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# Graph theory and selected applications

# Grafovi i neke njegove primjene

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**Abstract:** *Graph theory was founded with Euler's work in 1736. It is applied to numerous current real-world problems, such as social media networks, telecommunication networks, electrical networks etc. It is also used in web search engines and other important algorithms, especially in solving optimization problems. This paper will present theoretical basis and applications of graph theory used as a part of the two undergraduate professional studies at Virovitica University of Applied Sciences through courses that involve graph theory along with its applications. Learning outcomes and their evaluation will be presented. Also, students' opinions on the courses will be discussed.*

**Keywords:** *graph theory; learning outcomes; networks; algorithms*

**Sažetak:** *Teorija grafova započela je Eulerovim radom 1736. godine. Primjenjuje se na brojne trenutačne probleme iz stvarnoga svijeta, kao što su mreže društvenih medija, telekomunikacijske mreže ili električne mreže. Također se koristi u web tražilicama i drugim važnim algoritmima, posebno u rješavanju optimizacije problema. U ovom radu bit će prikazane teorijske osnove i primjene teorije grafova koja se koristi u sklopu dvaju preddiplomskih stručnih studija Veleučilišta u Virovitici u kolegijima koji uključuju teoriju grafova s primjenama. Prikazat će se ishodi učenja i njihova evaluacija. Također, raspravljat će se i o mišljenjima studenata o predmetima.*

**Ključne riječi:** *teorija grafova, ishodi učenja, mreže, algoritmi*

## 1. Introduction

Many real-world occurrences, even from private life, can be described using graphs. Relationships between people can be encoded via graphs, e.g., nodes representing friends would be connected by an edge while those who aren't wouldn't – the so-called social networks (Barnes and Harary, 1983, 235-244). Telecommunication, electrical and physical networks are described using graphs, and even Google's successful web search algorithms are based on using the graph of the world wide web with web sites as nodes and hyperlinks between them as edges (Chung, 2010, 726-732). In general, systems made of interconnected objects - interactions between objects or more complex systems - can be described using graphs. For example, a graph (network diagram) can be used to display events and activities of a project (Klobučar and Crnjac, 1997, 67 - 69). Beginnings of graph theory date back to 1736 when Leonhard Euler published his work solving the Königsberg bridge problem (Nakić and Pavčević, 2018). The theory has a wide scope in modern applications.

This paper is based upon the theoretical foundation which the students should adopt, allowing them to solve problems in their field of expertise.

## 2. Definitions and basic principles of graph theory

The main objective of graph theory is to help us understand the graph's structure and compare it to other graphs to see how they differ (Nakić and Pavčević, 2018).

Graph, or simple graph, is an ordered triple $G = (V, E, \varphi)$ ), where $V = V(G)$ is a non-empty set whose elements are called nodes or vertices, $E = E(G)$ is a set disjoint from $V$ whose elements are called edges, and $\varphi$ is a function which maps any $e$ from $E$ to a set $\{u, v\}$ of a pair of, not necessarily distinct, vertices in $V$. For a vertex $v$ connected to a neighbor vertex $u$ via an edge $e = \{u, v\}$ it is said that the vertex $v$ is incident with the edge $e$ (Golemac et al., 2012). Alternating sequence of incident vertices and edges $u, e_1, v, e_2, ...$ is called a path. A path is said to be simple if the alternating sequence does not contain any of its vertices more than once (Golemac et al., 2012). A vertex can be connected to itself, i.e. $e = \{v, v\}$, and such an edge is called a loop, circular path or a cycle (Golemac et al., 2012).

A connected graph is a graph in which every two vertices are connected by a path. A graph in which all the vertices are connected directly, i.e., every pair of vertices has an edge incident to both, is called the complete graph. If there is a distinction for all edges between vertices incident to a given edge such that one of them is its starting vertex and the other is its terminal vertex, then such a graph is called a directed graph or digraph. Weighted graphs are graphs in which every edge has an associated weight (e.g., time units)[1].

Degree of a vertex $v$ in graph $G$ is the number of edges incident with it, i.e., the degree of the vertex is equal to the number of edges which are incident with it. Sum of degrees of all vertices in a simple graph is even (Golemac et al., 2012). Vertices of degree 0 are called isolated vertices, and those of degree 1 are called terminal vertices. If a graph is directed, in-degree of a vertex is the number of edges for which the vertex is their terminal vertex, and it's out-degree is the number of edges for which the vertex is their starting vertex. A vertex of in-degree zero is called a source, and a vertex of out-degree zero is called a sink[1].

If all vertices of a graph have the same degree, the graph is called regular, and its degree of regularity is said to be equal to the degree of any vertex[1].

Adjacency matrix $A$ is an $n \times n$ matrix such that its entries $a_{ij}$ are equal to the number of edges which connect vertex $i$ to vertex $j$. For a simple graph, the adjacency matrix is a symmetric matrix in which the sum of values in a row is equal to the degree of the corresponding vertex (Golemac et al., 2012).

Incidence matrix $B$ is an $n \times m$ matrix such that its entries $b_{ij}$ are 1 if a vertex $i$ is incident with an edge $j$, and other entries are zero (Golemac et al., 2012).

Trees are a special type of directed graphs in which there exists a root, i.e., an initial vertex. In it, if two vertices are connected, the one which is closer to the root is called a parent vertex and the one which is farther is called a child vertex. All vertices without a child vertex are called leaves[1].

## 3. Graph applications

### 3.1. Courses which use graph theory at Virovitica University of Applied Sciences

The course of "Mathematics II" is taught during the second semester of the undergraduate professional study of the Electrical Engineering. Learning outcomes are tested through class

1 https://www.fer.unizg.hr/_download/repository/ Osnovni_pojmovi-teorija_grafova.pdf

activities, homework assignments, written and oral exams. Graph theory is allotted 4 hours of lectures and 4 hours of auditory exercises. Learning outcomes of the course connected to graph theory are:

- LO7: Familiarity with definitions and an ability to connect basic concepts and problems related to graph theory
- LO8: Modelling problems using the framework of graph theory.

LO7 is tested through oral exam and grants up to 6 points, while LO8 is tested through:

- activity and short tests during class – 2 points,
- homework - 2 points,
- written exam – 12 points.

Half of the maximum number of points is required to successfully pass each of the learning outcomes, i.e., 3 for LO7 and 8 for LO8.

Students enrolled in the Software engineering study program through the first year's "Algorithms and data structures" course, taught through the winter semester, study trees, a special type of graphs. Students are expected to be able to describe and identify the properties of trees and use them to create a solution to a given problem.

Verifying the adoption of learning outcomes is accomplished through tracking of four different activities: class attendance, Loomen test, homework, projects in pairs, test, and oral exam.

The course of Numerical Mathematics is taught during the second semester of the undergraduate professional study of the Computer Science. The learning objectives are to develop knowledge about basic ideas and methods of numerical mathematics, to analyse the real-life problems and create corresponding mathematical models and give a review of the results. Learning outcomes are tested through class activities, homework assignments, written and oral exams, group projects and Loomen quizzes.

PageRank algorithm, as an integral part of the Google's search engine, is briefly presented as a motivational example for the problem of eigenvalues in the 7th week of the classes during the lecture. As such, it is not included in testing of learning outcomes. PageRank algorithm is also very important project of the course, which is a favourite pick among the students. The Solving simple problems by using demonstrated methods learning outcome is tested through the group project. A maximum of three students can be assigned to the same group, and the project grants up to 20 points, with 10 points being required to pass the learning outcome. The students are expected to study the theoretical background of the problem on their own, write a script in the software package GNU Octave or Python as the solution to the problem and to demonstrate its functionality on academic and real-world example. This presents an opportunity for the students to experience collaboration and solving somewhat more complex problems than those usually presented through classes. Presentation of the projects is held in the last week of the semester. Also, at the end of the summer semester, an anonymous survey is carried out through the Loomen platform, through which the students can provide feedback about the course. Overwhelming majority of students rated the group project as not overly difficult, noting specific obstacles – mostly related to starting to work too late. More than half of the students emphasized that the group project has helped them better understand the subject matter or coding in general, while it seems that being required to collaborate was deemed as an obstacle rather than a boon, which is to be expected at that point in their studies.

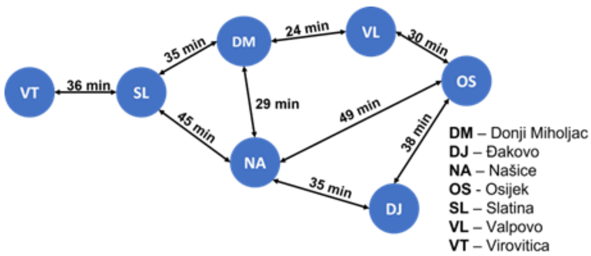
Two of the students had shown great interest in expanding their project related to the PageRank algorithm, which resulted in a professional paper published in the journal Math.e (Hajba et al., 2021, 18-32).

### 3.2. Introductory motivating examples

Motivation - arousing curiosity and interest from students - is one of the most important parts of a lecture. Students will learn more easily if they are actively engaged during class, depending upon their previous knowledge (Vizek-Vidović et al., 2005, 15-29).

Most students use some parts of graph theory without even realizing it. To familiarize the students of the "Telecommunications and informatics" study program with graph theory definitions, the introductory part of the lecture is used to pose the problem:

**Figure 1:** *Graph of the motivational example*




*Source: Author*

Look at the graph (shown at Fig. 1) which displays road connections between the city of Virovitica and Osijek, then answer the following questions:

1. What do the circles in the graph represent?
2. What are the arrows in the graph?
3. What are the neighbouring towns of Donji Miholjac (DM)?
4. List two ways in which you can get from Virovitica to Osijek?
5. Find the fastest way from Virovitica to Osijek.

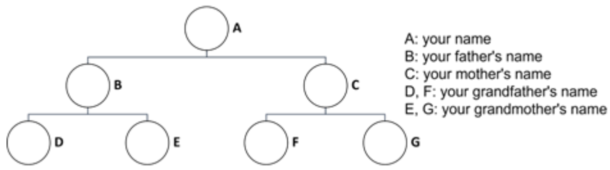
At the end of the lecture, after the definitions and principles of graph theory are presented, the students are asked to look at the problem again and apply the definition of a vertex, edge, path etc. to the same graph, now recognizing the circles as vertices, arrows as edges and so on.

A lecture from the "Algorithms and data structures" course has, as an introductory example for students, a family tree (Fig. 2). At the start of the lecture, each student writes down their own family tree, keeping it for reference during the lecture. After being presented with definitions and principles from graph theory, they are expected to apply them to their family tree answering the following questions:

1. Determine the level of the family tree?
2. Determine the nodes that are leaves?
3. Determine the length of the path ($A$, B, E)?
4. Calculate the internal length.

Then, after being done, they switch their work with each other to check for errors.

**Figure 2:** *Example of the lesson sheet "family tree"*




*Source: Author*

### 3.3. Applications of graph theory for the undergraduate professional study of the Electrical Engineering

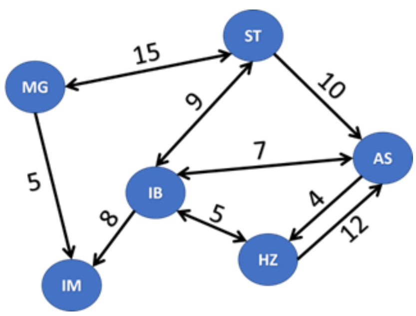
Lectures and exercises start with simple problems, like simple graphs with 4 or 5 nodes to demonstrate basics of graph theory. Weighted and directed graphs are introduces and explained to students, which open possibilities to demonstrate different applications of the graph theory. Some of the problems will be demonstrated.

**Example 1:** Let's assume that vertices are streets, and edges are electricity lines, shown at Fig. 3. Also, weights are representing maximal electricity capacity in appropriate units. Determine the set of vertices, edges and degrees of all vertices for a given graph. Does it contain loops? Find a path containing at least 4 vertices. Write down the graph's adjacency matrix and its incidence matrix that contains vertices MG, ST, IB, IM and corresponding edges. Calculate total capacity of the town. Which street(s) could receive most electricity? Which street(s) could produce most electricity? Is there any source or sink in this graph?
Let's assume that neighbourhoods are organized as follows:

- NB1: IB, ST, MG, IM
- NB2: AS, HZ.

Calculate in and out capacity of the NB1 and NB2.
Note that similar graph can be used to describe metro, tramways, pipelines (water, gas, oil), telecommunication networks etc.

**Figure 3:** *Graph of the example 1.*

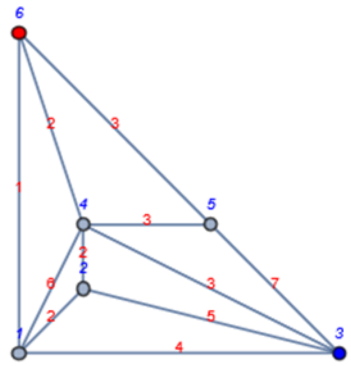



*Source: Author*

**Example 2:** For the graph $G$ which represents a resistive DC circuit shown at Fig. 4, where vertices represent resistor connections and their labels are denoted in blue, edges represent resistors and their resistance is denoted in red, the vertex 6 is the source and the vertex 3 is the sink of current, and the voltage between the source and the sink is equal to 1, determine the

potential of all vertices, value of the current flowing from the source, the circuit's effective resistance, and the value and direction of all currents.
Solution: First, we calculate conductance for all resistors as reciprocals of their resistance, set edge weights to be equal to their conductance, and set vertex weights to be equal to the sum of conductances of the edges incident to a vertex (Fig. 5, conductances denoted in green, vertex weights in blue). Call this graph $G_1$.

**Figure 4:** *Resistive circuit of example 2.*

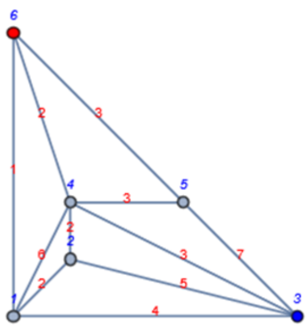


*Source: Author*

vertex 6 is the source and the vertex 3 is the sink of current, and the voltage between the source and the sink is equal to 1, determine the potential of all vertices, value of the current flowing from the source, the circuit's effective resistance, and the value and direction of all currents.
Solution: First, we calculate conductance for all resistors as reciprocals of their resistance, set edge weights to be equal to their conductance, and set vertex weights to be equal to the sum of conductances of the edges incident to a vertex (Fig. 5, conductances denoted in green, vertex weights in blue). Call this graph $G_1$.
Then, we create a directed weighted graph $G_2$ from $G_1$ such that they have the same vertices, if an edge connects a pair of vertices in $G_1$ then there are edges in both directions connecting the pair in $G_2$, and weight of a directed edge in $G_2$ is equal to the weight of an edge connecting the same vertices in $G_1$ divided by the weight of its starting vertex. We are only interested in the weighted adjacency matrix of $G_2$, reorder the vertices in such a way that the source and the sink are the last vertices in the weighted adjacency matrix. This will result in the matrix of the form

$$A = \begin{pmatrix} Q_{4\times4} & R_{4\times2} \\ 0_{2\times4} & I_{2\times2} \end{pmatrix}$$

**Figure 5:** *Vertex and edge weights*



*Source: Author*

Now, using the potential theory for Markov chains, which won't be explained to students, it is known that one of the columns of the matrix $(I-Q)^{-1}R$ is the column of potentials for the vertices in $G$, in the same order in which they were used for writing down $A$ (Golemac et al., 2012.). Now it's easy to calculate that the current flowing from the source is

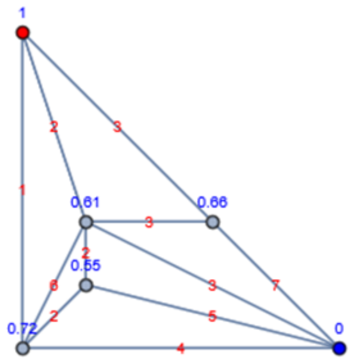
$$I = \frac{1-0.72}{1} + \frac{1-0.66}{3} + \frac{1-0.61}{2} = 0.59 \qquad (1)$$

The effective resistance of the resistive circuit is

$$R = \frac{U}{I} = \frac{1}{0.59} = 1.70 \qquad (2)$$

The students can calculate the rest of the current values and determine their directions. Note that if the voltage between the source and the sink is not equal to one, all the required values are simply scaled by the real value of the voltage.

**Figure 6:** *Edge resistances (red) and vertex potentials (blue)*

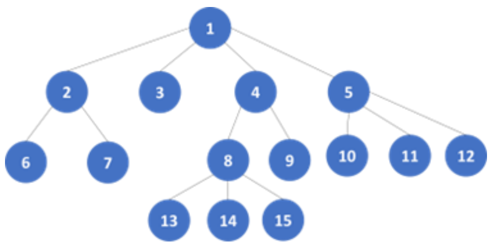



*Source: Author*

### 3.4. Application of graph theory for the undergraduate professional study of the Computer Science

First, few easy academic problems are used to demonstrate theoretical background of the trees. Then, more advanced problems are presented to students, which usually require to apply different algorithms.

**Figure 7:** *Graph of the example 3.*




*Source: Author*

**Example 3:** For the tree shown on the Fig. 7, determine:

a) Parent $\pi[x]$
b) $Left - Child[x]$

c) $Right - Sibling[x]$
d) $leaves[T]$
e) $depth[T]$

where $x = 6$.
*Solution:*

a) parent: $\pi[6] = 2$
b) $Left - Child[6] = NIL$
c) $Right - Sibling[6] = 7$
d) $leaves[T] = \{3, 6, 7, 9, 10, 11, 12, 13, 14, 15\}$
e) $depth[T] = 3$

## 4. Conclusion

Graph theory can be applied to numerous engineering problems. In the paper some of the examples and applications used during classes at first year of the undergraduate professional studies at Virovitica University of Applied Sciences are presented. These courses, which contain graph theory, with its corresponding learning outcomes and grading are presented. Interdisciplinary approach, using applications in real-world are more interesting to students and they in general achieve better results on corresponding learning outcomes. Taking in account their knowledge, problems should be of appropriate difficulty to challenge them, but not demotivate them. Graph theory offers broad range of interesting applications in engineering, which will be implemented in other courses, and more often as a task in thesis.

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