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QUASI-SYMMETRIC 2-$(28, 12, 11)$ DESIGNS WITH AN AUTOMORPHISM OF ORDER 5

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Dedicated to the memory of Professor Zvonimir Janko

Abstract. A design is called quasi-symmetric if it has only two block intersection numbers. Using a method based on orbit matrices, we classify quasi-symmetric 2-$(28, 12, 11)$ designs with intersection numbers 4, 6, and an automorphism of order 5. There are exactly 31 696 such designs up to isomorphism.

1. Introduction

A 2-(v, k, λ) *design* is a set of v points together with a collection of k-element subsets called *blocks* such that every pair of points is contained in exactly λ blocks. The number of blocks through a single point r and the total number of blocks b can be computed from v, k, and λ. A design is *quasi-symmetric* if any two blocks intersect in either x or y points, for non-negative integers $x < y$. The numbers x and y are called *intersection numbers*. We refer to [29, 30] for definitions and results about quasi-symmetric designs (QSDs), and to [1] for designs in general.

The first 2-$(28, 12, 11)$ QSDs with $x = 4$ and $y = 6$ were constructed as derived designs of symplectic symmetric 2-$(64, 28, 12)$ designs [20]. The symplectic group $Sp(6, 2)$ of order 1 451 520 acts on these designs and they have the *symmetric difference property (SDP)*. This means that the symmetric difference of any three blocks is either a block or the complement of a block.

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The parameters of symmetric SDP designs with $k < v/2$ are of the form

$$v = 2^{2m}, \qquad k = 2^{m-1}(2^m - 1), \qquad \lambda = 2^{m-1}(2^{m-1} - 1).$$

A *nonsymmetric SDP design* [18] is a 2-design with $v < b$ such that the symmetric difference of any two blocks is either a block or the complement of a block. Any such design is quasi-symmetric. The derived and residual designs of symmetric SDP designs have this property [19]. Furthermore, the derived and residual designs of nonisomorphic symmetric SDP designs are also nonisomorphic [18, Theorem 2.5]. There are four symmetric 2-$(64, 28, 12)$ SDP desings and each of them yields one derived 2-$(28, 12, 11)$ design up to isomorphism. Full automorphism groups of these quasi-symmetric SDP designs are of orders 1 451 520, 10 752, 1 920, and 672 [28].

In [25], the first examples of 2-$(28, 12, 11)$ QSDs without the SDP property were constructed. Then in [4], 2-$(28, 12, 11)$ QSDs with an automorphism of order 7 without fixed points and blocks were classified. There are exactly 246 such designs. The enumeration was performed with the help of orbit matrices. In [23], the number of known 2-$(28, 12, 11)$ QSDs was increased to 58 891. Some of these designs were constructed using the Kramer-Mesner method [21] adapted to QSDs. A direct construction based on Hadamard matrices and mutually orthogonal Latin squares from [2, 26] was used to find more examples, some with trivial full automorphism groups. The distribution of the known designs by order of full automorphism group is given in Table 1.

Table 1. Distribution of the known 2-$(28, 12, 11)$ QSDs by order of full automorphism group.

|Aut|	#	|Aut|	#	|Aut|	#	|Aut|	#	|Aut|	#
1451520	1	512	14	144	12	42	3	12	12908
10752	1	384	102	128	4745	40	2	10	28
4608	3	360	1	120	17	36	33	7	47
1920	4	320	4	96	26039	32	1299	3	172
1536	13	288	10	84	15	28	12	2	62
1344	4	256	258	80	372	24	360	1	9554
768	18	224	8	72	11	21	95		
672	8	192	652	64	110	20	26		
640	1	168	2	60	8	18	7		
576	12	160	564	48	1224	14	50		

Table 1 shows that no 2-$(28, 12, 11)$ QSDs with full automorphism group of order 5 are known. The purpose of this paper is to perform a complete classification of 2-$(28, 12, 11)$ QSDs with automorphisms of order 5. This is the next open case, designs with automorphisms of order 7 and larger prime orders having already been classified.

The layout of the paper is as follows. In Section 2 we describe the construction method based on orbit matrices. In Section 3 we study the number of points and blocks fixed by an automorphism of order 5. Results of the classification and details of the computation are described in Section 4.

2. Orbit matrices and indexing

Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ be the point orbits and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ the block orbits of a 2-(v, k, λ) design with respect to an automorphism group G. Denote the orbit sizes by $\nu_i = |\mathcal{V}_i|$ and $\beta_i = |\mathcal{B}_i|$. Then, $\sum_{i=1}^m \nu_i = v$ and $\sum_{j=1}^n \beta_j = b$. Let $a_{ij} = |\{P \in \mathcal{V}_i \mid P \in B\}|$, for some $B \in \mathcal{B}_j$. This number does not depend on the choice of B because the orbits form a tactical decomposition. The matrix $A = [a_{ij}]$ has the following properties:

1. $\sum\limits_{i=1}^{m} a_{ij} = k$,
2. $\sum\limits_{j=1}^{n} \frac{\beta_j}{\nu_i}\, a_{ij} = r$,
3. $\sum\limits_{j=1}^{n} \frac{\beta_j}{\nu_{i'}}\, a_{ij}a_{i'j} = \begin{cases} \lambda\nu_i, & \text{for } i \neq i', \\ \lambda(\nu_i - 1) + r, & \text{for } i = i'. \end{cases}$

A matrix with these properties is called an *orbit matrix* for 2-(v, k, λ) and G. Furthermore, for a QSD with intersection numbers x and y, the matrix A has the additional properties

4. $\sum\limits_{i=1}^{m} \frac{\beta_j}{\nu_i}\, a_{ij}a_{ij'} = \begin{cases} sx + (\beta_j - s)y, & \text{for } j \neq j',\ 0 \leq s \leq \beta_j, \\ sx + (\beta_j - 1 - s)y + k, & \text{for } j = j',\ 0 \leq s \leq \beta_j. \end{cases}$

An orbit matrix satisfying these equations is called *good*. The construction of designs based on orbit matrices proceeds in two steps.

1. Find all (good) orbit matrices A up to rearrangements of rows and columns.
2. Refine each matrix A in all possible ways to an incidence matrix of a design. Each refinement step replaces an entry a_{ij} with a 0-1 matrix N_{ij} of size $\nu_i \times \beta_j$ invariant under the action of the group G, such that the column sum of N_{ij} is a_{ij}. This is called *indexing*.

In the early 1980s Z. Janko and T. V. Tran used orbit matrices in their investigations of hypothetical projective planes of order 12 and other non prime power orders; see e.g. [11–13]. In the mid 1980s they proved existence of a number of symmetric designs by this method [14–17]. Janko continued using orbit matrices throughout the 1990s in constructions of symmetric designs [5–8] and some other designs [9,10]. The method became affectionately known as "Janko's method" among his Croatian collaborators, although orbit matrices had already been used by Dembowski [3]. The method was applied to quasi-symmetric designs in [4,24] and will be our main tool in Section 4. We first study fixed elements of an automorphism of order 5.

3. The number of fixed points and blocks

In the sequel, let α be an automorphism of order 5 of a 2-$(28, 12, 11)$ QSD with intersection numbers $x = 4$ and $y = 6$. We prove a succession of lemmas to determine the number of fixed points and blocks of α.

Lemma 3.1. *If a block contains five fixed points, then it is fixed.*

Proof. Let B be a non-fixed block containing five fixed points $F_1, \ldots, F_5$. There can be at most one other fixed point on B, because otherwise it would intersect its images B^α in more than six points. If there is a sixth fixed point on B, then the remaining points on B belong to different orbits of α, for the same reason. Thus, there would be at least six orbits of length 5, which is not possible because there are only 28 points. A similar argument shows that exactly five fixed points on B are also not possible. The block B could contain at most two points from two orbits of length 5 and one point from each of three other orbits. Then there would be at least five orbits of length 5, which together with $F_1, \ldots, F_5$ is more than 28. ☐

Lemma 3.2. *No block is fixed pointwise.*

Proof. Let B be a block with all of its points fixed. A divisibility argument shows that there must be at least one fixed point F not belonging to B. The $r = 27$ blocks through F intersect B in at least $x = 4$ points and are therefore fixed by Lemma 3.1. Again, this leads to a contradiction with there being only 28 points. ☐

Lemma 3.3. *The intersection of two fixed blocks contains exactly one fixed point.*

Proof. Let B_1 and B_2 be two fixed blocks with fixed points F_1 and F_2 in their intersection. The set of $\lambda = 11$ blocks through F_1 and F_2 is mapped onto itself by α, hence there are at least four other fixed blocks $B_3, \ldots, B_6$ among them. Each of these fixed blocks contains an orbit of length 5 by Lemma 3.2, contradicting $v = 28$. ☐

Lemma 3.4. *For any two fixed points there is exactly one fixed block containing them.*

Proof. This follows directly from the previous lemma. ☐

Theorem 3.5. *The automorphism α has three fixed points and three fixed blocks.*

Proof. By the lemmas, the set of all fixed points and fixed blocks of α has the following properties. There are exactly two points on each block, any two blocks intersect in one point, and there is one block through any pair of points. This can only be a triangle. ☐

4. Classification of quasi-symmetric 2-(28, 12, 11) designs

By Theorem 3.5, the orbit size distribution of the automorphism α is $\nu = (1,1,1,5,5,5,5,5)$ and $\beta = (1,1,1,5,5,5,5,5,5,5,5,5,5,5,5)$. The orbit matrix can be put in one of the following two forms by rearranging rows and columns:

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & & & & & & & & & & & & \\ 5 & 5 & 5 & & & & & & & & & & & & \\ 5 & 0 & 0 & & & & & & & & & & & & \\ 0 & 5 & 0 & & & & & & ? & & & & & & \\ 0 & 0 & 5 & & & & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & & & & \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & & & & & & & & & & & & \\ 5 & 5 & 0 & & & & & & & & & & & & \\ 5 & 0 & 5 & & & & & & & & & & & & \\ 0 & 5 & 5 & & & & & & ? & & & & & & \\ 0 & 0 & 0 & & & & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & & & & \end{bmatrix}.$$

We did a complete classification of such matrices by an orderly Read-Faradžev type algorithm described in [22], using a computer. The number of orbit matrices of type A_1 is 62 370, but only 198 of these matrices are good, i.e. have property 4. from Section 2. The number of orbit matrices of type A_2 is 55 573, but only 241 of them are good. Since we are looking for quasi-symmetric designs, we can discard matrices that are not good. This leads to a significant reduction of computation time in the second step of the classification, indexing.

For every good orbit matrix, we try to refine it in all possible ways to an incidence matrix of a 2-(28, 12, 11) QSD. The replacement of entries a_{ij} by 0-1 matices N_{ij} is done column-by-column, using a backtracking program checking the dot products along the way. Dot products of the columns correspond to block intersection sizes and must be $x = 4$ or $y = 6$. At the end we check if the constructed incidence matrices correspond to 2-designs and eliminate isomorphic copies using nauty [27]. This way of indexing proved more efficient than the usual row-by-row approach used e.g. in [22]. A similar conclusion was reached in [24] for a different computational method; instead of solving the Kramer-Mesner system, we found it was more efficient to search for cliques in the compatibility graph.

In total we found 3 449 nonisomorphic QSDs from orbit matrices of type A_1 and 28 247 nonisomorphic QSDs from orbit matrices of type A_2. Using nauty [27], we checked that designs coming from orbit matrices of different type are not isomorphic. This proves our main result.

THEOREM 4.1. *There are exactly* 31 696 *quasi-symmetric* 2-$(28, 12, 11)$ *designs with intersection numbers* $x = 4$*,* $y = 6$*, and an automorphism of order* 5*.*

The distribution of the constructed designs by order of full automorphism group is given in Table 2. There are exactly 878 such designs with full automorphism group of order 5.

TABLE 2. The distribution of 2-$(28, 12, 11)$ QSDs with an automorphism of order 5 by order of full automorphism group.

|Aut|	#	|Aut|	#	|Aut|	#	|Aut|	#
1451520	1	320	4	60	8	5	878
1920	4	160	564	40	2		
640	1	120	17	20	26		
360	1	80	372	10	29818		

By summarizing the previously known designs from [4, 23, 25] and the designs from Theorem 4.1, we can give a new lower bound on the number of quasi-symmetric 2-$(28, 12, 11)$ designs.

THEOREM 4.2. *There are at least* 89 559 *nonisomorphic quasi-symmetric* 2-$(28, 12, 11)$ *designs with intersection numbers* $x = 4$ *and* $y = 6$*.*

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