-a, \frac{1}{b}-\frac{1}{a}\right) = \frac{b-a}{ab}(ab,-1),$$

$$\overrightarrow{P_aP_b} = \frac{1}{2(p-1)}\left(ab^2cd-a-a^2bcd+b, bcd-\frac{1}{b}-acd+\frac{1}{a}\right)$$
$$= \frac{(b-a)(p+1)}{2ab(p-1)}(ab,1).$$

As the vectors $[ab,-1]$ and $[ab,1]$ have the same square of the lengths equals to a^2b^2+1, then from previous mentioned two equalities it follows that the ratio of the lengths P_aP_b and AB equals to $\frac{p+1}{2(p-1)}$, the same is valid for the rest of the corresponding sides of $ABCD$ and $P_aP_bP_cP_d$. So, we can conclude

Theorem 13 *The quadrangles $ABCD$ and $P_aP_bP_cP_d$ are similar and the coefficient of the similarity is $\frac{p+1}{2(p-1)}$.*

This result can be reached in [8].

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