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BUSSEY SYSTEMS AND STEINER'S TACTICAL PROBLEM

Charles J. Colbourn, Donald L. Kreher and
Patric R. J. Östergård
Arizona State University (USA), Michigan Technological University (USA),
Aalto University (Finland)

In memory of Zvonimir Janko

Abstract. In 1853, Steiner posed a number of combinatorial (tactical) problems, which eventually led to a large body of research on Steiner systems. However, solutions to Steiner's questions coincide with Steiner systems only for strengths two and three. For larger strengths, essentially only one class of solutions to Steiner's tactical problems is known, found by Bussey more than a century ago. In this paper, the relationships among Steiner systems, perfect binary one-error-correcting codes, and solutions to Steiner's tactical problem (Bussey systems) are discussed. For the latter, computational results are provided for at most 15 points.

1. Introduction

Let V be a set of v elements. Let $\mathcal{B}$ be a collection of subsets (*blocks*) of V. Then $(V, \mathcal{B})$ is a *set system*; it is *simple* if the blocks are pairwise distinct. A t-*block* is a block of size t and $\mathcal{B}_t = \{B \in \mathcal{B} : |B| = t\}$ is the collection of t-blocks. The 3-, 4-, 5-blocks are sometimes called *triples*, *quadruples*, *quintuples*, and so forth. The maximum permitted size of a block is denoted by k.

Early geometrical research of Plücker [51, 52] and algebraic research of Sylvester [61] made the climate conducive to formulating "tactical" questions [23]. In 1844, W. S. B. Woolhouse [67] posed a Prize Question in the *Lady's and Gentleman's Diary*. In modern terminology, Woolhouse asked: For which positive integers $v \geq k \geq t$ does there exist a set system $(V, \mathcal{B})$ with $v = |V|$

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and $\mathcal{B} = \mathcal{B}_k$ so that every t-subset of V occurs in exactly one block? When no solutions were forthcoming from readers of the *Diary*, Woolhouse restated the question in the special case when $k = 3$ and $t = 2$. In 1847, Kirkman [36] established the necessary and sufficient condition that, in this case, $v \equiv 1, 3 \pmod 6$.

Soon thereafter, apparently unaware of Woolhouse's question and Kirkman's solution to the special case, Steiner [57] asked for a set system $(V, \mathcal{B})$, where

1. $\mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2 = \emptyset$,
2. for $2 \leq t < k$ every t-element subset of V either contains at least one block of $\cup_{i=3}^{t} \mathcal{B}_t$ or is contained in exactly one block of B_{t+1}, but not both.

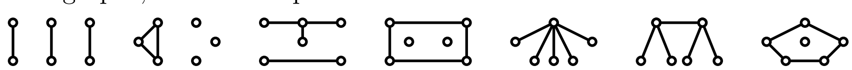
Steiner's decision to start with blocks of size 3 appears to have been motivated by his geometrical research [58,59] on the double tangents of quartic curves; we mention variants that start with other block sizes in Section 5.1.

A subset of V is *blocked* when it contains a block, *free* otherwise. It is *available* when every proper subset of it is free. The set system $(V, \mathcal{B})$ is *closed* when every k-subset of V is blocked. In this language, Steiner asks that every block be available, and every free t-set for $2 \leq t < k$ be in exactly one $(t+1)$-block. Being closed is equivalent to no k-set being free.

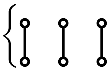
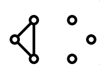
It can then be understood either that Steiner asked one question concerning the existence of a closed set system whose largest block size is k, or that he asked a sequence of $k - 2$ questions concerning (not necessarily closed) set systems of maximum block size k' for each $3 \leq k' \leq k$. In keeping with more recent usage, we adopt the latter understanding, but comment on the former when it is relevant.

In order to illustrate Steiner's tactical problem, we give a graphical example.

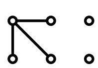
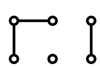
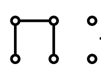
Example 1.1. Let $X = E(K_6)$, the edge set of K_6, and let $\mathcal{B}$ consist of all subgraphs, each isomorphic to one of:

 .

According to [13], every pair of edges occurs in exactly one subgraph of

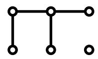
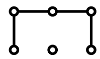
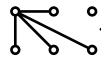
$\mathcal{B}_3 = \Big\{$  ,  $\Big\}$.

The (orbits of) 3-edge subgraphs that are not in $\mathcal{B}_3$ are:

 ,  and  .

Each is contained in exactly one block in $\mathcal{B}_4$.

The (orbits of) 4-edge subgraphs that do not contain a block in $\mathcal{B}_3 \cup \mathcal{B}_4$ are:

 ,  and .

Each is contained in exactly one block in $\mathcal{B}_5$. Because there is no 5-set that is not blocked, this system is closed.

In the remainder of the paper, we focus on Steiner's tactical problem for small orders. In Section 2, we explore similarities and differences between solutions to Woolhouse's problem (now known as Steiner systems) and solutions to Steiner's tactical problem (which we call Bussey systems). In Section 3, we explore the necessary conditions stated by Steiner, Bussey, and later authors, concluding that additional restrictions are needed in order to make the conditions valid. In Section 4.1, computational results when $k \in \{3, 4\}$ are given for $v \leq 15$. Then in Section 4.2 we constructively enumerate Bussey systems with $k = 5$ for $v \leq 15$. We also develop a doubling construction. In Section 5 we briefly explore two generalizations of Bussey systems.

2. Steiner systems and Steiner's tactical problems

At first, it appears that Steiner's question is the same as Woolhouse's when $k = t + 1$. Indeed, when $k = 3$ and $t = 2$, Steiner's question asks only for a set of triples for which every pair occurs in exactly one triple. This is precisely what Woolhouse asked for and Kirkman provided. When $k = 4$ and $t = 3$, Steiner's question asks for a set system $(V, \mathcal{B})$ containing triples $\mathcal{B}_3$ and quadruples $\mathcal{B}_4$ for which every pair occurs in exactly one triple, and every 3-set (which is not a triple) appears in exactly one 4-block. Forming $(V \cup \{\infty\}, \mathcal{B}')$ with $\mathcal{B}' = \mathcal{B}_4 \cup \{B \cup \{\infty\} : B \in \mathcal{B}_3\})$ yields a solution to Woolhouse's problem with $k = 4$ and $t = 3$. Moreover, deleting any point from such a solution to Woolhouse's problem, the set system obtained answers Steiner's question with $k = 4$. Again, existence has been completely settled in this case [27].

Substantial research in the century after Woolhouse's problem focussed on solutions with $t = 2$ and $t = 3$, usually stated as solutions to Steiner's question. When $t = 2$ and $k = 3$, solutions are *Steiner triple systems*. When $t = 3$ and $k = 4$, solutions are *Steiner quadruple systems*. For many decades, few authors referenced Kirkman or Woolhouse together with Steiner. Moore's 1896 *Tactical Memoranda* [42] essentially restates Woolhouse's problem without attribution. Bose's 1939 paper [8] forged strong connections with experimental design in statistics, but mentions only Steiner. Various editions of Rouse Ball's influential *Mathematical Recreations and Essays* [4] mention both Kirkman and Steiner but carefully avoid conflating Woolhouse's and Steiner's problems. Nevertheless a clear statement of the differences between them seems not to appear in the early literature. The first solutions to Woolhouse's problem with $t \in \{4, 5\}$ were given by Witt [65, 66] in 1937. For understandable reasons, Witt named these solutions after Steiner. Given integers $1 \leq t \leq k \leq v$ a *Steiner system* $S(t, k, v)$ is a set system $(V, \mathcal{B})$, where $v = |V|$, $\mathcal{B}_k = \mathcal{B}$, and every t-subset of points is contained in exactly one block. This nomenclature has become completely standard, and history has been rewritten (even, in places, in [15]) to attribute the question of the existence of Steiner systems to Steiner rather than Woolhouse or Kirkman.

However, the situation is more interesting. Consider the case of $k = 5$ and $t = 4$. As a special case, Woolhouse's question asks for which v an $S(4, 5, v)$ exists. Keevash [35] shows that the elementary necessary conditions are asymptotically sufficient. Steiner's question asks for a set system $(V, \mathcal{B})$ containing triples $\mathcal{B}_3$, quadruples $\mathcal{B}_4$, and quintuples $\mathcal{B}_5$ for which every pair occurs in exactly one triple, every free 3-set appears in exactly one 4-block, and every free 4-set appears in exactly one 5-block. Bussey [10, 11] gives a closed solution to Steiner's question for $k = 5$ on 15 points. By our earlier remarks, the triples in such a solution give a Steiner triple system $S(2, 3, 15)$; together with the quadruples we obtain a Steiner quadruple system $S(3, 4, 16)$. However, there is no $S(4, 5, 17)$ [45]. Hence the existence of an $S(4, 5, v + 2)$ given a solution to Steiner's problem with $k = 5$ on v points fails.

On the other hand, there exists an $S(5, 6, 72)$ [25]; each derived design is an $S(4, 5, 71)$. For Steiner's problem on 69 points with $k = 5$, elementary counting shows that we must have $|\mathcal{B}_3| = 782$, $|\mathcal{B}_4| = 12903$, and $|\mathcal{B}_5| = \frac{69\cdot 68\cdot 66\cdot 62}{5!} = 159997\frac{1}{5}$. However, the number of 5-blocks must be integral, so Steiner's problem admits no solution. Hence the existence of a solution to Steiner's problem on v points with $k = 5$ given an $S(4, 5, v + 2)$ fails.

The existence of an $S(k - 1, k, v)$ is neither necessary nor sufficient to answer Steiner's question with $k \geq 5$.

We have arrived at a terminological impasse, given the divergence between Steiner systems and Steiner's problem. We are by no means the first to observe this disparity; see [2, 18]. Hanani and Schönheim [29] and Assmus and Mattson [3], for example, discuss Steiner's question and establish the existence of closed solutions in cases when the corresponding Steiner system does not exist. However, much of their work was anticipated by Bussey [10,11] in the early twentieth century.

Citing Steiner's research [58, 59], Bussey [10] writes:

> When $N = 2^6 - 1 = 63$, it is possible to arrange the elements in triads, tetrads, pentads, hexads, and heptads. There is no arrangement of the 63 elements in ℓ-ads for $\ell > 7$. This special case was involved in Steiner's investigation of the configuration of the 28 double tangents of a quartic curve and led him to propose for solution the "Combinatorische Aufgabe" which I have called "The tactical problem of Steiner."

Confusion can only increase if we refer to such solutions as Steiner systems, or even as solutions to Steiner's problem. In an attempt to correct the attribution, Cummings [20] suggests that Steiner's problem be called the "modified Kirkman combination problem." Hanani [28] proposes the name "original Steiner systems" in order to disambiguate. We prefer a new term: A *Bussey system*, $\mathsf{Bus}(k, v)$, is a set system on v points that is a solution to Steiner's question for the given value of k. It is *closed* if no k-set is free.

CONSTRUCTION 2.1 ([10, 11]). *For $k \geq 3$, let X be the $2^{k-1} - 1$ nonzero vectors in a $(k-1)$-dimensional vector space over $\mathbb{F}_2$ and let $\mathcal{B} = \{A \cup \{\sum_{\mathbf{x} \in A} \mathbf{x}\} : A \subset X$ is linearly independent and $|A| \geq 2\}$. Then $(X, \mathcal{B})$ is a closed $\mathsf{Bus}(k, 2^{k-1} - 1)$.*

The same construction was later given in [3, 28], apparently unaware of Bussey's earlier work. It therefore seems most appropriate to use the term of Bussey system, especially in view of the fact that after a century no other parameters of Bussey systems with $k \geq 5$ seem to appear in the literature. This is not to say that no Bussey systems other than those from Construction 2.1 have been found. In order to explore this, we need further definitions.

A *binary code* of *length* n and *distance* d is a set of vectors (*codewords*) $\mathcal{C}$ each in $\{0, 1\}^n$ so that the Hamming distance between every two distinct vectors in $\mathcal{C}$ is at least d. It is *perfect* if, for some integer e, every vector in $\{0, 1\}^n$ is at distance at most e from exactly one codeword. This is a very well-studied topic; see [30, 62], for example. We focus on the case when $d = 3$ (i.e., $e = 1$). This leads to certain perfect binary (one-error-correcting) codes, the Hamming codes [24, 26]. Compare Construction 2.1 with Hamming's construction. They are essentially the same, although Bussey was concerned only with codewords of small weight. Nevertheless, in one sense Bussey's construction anticipates that of Hamming by forty years. The similarity of the constructions points to connections in the underlying problems that they solve. How are they related?

A binary code $\mathcal{C}$ of length n and distance 3 is *perfect ℓ-limited* when each codeword has Hamming weight at most ℓ, $0^n \in \mathcal{C}$, and for every vector $\mathbf{v} \in \{0, 1\}^n$ of weight at most $\ell - 1$, there exists exactly one $\mathbf{w} \in \mathcal{C}$ for which $dist(\mathbf{w}, \mathbf{v}) \leq 1$. When a binary perfect code of length n and distance 3 exists, for every $0 < \ell < n$, removing all vectors of weight greater than ℓ yields a perfect ℓ-limited binary code of length n. However, not every perfect ℓ-limited code arises in this way; for example, the incidence vectors of blocks of a Steiner triple system of order 9, together with the all-0 vector, form a perfect 3-limited code, but no perfect binary one-error-correcting code of length 9 exists. Moreover, although perfect binary one-error-correcting codes of length 15 exist, there is a Steiner triple system of order 15 that does not appear in any of the codes [44]. More precisely, exactly 33 of the 80 nonisomorphic $S(2, 3, 15)$s appear in a perfect binary one-error-correcting code [47]. Then because every $S(2, 3, 15)$ is the derived design of some Steiner quadruple system of order 16 [22, 33], some $\mathsf{Bus}(4, 15)$ does not extend to a perfect binary one-error-correcting code. Indeed, exactly 15,590 of the 1,054,163 nonisomorphic

$S(3,4,16)$s yield a $\mathsf{Bus}(4,15)$ that lives in a perfect binary one-error-correcting code of length 15, see [47].

LEMMA 2.2. *For $3 \le \ell \le 5$, a $\mathsf{Bus}(\ell,n)$ is equivalent to a binary code of length n that is perfect ℓ-limited.*

PROOF. The blocks of the $\mathsf{Bus}(\ell,n)$ are the supports of the nonzero codewords of the perfect ℓ-limited binary code $\mathcal{C}$. We verify that the $\mathsf{Bus}(\ell,n)$ gives the code by adjoining the all-0 vector; the converse proceeds in a similar manner. Consider a vector $\mathbf{v}$ of weight between 0 and $\ell-1$. Denote by $\delta_\downarrow(\mathbf{v})$ all vectors obtained from $\mathbf{v}$ by replacing a 1 by a 0, and by $\delta_\uparrow(\mathbf{v})$ all vectors obtained from $\mathbf{v}$ by replacing a 0 by a 1.

- $wt(\mathbf{v})=0$: $\mathbf{v}\in\mathcal{C}$, $\delta_\downarrow(\mathbf{v})=\emptyset$ and $\delta_\uparrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no block has size 1);
- $wt(\mathbf{v})=1$: $\mathbf{v}\notin\mathcal{C}$, $\delta_\downarrow(\mathbf{v})=\{\mathbf{0}\}$ and $\mathbf{0}\in\mathcal{C}$ and $\delta_\uparrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no block has size 2);
- $wt(\mathbf{v})=2$: $\mathbf{v}\notin\mathcal{C}$, $\delta_\downarrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no block has size 1) and $|\delta_\uparrow(\mathbf{v})\cap\mathcal{C}|=1$ (every pair of points occurs in exactly one 3-block);
- $wt(\mathbf{v})=3$: $\delta_\downarrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no block has size 2), and
  - if $\mathbf{v}\in\mathcal{C}$, $\delta_\uparrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no 3-block is contained in a 4-block);
  - if $\mathbf{v}\notin\mathcal{C}$, $|\delta_\uparrow(\mathbf{v})\cap\mathcal{C}|=1$ (a free 3-set is contained in exactly one 4-block);
- $wt(\mathbf{v})=4$:
  - if $\mathbf{v}\in\mathcal{C}$, $\delta_\downarrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no 3-block is contained in a 4-block), and $\delta_\uparrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no 4-block is contained in a 5-block);
  - if $\mathbf{v}\notin\mathcal{C}$ and $\delta_\downarrow(\mathbf{v})\cap\mathcal{C}\neq\emptyset$, $|\delta_\downarrow(\mathbf{v})\cap\mathcal{C}|=1$ (at most one triple on four points). Moreover, $\delta_\uparrow(\mathbf{v})\cap\mathcal{C}=\emptyset$ (no 3-block is contained in a 5-block);
  - if $\mathbf{v}\notin\mathcal{C}$ and $\delta_\downarrow(\mathbf{v})\cap\mathcal{C}=\emptyset$, then $|\delta_\uparrow(\mathbf{v})\cap\mathcal{C}|=1$ (every free 4-set is in exactly one 5-block).

This completes the proof. □

The correspondence in Lemma 2.2 breaks down at $\ell=6$. To see this, suppose that a $\mathsf{Bus}(6,n)$ with $n>7$ contains a 3-block $\{a,b,c\}$. Choose $d\notin\{a,b,c\}$ and let $\{a,b,d,x\}$, $\{a,c,d,y\}$, or $\{b,c,d,z\}$ be 4-blocks. Choose $e\notin\{a,b,c,d,x,y,z\}$. Then $\{a,b,c,d,e\}$ cannot be a 5-block, or contained in a 6-block, because it contains the 3-block $\{a,b,c\}$. Every 4-subset of $\{a,b,c,d,e\}$ either contains the 3-block or shares three but not four elements with one of the specified 4-blocks, so it cannot contain a 4-block. But then it is not at distance 1 from any block at all, and the corresponding code is not perfect 6-limited.

3. The structure of Bussey systems

Lemma 3.1. *Let* $(X, \mathcal{B})$ *be a* $\mathsf{Bus}(k, v)$. *Then*

$$\begin{aligned} |\mathcal{B}_3| &= \tfrac{1}{3!} v(v-1), \\ |\mathcal{B}_4| &= \tfrac{1}{4!} v(v-1)(v-3), \\ |\mathcal{B}_5| &= \tfrac{1}{5!} v(v-1)(v-3)(v-7). \end{aligned}$$

Proof. The value for $|\mathcal{B}_3|$ follows from the fact that $(X, \mathcal{B}_3)$ is a Steiner triple system of order v. Moreover, $(X \cup \{\infty\}, \{B \cup \{\infty\} : B \in \mathcal{B}_3\} \cup \mathcal{B}_4)$ is a Steiner quadruple system of order $v+1$, and hence $|\mathcal{B}_3| + |\mathcal{B}_4| = \binom{v+1}{3}/4$, so $|\mathcal{B}_4| = v(v-1)(v-3)/24$.

Now consider $\mathcal{B}_5$. Because no quadruple in $\mathcal{B}_4$ contains a triple in $\mathcal{B}_3$, and on four points there can be at most one triple, the number of blocked 4-subsets of X is $(v-3)|\mathcal{B}_3| + |\mathcal{B}_4|$. Then

$$|\mathcal{B}_5| = \frac{1}{5}\left(\binom{v}{4} - (v-3)|\mathcal{B}_3| + |\mathcal{B}_4|\right) = v(v-1)(v-3)(v-7)/120,$$

because every free 4-subset must be in exactly one block in $\mathcal{B}_5$. □

Can one obtain the generalization that

$$|\mathcal{B}_t| = \frac{1}{t!}(v(v-1)(v-3)\cdots(v-[2^{t-2}-1])?$$

Steiner [57] appears to assume that this is necessary. In Netto's 1927 text [43], an incomplete argument for necessity is given. Some subsequent work has stated this as a necessary condition [10,29]. However, Assmus and Mattson [3] comment that "in stating the problem, Steiner drew out some questionable necessary conditions." Indeed, using the connection to perfect binary one-error-correcting codes, they show that there are $\mathsf{Bus}(5, 15)$s in which certain 5-subsets of points are free, while Bussey's $\mathsf{Bus}(5, 15)$ is closed. Were Steiner's condition to hold in general, no 5-set could be free because $|\mathcal{B}_6| = 0$.

Among the earlier work, only Bussey [11] purports to prove Steiner's necessary conditions. In light of [3], Bussey's argument must either entail further assumptions, or contain a flaw. It is nonetheless instructive to examine Bussey's argument (and contrast it with [3, Section 5]).

Bussey [11] proceeds as follows. Consider a $\mathsf{Bus}(k, n)$, $(X, \mathcal{B})$. For $2 \le \ell \le k$, let $N_\ell = |\mathcal{B}_\ell|$, $\mathcal{A}_\ell$ be the free ℓ-sets of points, and $O_\ell = |\mathcal{A}_\ell|$.

The definition of Bussey systems ensures that $O_{\ell-1} = \ell N_\ell$. To determine $N_{\ell+1}$, we determine O_ℓ by examining $\mathcal{A}_\ell$. Whenever a set $A = \{a_1, \ldots, a_\ell\}$ satisfies $A \setminus \{a_i\} \in \mathcal{A}_{\ell-1}$ for each $1 \le i \le \ell$, so $A \in \mathcal{A}_\ell$ can be made from ℓ distinct sets of $\mathcal{A}_{\ell-1}$.

Let $3 \le \ell \le k$ and $A' \in \mathcal{A}_{\ell-1}$. Which elements can be adjoined to A' to produce a set of $\mathcal{A}_\ell$? (This process must produce O_ℓ sets, ℓ times each.)

Partition X into two classes, G and H. When $A' \cup \{x\} \in \mathcal{A}_\ell$, place $x \in G$; otherwise place $x \in H$. Then A' extends to $|G|$ sets in $\mathcal{A}_\ell$, and $|G| = v - |H|$. We determine H for A':

1. Place all elements of A' in H because $A' \cup \{x\}$ contains only $\ell - 1$ elements when $x \in A'$.
2. For every $C \subseteq A'$ with $|C| \geq 2$, there is a unique element y for which $C \cup \{y\} \in B_{|C|+1}$, so place y in H (because $C \cup \{y\}$ contains a block).

In this process, we have placed $(\ell - 1) + \sum_{\ell=2}^{\ell-1} \binom{\ell-1}{\ell} = 2^{\ell-1} - 1$ elements in H. One obtains a lower bound on $N_{\ell+1}$, as follows. Certainly $|H| \leq 2^{\ell-1} - 1$, so $|G| \geq v - 2^{\ell-1} + 1$. Therefore $(v - 2^{\ell-1} + 1)O_{\ell-1} \leq \ell O_\ell$. But $O_{\ell-1} = \ell N_\ell$, so $(v - 2^{\ell-1} + 1)N_\ell \leq O_\ell$. Because $O_\ell = (\ell + 1)N_{\ell+1}$, $(v - 2^{\ell-1} + 1)N_\ell \leq (\ell + 1)N_{\ell+1}$. Rewriting, $N_{\ell+1} \geq (v - 2^{\ell-1} + 1)N_\ell/(\ell + 1)$.

Bussey claims equality, by asserting that the $2^{\ell-1} - 1$ elements in H are (always) all distinct. If this held, Steiner's necessary condition would apply. But consider an $S(3, 4, v+1)$ yielding a $\mathsf{Bus}(4, v)$ in which $\{a, b, e\}$ and $\{c, d, e\}$ are triples, but their symmetric difference $B = \{a, b, c, d\}$ is not a quadruple. No 3-subset of B can be a triple, so $B \in \mathcal{A}_4$. Bussey's assertion requires that there be exactly $v - 15$ points that can be added to B to get a set in $\mathcal{A}_5$, and hence 15 that cannot. Yet this count of 15 includes e twice, and the assertion fails. In a $\mathsf{Bus}(4, 9)$, for example, this situation is unavoidable.

Using the language of perfect codes and examples from [63], a similar problem is observed in [3]. They remark that, as in our example, the Bussey system is not closed. If Bussey's assertion holds for closed systems, his proof as given is incomplete. It may be that Bussey's assertion fails even for closed Bussey systems; proving this would necessitate the existence of closed Bussey systems other than those from Construction 2.1.

It is plausible that Steiner, Bussey, and others left an essential necessary condition unstated, and it is natural to seek such a condition. Considering the manner in which Bussey counts, it suffices to adjoin the requirement that the symmetric difference of two blocks always contains a block. Although this would lead to Steiner's block counts and fill the gap in Bussey's proof, it may be too restrictive. It remains possible that a weaker condition suffices if k is large enough, or if we require the solution to be closed.

4. Bussey systems for small values of k

4.1. *$\mathsf{Bus}(3, v)$s and $\mathsf{Bus}(4, v)$s.* The correspondence of $\mathsf{Bus}(3, v)$s with Steiner triple systems and that of $\mathsf{Bus}(4, v)$s with Steiner quadruple systems (of order $v + 1$) provide a substantial amount of data on the number of non-isomorphic Bussey systems:

Order v	# $S(2,3,v)$	# $\mathsf{Bus}(3,v)$	# $S(3,4,v+1)$	# $\mathsf{Bus}(4,v)$
7	1	1	1 [5]	1
9	1	1	1 [5]	1
13	2 [9, 21]	2	4 [41]	8
15	80 [19]	80	1054163 [33]	7972282
19	11084874829 [32]	11084874829		

$\mathsf{Bus}(3,v)$s and Steiner triple systems of order v are in one-to-one correspondence, so the table simply restates published existence results. To obtain a $\mathsf{Bus}(4,v)$ from a Steiner quadruple system one deletes a point; hence the number of Bussey systems obtained is equal to the number of point orbits of the quadruple system. A lengthy but easy computation provides the last column.

4.2. $\mathsf{Bus}(5,v)$*s.* We refine Lemma 3.1, employing techniques similar to [7, 14] applied to point sets rather than block (line) sets. For a set P of $\ell \le 5$ points, define the *type* of P to be the multiset of the cardinalities of blocks lying entirely on P. We use the abbreviation F for free, B when P is a block, and one of $\{T, Q, TT, QT, QTT\}$ when there is one triple, one quadruple, two triples, and so on. Then for a type Y, Y_ℓ counts the ℓ-sets of type Y.

THEOREM 4.1. *In a* $\mathsf{Bus}(5,v)$ *we have*

$$
\begin{array}{rcl}
B_3 &=& v(v-1)/6,\\
F_3 &=& v(v-1)(v-3)/6,\\
B_4 &=& v(v-1)(v-3)/24,\\
T_4 &=& v(v-1)(v-3)/6,\\
F_4 &=& v(v-1)(v-3)(v-7)/24,\\
B_5 &=& v(v-1)(v-3)(v-7)/120,\\
Q_5 &=& v(v-1)(v-3)(v-10)/24 + QTT5,\\
QT_5 &=& v(v-1)(v-3)/4 - 2QTT_5,\\
QTT5 &=& QTT5,\\
TT_5 &=& v(v-1)(v-3)/8 - QTT_5,\\
T_5 &=& v(v-1)(v-3)(v-10)/12 + 2QTT_5,\\
F_5 &=& v(v-1)(v-3)(v-10)(v-12)/120 - QTT_5,\\
F_5 &=& v(v-1)(v-3)(v-7)(v-15)/120 + TT_5.
\end{array}
$$

PROOF. Lemma 3.1 gives B_3, B_4, and B_5. For $\ell \in \{3,4\}$, $F_\ell = (\ell+1)B_{\ell+1}$. Compute T_4 as $\binom{v}{4} - B_4 - F_4$.

Further classify quadruples as follows. Each quadruple counted by B_4 contains six pairs, each of which forms a triple with an element not in the quadruple. Among the six triples, there can be 3, 4, 5, or 6 distinct additional elements. We set $B_4^{[c]}$ equal to the number of quadruples in which c distinct elements arise in the six triples, for $c \in \{3,4,5,6\}$. Then

$$\begin{array}{rclllll} QTT_5 & = & 3B_4^{[3]}+ & 2B_4^{[4]}+ & B_4^{[5]}, & \\ QT_5 & = & & 2B_4^{[4]}+ & 4B_4^{[5]}+ & 6B_4^{[6]}, \\ Q_5 & = & (v-7)B_4^{[3]}+ & (v-8)B_4^{[4]}+ & (v-9)B_4^{[5]}+ & (v-10)B_4^{[6]}, \\ B_4 & = & B_4^{[3]}+ & B_4^{[4]}+ & B_4^{[5]}+ & B_4^{[6]}. \end{array}$$

It follows that $Q_5 = (v-10)B_4 + QTT5$ and $QT_5 = 6B_4 - 2QTT_5$.

The number of intersecting pairs of triples is $v\binom{(v-1)/2}{2}$, so $TT_5 + QTT_5 = v(v-1)(v-3)/8$. Each triple can be extended to a 5-set by adding any of $\binom{v-3}{2}$ pairs of points and hence

$$2QTT_5 + QT_5 + 2TT_5 + T_5 = v(v-1)(v-3)(v-4)/12.$$

So $T_5 = v(v-1)(v-3)(v-10)/12 + 2QTT_5$. Simplify $F_5 = \binom{v}{5} - B_5 - QTT_5 - QT_5 - Q_5 - TT_5 - T_5$ to obtain the penultimate equality, and substitute $QTT_5 = v(v-1)(v-3)/8 - TT_5$ to get the final one. □

4.3. *Small Orders.* Construction 2.1 gives a closed $\mathsf{Bus}(4,7)$; because it is closed, it meets the condition to be a $\mathsf{Bus}(5,7)$ with no 5-blocks. By Lemma 3.1 a putative $\mathsf{Bus}(5,9)$ must have a number of 5-blocks that is not integral, so none can exist.

THEOREM 4.2. *There is no* $\mathsf{Bus}(5,13)$.

PROOF. A $\mathsf{Bus}(4,13)$ has 390 free 4-sets. For a free 4-set S, x is *useless* when $x \notin S$ and $S \cup \{x\}$ carries a triple or quadruple. Each of the eight $\mathsf{Bus}(4,13)$s can be classified according to the numbers of free 4-sets having 5, 6, 7, 8, or 9 useless elements:

5	6	7	8	9
0	0	105	165	120
0	5	55	205	125
0	16	115	177	82
0	18	102	192	78
0	18	106	184	82
3	3	156	147	81
4	10	119	186	71
5	14	114	190	67

When a free 4-set has nine useless elements, it appears in no available 5-set. Because each of the $\mathsf{Bus}(4,13)$s has such free 4-sets, no $\mathsf{Bus}(5,13)$ can exist. □

In addition to the $\mathsf{Bus}(5,15)$ from Construction 2.1, one can construct further Bussey systems using Lemma 2.2 from perfect binary one-error-correcting codes. Starting with any perfect binary one-error-correcting code $\mathcal{C}$, for every codeword $c \in \mathcal{C}$, the code $\mathcal{C}_c = \{c' \oplus c : c' \in \mathcal{C}\}$ is a perfect binary one-error-correcting code containing the all-0 codeword. Two codes $\mathcal{C}$ and $\mathcal{C}'$, each containing the all-0 codeword, are *equivalent* if there exists $c \in \mathcal{C}$ for which $\mathcal{C}'$ is equal to one of the 15! coordinate permutations of $\mathcal{C}_c$; they are *inequivalent* otherwise. There are exactly 5983 inequivalent perfect binary one-error-correcting codes of length 15 [46]. Now for $c, d \in \mathcal{C}$ with $c \neq d$, $\mathcal{C}_c$ and $\mathcal{C}_d$ are certainly equivalent, but the sets of 3-, 4-, and 5-blocks may form nonisomorphic perfect 5-limited codes, or $\mathsf{Bus}(5, 15)$s. Each perfect binary one-error-correcting code of length 15 contains 2048 codewords. Hence one produces $2048 \cdot 5983$ $\mathsf{Bus}(5, 15)$s, among which many are isomorphic. A routine computation using `nauty` [39] establishes the following.

LEMMA 4.3. *There are* $139,247$ *nonisomorphic* $\mathsf{Bus}(5, 15)$*s that live in perfect binary one-error-correcting codes of length* 15.

Because there are $\mathsf{Bus}(3, 15)$s and $\mathsf{Bus}(4, 15)$s that do not live in perfect binary one-error-correcting codes, one should anticipate that the same holds for $\mathsf{Bus}(5, 15)$s. In order to find these, we start with the 1,054,163 nonisomorphic $S(3, 4, 16)$s [33]. Deleting a point from each point orbit in turn in an $S(3, 4, 16)$, one produces all nonisomorphic $\mathsf{Bus}(4, 15)$s. This process yields 7,972,282 nonisomorphic $\mathsf{Bus}(4, 15)$s, which is exhaustive and in agreement with [33, Table 2].

From a $\mathsf{Bus}(4, 15)$, a $\mathsf{Bus}(5, 15)$ is found by first computing the set $\mathcal{F}$ of all free 4-sets. The number of 5-blocks must be $|\mathcal{F}|/5$, so $|\mathcal{F}|$ is divisible by 5. Form the set of available 5-sets

$$\mathcal{A} = \{S : |S| = 5 \text{ and } S \setminus \{x\} \in \mathcal{F} \text{ for each } x \in S\}.$$

A lengthy computation using `exact_solve` from the package `libexact` [34] and `nauty` [39] establishes the following.

THEOREM 4.4. *There are* $174,691$ *nonisomorphic* $\mathsf{Bus}(5, 15)$*s.*

As a sanity check, we verified that each of the $139,247$ nonisomorphic $\mathsf{Bus}(5, 15)$s that live in perfect binary one-error-correcting codes arises exactly once among the $174,691$. Hence there are 35,444 nonisomorphic $\mathsf{Bus}(5, 15)$s that do not live in perfect binary one-error-correcting codes.

Table 1 tabulates the number of nonisomorphic $\mathsf{Bus}(5, 15)$s for each possible number of free 5-sets. Because there is a unique solution with $F_5 = 0$, there is only one closed $\mathsf{Bus}(5, 15)$ up to isomorphism. It is the system from Construction 2.1, and is the same as Example 1.1.

This is not surprising. When $v = 15$, $0 \leq QTT_5 \leq 315$ because each of the 105 quadruples can contribute 0, 1, 2, or 3 to QTT_5. But $F_5 + QTT_5 = 315$ so $0 \leq F_5 \leq 315$ as well. For the system to be closed, $F_5 = 0$ and hence $B_4^{[3]} = 105$ and $B_4^{[4]} = B_4^{[5]} = B_4^{[6]} = 0$. A *Pasch configuration* in

F_5	#	F_5	#	F_5	#	F_5	#	F_5	#	F_5	#	F_5	#
0	1	24	1	48	4	56	1	64	2	68	2	72	19
76	4	80	7	84	7	88	21	92	18	96	78	100	27
104	49	108	89	109	1	112	133	113	3	116	123	117	3
120	285	121	3	124	159	125	13	126	3	128	297	129	15
130	9	132	446	133	17	134	22	135	1	136	528	137	12
138	21	140	465	141	32	142	59	144	975	145	59	146	78
147	5	148	630	149	41	150	141	151	18	152	906	153	35
154	159	155	13	156	1289	157	53	158	270	159	19	160	1358
161	92	162	381	163	41	164	1342	165	86	166	455	167	55
168	2532	169	94	170	657	171	48	172	1829	173	74	174	843
175	84	176	2248	177	84	178	1039	179	110	180	3553	181	153
182	1198	183	116	184	2927	185	143	186	1810	187	153	188	3080
189	123	190	1983	191	192	192	5636	193	146	194	2323	195	161
196	4160	197	211	198	2809	199	198	200	4878	201	173	202	2800
203	232	204	7881	205	228	206	3589	207	202	208	6531	209	292
210	3880	211	225	212	6541	213	234	214	3912	215	251	216	9557
217	298	218	3856	219	257	220	6421	221	296	222	4228	223	227
224	6219	225	291	226	4082	227	284	228	7785	229	251	230	3443
231	178	232	5682	233	282	234	2690	235	204	236	4814	237	179
238	2284	239	210	240	5516	241	206	242	1556	243	137	244	2727
245	99	246	792	247	151	248	1900	249	171	250	438	251	42
252	785	253	64	254	197	255	104	256	374	257	43	258	137
259	20	260	207	261	102	262	83	263	40	264	82	265	21
266	115	267	11	268	103	269	27	270	32	271	37	272	44
273	31	274	49	275	2	276	54	277	22	278	1	279	48
280	5	282	37	285	3	288	3	291	1	294	7		

TABLE 1. Free 5-sets in nonisomorphic $\mathsf{Bus}(5, 15)$s

a triple system is a set of four blocks on six points. An easy count shows that the number of Pasch configurations is bounded below by $B_4^{[3]} + \frac{1}{3}B_4^{[4]}$ and above by $v(v-1)(v-3)/24$. Hence in a closed $\mathsf{Bus}(5, 15)$ the Steiner triple system admits the maximum number of Pasch configurations. Using the Veblen-Young axiom [64], Stinson and Wei [60] establish that this maximum is realized only when the Steiner triple system is projective, and hence v is of the form $2^k - 1$. As expected, the Steiner triple system in Construction 2.1 is the projective one.

THEOREM 4.5. *There is only one* $\mathsf{Bus}(6, 15)$ *up to isomorphism, and it contains no 6-blocks.*

Proof. Let $(V, \mathcal{B})$ be a $\mathsf{Bus}(5, 15)$ and let $\mathcal{F}_5$ be its set of free 5-sets. In order to extend to a $\mathsf{Bus}(6, 15)$, every free 5-set must appear in exactly one 6-block. Hence

$$(4.1) \quad |\{F \in \mathcal{F}_5 : X \subset F\}| \equiv 0 \pmod{6 - s} \text{ when } X \subset V,\ 0 \le |X| = s < 5.$$

Of the 174,691 nonisomorphic $\mathsf{Bus}(5, 15)$s, exactly 110,302 violate (4.1) when $s = 0$. Among the 64,389 with $F_5 \equiv 0 \pmod 6$, exactly 64,381 violate (4.1) when $s = 1$. Eight $\mathsf{Bus}(5, 15)$s remain, one with $F_5 = 0$ and each of the other seven with $F_5 = 168$. Of these eight, six violate (4.1) when $s = 2$. Only one $\mathsf{Bus}(5, 15)$ with $F_5 > 0$ remains, having one point in no free 5-sets and each of the other 14 points in 60 free 5-sets. (Every pair is in 0 or 20 free 5-sets.) This system satisfies (4.1) with $s = 3$, as every triple is in 0, 6, or 12 free 5-sets. However, it violates (4.1) with $s = 4$, and hence no $\mathsf{Bus}(5, 15)$ with $F_5 > 0$ extends to a $\mathsf{Bus}(6, 15)$.

The unique $\mathsf{Bus}(5, 15)$ with $F_5 = 0$ forms a unique $\mathsf{Bus}(6, 15)$ (with no 6-blocks). □

Although Steiner triple systems of order 19 have been exhaustively enumerated [16, 32], Steiner quadruple systems of order 20 have not. In any event, Lemma 3.1 establishes nonexistence of a $\mathsf{Bus}(5, 19)$. Examining possible $\mathsf{Bus}(5, 21)$s, $\mathsf{Bus}(4, 21)$s, or even $\mathsf{Bus}(3, 21)$s, is beyond the range of current computation. For reference, we tabulate the number of blocks in a putative $\mathsf{Bus}(5, v)$ for some values of v.

v	$\lvert\mathcal{B}_3\rvert$	$\lvert\mathcal{B}_4\rvert$	$\lvert\mathcal{B}_5\rvert$	v	$\lvert\mathcal{B}_3\rvert$	$\lvert\mathcal{B}_4\rvert$	$\lvert\mathcal{B}_5\rvert$
13	26	65	78	27	117	702	2808
15	35	105	168	31	155	1085	5208
21	70	315	882	33	176	1320	6864
25	100	550	1980	37	222	1887	11322

4.4. *A Doubling Construction.* In the 1840s, Kirkman [36] showed that one can double an $S(2, 3, v)$ to form an $S(2, 3, 2v + 1)$. In the 1930s, various researchers [6, 12, 65] showed that one can double an $S(3, 4, v + 1)$ to form an $S(3, 4, 2v + 2)$. For binary codes, Plotkin [50], Vasil'ev [63], and Sloane and Whitehead [54] showed (among other things) that one can double a perfect binary one-error-correcting code of length v to produce one of length $2v + 1$. Solov'eva [55] and Phelps [48, 49] (see also [30]) give a framework for such doubling, which we adopt here.

Theorem 4.6. *Whenever a* $\mathsf{Bus}(5, v)$ *exists, a* $\mathsf{Bus}(5, 2v + 1)$ *exists.*

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_v$ be the binary vectors of length v and Hamming weight 1. Let $\mathcal{C}_0$ be a perfect 5-limited code of length v. Let $\mathcal{C}_i = \{\mathbf{c} \oplus \mathbf{e}_i : \mathbf{c} \in \mathcal{C}_0\}$ for $1 \le i \le v$. Let $\mathcal{R}_0$ be a (possibly different) 5-limited code of length v, and define $\mathcal{R}_i$ for $1 \le i \le v$ in the same manner. Then for $0 \le i \le v$ define

$\mathcal{D}_i = \{(x_0, \ldots, x_{v-1}, \sum_{i=0}^{v-1} x_i \pmod 2) : (x_0, \ldots, x_{v-1}) \in \mathcal{R}_i\}$. Because $\{\mathcal{C}_i : 0 \le i \le v\}$ is a partition of all binary vectors in $\{0,1\}^v$, distinct codewords $\mathbf{c}$ and $\mathbf{c}'$ are at distance at least 3 if $\mathbf{c}, \mathbf{c}' \in \mathcal{C}_i$ for some i, distance at least 1 if $\mathbf{c} \in \mathcal{C}_i$ and $\mathbf{c}' \in \mathcal{C}_j$ for some $i \neq j$. Because $\{\mathcal{D}_i : 0 \le i \le v\}$ is a partition of all even weight binary vectors in $\{0,1\}^{v+1}$, distinct codewords $\mathbf{d}$ and $\mathbf{d}'$ are at distance at least 4 if $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_i$ for some i, distance at least 2 if $\mathbf{d} \in \mathcal{D}_i$ and $\mathbf{d}' \in \mathcal{D}_j$ for some $i \neq j$.

For $0 \le i \le v$ define $\mathcal{E}_i = \{\mathbf{cd} : \mathbf{c} \in \mathcal{C}_i, \mathbf{d} \in \mathcal{D}_i\}$. (Here $\mathbf{cd}$ is the concatenation of $\mathbf{c}$ and $\mathbf{d}$.) Set $\mathcal{E} = \cup_{i=0}^{v} \mathcal{E}_i$. Distinct codewords $\mathbf{cd}$ and $\mathbf{c}'\mathbf{d}'$ in $\mathcal{E}_i$ must be at distance at least 4 if $\mathbf{c} = \mathbf{c}'$, at least 3 if $\mathbf{d} = \mathbf{d}'$, and at least 7 otherwise. Distinct codewords $\mathbf{cd}$ in $\mathcal{E}_i$ and $\mathbf{c}'\mathbf{d}'$ in $\mathcal{E}_j$ for $i \neq j$ must be at distance at least 3. Now $\mathcal{C}_0$ has $\frac{1}{120}v(v-1)(v-3)(v-7)$ codewords of weight 5, while $\mathcal{D}_0$ has none. The numbers of codewords of smaller Hamming weight in $\mathcal{C}_i$ and $\mathcal{D}_i$ are tabulated here:

weight $\rightarrow$	0	1	2	3	4
$\mathcal{C}_0$	1	0	0	$\frac{1}{6}v(v-1)$	$\frac{1}{24}v(v-1)(v-3)$
$\mathcal{D}_0$	1	0	0	0	$\frac{1}{24}(v+1)v(v-1)$
$\mathcal{C}_i$, $i \neq 0$	0	1	$\frac{1}{2}(v-1)$	$\frac{1}{6}(v-1)(v-3)$	$\frac{1}{24}(v-1)(v-3)^2$
$\mathcal{D}_i$, $i \neq 0$	0	0	$\frac{1}{2}(v+1)$	0	$\frac{1}{24}(v+1)(v-1)(v-3)$

It follows that $\mathcal{E}$ has 1 codeword of weight 0, $\frac{1}{6}(2v+1)(2v)$ of weight 3, $\frac{1}{24}v(v-1)(v-3) + \frac{1}{24}(v+1)v(v-1) + \frac{1}{4}(v+1)v(v-1) = \frac{1}{24}(2v+1)(2v)(2v-2)$ of weight 4, and $\frac{1}{120}(2v+1)(2v)(2v-2)(2v-6)$ of weight 5. Hence the codewords of $\mathcal{E}$ of weight at most 5 form a perfect 5-limited code of length $2v+1$, and a $\mathsf{Bus}\big(5, 2v+1\big)$ exists. □

Although the construction of [48,49,55] provides a doubling for $\mathsf{Bus}\big(5, v\big)$s that live in perfect binary one-error-correcting codes, Theorem 4.6 establishes that their construction works for all $\mathsf{Bus}\big(5, v\big)$s. It also provides numerous nonisomorphic $\mathsf{Bus}\big(5, 2^k-1\big)$s not living in perfect binary one-error-correcting codes.

4.5. *Questions and speculations.* Although $\mathsf{Bus}\big(4, v\big)$s exist whenever $v \equiv 1, 3 \pmod 6$, no $\mathsf{Bus}\big(5, v\big)$ is known for which v is not of the form $2^k - 1$. In contrast with Steiner 4-designs [35], asymptotic existence for $\mathsf{Bus}\big(5, v\big)$s is not settled. It would be of substantial interest to produce any example for any v not of the form $2^k - 1$, or perhaps to prove that none exists. We speculate that one exists.

At this time, the only known $\mathsf{Bus}\big(6, v\big)$s arise from Construction 2.1. When $v = 2^k - 1 \ge 31$, truncating a perfect binary one-error-correcting code need not produce a $\mathsf{Bus}\big(6, v\big)$. Is there a $\mathsf{Bus}\big(6, 2^k - 1\big)$ not isomorphic to one from Construction 2.1? Again we speculate that one exists.

The only known closed $\mathsf{Bus}(k,v)$s have $v = 2^{k-1} - 1$ and arise from Construction 2.1. Does there exist a closed $\mathsf{Bus}(k,v)$ not isomorphic to one from Construction 2.1? If the answer is negative, it would prove the necessary conditions given by Steiner and others if one supposes that Steiner's question demands a closed solution. We do not find sufficient evidence to advance a speculation.

5. Extending the concept

We consider two ways in which Bussey systems (Steiner's questions) can be generalized. Assmus and Mattson [3] propose a different generalization motivated by nonbinary perfect codes, which we do not pursue here.

5.1. *Different Minimum Block Sizes.* Steiner's question supposes that $\mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2 = \emptyset$, and hence the smallest blocks are triples. A natural generalization is obtained when the smallest blocks have size m.

A Bussey system of *base* m, $\mathsf{Bus}_m(k,v)$, is a set system $(X,\mathcal{B})$ where

1. $\mathcal{B}_0 = \mathcal{B}_1 = \cdots = \mathcal{B}_{m-1} = \emptyset$, and
2. for $m-1 \le t < k$, every free t-set of points is contained in exactly one block of B_{t+1}.

A $\mathsf{Bus}_m(k,v)$ is *proper* when $\mathcal{B}_k \ne \emptyset$, and *closed* when no k-set is free.

We only consider cases when $m > 0$, for if $\mathcal{B}_0 \ne \emptyset$, there could be no nonempty blocks. A $\mathsf{Bus}_1(k,v)$ is equivalent to a $\mathsf{Bus}_2(k,v-1)$, obtained by deleting the unique singleton block and the point contained in it. For larger values of m, when $k \in \{m, m+1\}$ the correspondence with Steiner systems is straightforward.

Lemma 5.1. *Let $m \ge 2$. The set system $(V,\mathcal{B})$ is a $\mathsf{Bus}_m(m,v)$ if and only if it is an $S(m-1,m,v)$. It is a $\mathsf{Bus}_m(m+1,v)$ if and only if $(V \cup \{\infty\}, \mathcal{B}_{m+1} \cup \{B \cup \{\infty\} : B \in \mathcal{B}_m\})$ is an $S(m,m+1,v+1)$.*

When $m = 3$, a $\mathsf{Bus}_3(k,v)$ is precisely a $\mathsf{Bus}(k,v)$; there, Construction 2.1 and the results of Section 4.2 provide examples with $k \ge m+2$. One interesting example with $m = 5$ follows.

Lemma 5.2. *When t is odd and an $S(t-1,t,2t+1)$ exists, a closed $\mathsf{Bus}_t(k,2t+1)$ exists for all $k \ge t+1$. In particular, a closed $\mathsf{Bus}(k,7)$ exists when $k \ge 4$ and a closed $\mathsf{Bus}_5(k,11)$ exists when $k \ge 6$.*

Proof. An $S(t-1,t,2t+1)$ can always be extended to an $S(t,t+1,2t+2)$ [1, 40], and hence a $\mathsf{Bus}_t(t+1,2t+1)$, $(V,\mathcal{B})$, exists. Suppose to the contrary that some $(t+1)$-set S is free. There are $\binom{t+1}{t-1} = \frac{1}{2}t(t+1)$ free $(t-1)$-sets $\mathcal{F}$ in S, each of which must appear in a t-block. For each $F \in \mathcal{F}$ let x_F be the unique element for which $F \cup \{x_f\}$ is a t-block. Because S is free, $x_F \in V \setminus S$. In the multiset $M = \{x_F : F \in \mathcal{F}\}$, the multiplicity of any element cannot exceed $\frac{t+1}{2}$ because no two t-blocks can share $t-1$ points. Hence M consists of the t elements of $V \setminus S$, each with multiplicity $\frac{t+1}{2}$. Now consider a free t-set $F' \subset S$. Exactly one $(t+1)$-block S' contains F' and the element in $S' \setminus F'$ cannot be in F' or in M. Then $S = S'$ and S is a $(t+1)$-block and is not free, a contradiction. Because there are no free $(t+1)$-sets, $(V, \mathcal{B})$ is a proper, closed $\mathsf{Bus}_t\big(t+1, 2t+1\big)$ and a closed $\mathsf{Bus}_t\big(k, 2t+1\big)$ when $k \geq t+1$. The particular examples arise from an $S(2,3,7)$ and an $S(4,5,11)$. □

The closed $\mathsf{Bus}\big(k,7\big)$ from Lemma 5.2 is the one from Construction 2.1 again. The $\mathsf{Bus}_5\big(k,11\big)$ provides an example of a closed Bussey system with base m not arising from Bussey's construction, but it is proper only when $k=6$. We do not know any proper $\mathsf{Bus}_m\big(m+2, v\big)$ with $m \geq 2$ but $m \neq 3$. A necessary, but not sufficient, condition for the explicit construction of a $\mathsf{Bus}_m\big(m{+}2, v\big)$ is that there be an explicit construction of an $S(m, m{+}1, v{+}1)$. Although asymptotic existence has been established for Steiner systems [35], infinitely many explicitly constructed $S(m, m{+}1, v{+}1)$s are known only when $m \in \{2,3\}$, and a small finite collection is known when $m \in \{4,5\}$ (see [17]). For larger m, explicit examples remain elusive. Hence it is reasonable to focus on small values of m. To eliminate many of the parameter sets for putative $\mathsf{Bus}_m\big(m+2, v\big)$s, we establish useful necessary conditions patterned on the divisibility conditions for Steiner systems (see [17]) and related designs [38].

Lemma 5.3 (Divisibility). *In a proper $\mathsf{Bus}_m\big(k, v\big)$, the number F_s of free s-sets for $0 \leq s < \min(k, m+2)$ is*

$$F_s = \begin{cases} \frac{1}{s!}\prod_{i=0}^{s-1}(v-i), & \textit{if} \quad 0 \leq s < m, \\ \frac{1}{m!}\left[\prod_{i=0}^{m-2}(v-i)\right](v-m), & \textit{if} \quad s = m, \\ \frac{1}{(m+1)!}\left[\prod_{i=0}^{m-2}(v-i)\right](v-m)(v-2m-1), & \textit{if} \quad s = m+1, \end{cases}$$

and the number B_r of r-blocks for $0 \leq r \leq \min(k, m+2)$ is

$$B_r = \begin{cases} 0, & \textit{if} \quad 0 \leq r < m, \\ \frac{1}{m!}\left[\prod_{i=0}^{m-2}(v-i)\right], & \textit{if} \quad r = m, \\ \frac{1}{(m+1)!}\left[\prod_{i=0}^{m-2}(v-i)\right](v-m), & \textit{if} \quad r = m+1, \\ \frac{1}{(m+2)!}\left[\prod_{i=0}^{m-2}(v-i)\right](v-m)(v-2m-1), & \textit{if} \quad r = m+2, \end{cases}$$

whenever $0 \leq s < r \leq \min(k, m+2)$, $\frac{\binom{r}{s}B_r}{F_s}$ is an integer.

Proof. The counts of free sets and blocks are straightforward. Every subset of a free set is free, and every proper subset of a block is free. Hence for $s < r$, every r-block contains $\binom{r}{s}$ free s-sets. For $0 \leq s < m$, every free s-set is contained in exactly $\binom{v}{m-1-s}/\binom{m-1}{m-1-s}$ free $(m-1)$-sets. Every free $(m-1)$-set is contained in one m-block and $v-m$ free m-sets. Every free

m-set is contained in one $(m+1)$-block and $v-2m-1$ free $(m+1)$-sets. Every free $(m+1)$-set is contained in one $(m+2)$-block. ◻

Applying the lemma when $r \in \{m, m+1\}$ yields the basic divisibility conditions for Steiner systems, for example:

COROLLARY 5.4. *A* $\mathsf{Bus}_m(k,v)$ *with* $k \in \{m, m+1\}$ *can exist only when*

1. $v \equiv m \pmod 2$ *when* $m \geq 2$,
2. $v \equiv m, m+1 \pmod 3$ *when* $k \geq 3$, *and*
3. $v \equiv m, m+1, m+2, m+3 \pmod 5$ *when* $k \geq 5$.

PROOF. Apply Lemma 5.3 with $r = m$ using $s = m-2$ for the first statement, $s = m-3$ for the second statement when $m \geq 3$, and $s = m-5$ for the third statement when $m \geq 5$. Apply Lemma 5.3 with $r = m+1$ using $s = m-2$ for the second statement when $m = 2$, and $s = m-4$ for the third when $m = 4$. ◻

COROLLARY 5.5. *A proper* $\mathsf{Bus}_m(k,v)$ *with* $k \geq m+2$ *cannot exist when* m *is even. When* $m \geq 3$ *is odd and* $m \equiv 1, 5 \pmod 6$, *one can exist only if* $v \equiv m \pmod 6$. *Moreover, when* $m \geq 3$ *is odd, one can exist only if* $v \equiv m, m+2, m+3 \pmod 5$ *when* $m \equiv 1, 7, 9 \pmod{10}$, $v \equiv m, m+2, m+3, m+4 \pmod 5$ *when* $m \equiv 3 \pmod{10}$, *or* $v \equiv m, m+1, m+2, m+3 \pmod 5$ *when* $m \equiv 5 \pmod{10}$.

PROOF. When m is even, apply Lemma 5.3 with $r = m+2$ and $s = m$ to determine that $\frac{1}{2}(v-2m-1)$ is an integer, so $v \equiv 1 \pmod 2$. This contradicts Corollary 5.4(1). When $m \geq 3$ is odd, first apply Lemma 5.3 with $r = m+2$ and $s = m-1$; then $\frac{1}{6}(v-m)(v-2m-1)$ must be integral. Corollary 5.4(2) ensures that $v \equiv m, m+4 \pmod 6$, so either $m \equiv 3 \pmod 6$ or $v \equiv m \pmod 6$. Finally apply Lemma 5.3 with $r = m+2$ and $s = m-3$; then $\frac{1}{120}(v-m+3)(v-m+2)(v-m)(v-2m-1)$ must be integral. ◻

Summarizing the divisibility conditions for $2 \leq m \leq 5$, one obtains:

m	$\mathsf{Bus}_m(m,v)$	$\mathsf{Bus}_m(m+1,v)$	$\mathsf{Bus}_m(m+2,v)$
2	0 (mod 2)	0, 2 (mod 6)	$\emptyset$
3	1, 3 (mod 6)	1, 3 (mod 6)	1, 3, 7, 13, 15, 21, 25, 27 (mod 30)
4	2, 4 (mod 6)	2, 4, 10, 14, 16, 20, 22, 26 (mod 30)	$\emptyset$
5	3, 5, 11, 15, 17, 21, 23, 27 (mod 30)	3, 5, 11, 15, 17, 21, 23, 27 (mod 30)	5, 17, 23, 35, 47, 53, 65, 77, 95, 107, 113, 137, 143, 155, 173, 185, 197, 203 (mod 210)

Even for small values of m, the divisibility conditions are restrictive. Indeed, cases with even values of m cannot lead to proper Bussey systems other than $\mathsf{Bus}_m(k,v)$s with $k \in \{m, m+1\}$, which are obtained directly from Steiner systems (Lemma 5.1). Nevertheless, this generalization of Steiner's question may be fruitful for odd values of m.

5.2. *Higher index.* A Bussey system of *index* λ, $\mathsf{Bus}(k,v,\lambda)$, is a set system $(X,\mathcal{B})$, where

1. $\mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2 = \emptyset$,
2. for $2 \le t < k$, each t-subset of points containing $x \ge 0$ blocks of $\cup_{j=2}^{t}\mathcal{B}_j$ is contained in $\max(0, \lambda - x)$ blocks in $\mathcal{B}_{t+1}$.

This coincides with the definition when $\lambda = 1$, where a blocked set may contain many blocks.

We produce $\mathsf{Bus}(5,v,\lambda)$s by adapting the well-known Kramer-Mesner technique [37]. We consider only systems in which, for every t-set T with $2 \le t < k$, the number of blocks contained in T plus the number that properly contain T is equal to λ. Define the matrix $W : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}$ by

$$W[T,K] = \begin{cases} 1, & \text{if } T \subseteq K, \\ 0, & \text{otherwise,} \end{cases}$$

and for integers $0 \le t, k \le v$ let W_{tk} be the restriction of W to $\binom{X}{t} \times \binom{X}{k}$.

To construct a $\mathsf{Bus}(5,v,\lambda)$ we let

$$M = \begin{bmatrix} W_{23} & \mathbb{O}_{24} & \mathbb{O}_{25} \\ \mathrm{I}_3 & W_{34} & \mathbb{O}_{35} \\ W_{34}^{\mathsf{T}} & \mathrm{I}_4 & W_{45} \end{bmatrix},$$

where I_t is the $\binom{v}{t}$ by $\binom{v}{t}$ identity matrix, and $\mathbb{O}_{tk}$ is the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of zeros. A $\mathsf{Bus}(5,v,\lambda)$ that may have repeated blocks is found when there is a non-negative valued solution U to $MU = \lambda J$, where $J = [\underbrace{1,1,\ldots,1}_{\binom{v}{3}+\binom{v}{4}+\binom{v}{5}}]^{\mathsf{T}}$.

A permutation subgroup G of $\mathsf{Sym}(X)$ (the symmetric group on X) acts on the subsets of X in a natural way:

$$g(S) = \{g(x) : x \in X\}, \text{ for } g \in G \text{ and } S \subseteq X,$$

partitioning the subsets into orbits. Let $,\mathcal{P}(X)/G$ be the collection of G-orbits of subsets, and define the matrices $A :, \mathcal{P}(X)/G \times, \mathcal{P}(X)/G \to \mathbb{Z}$ and $B :, \mathcal{P}(X)/G \times, \mathcal{P}(X)/G \to \mathbb{Z}$ by

$$A[\Delta,\Gamma] = |\{K \in \Gamma : K \supseteq T_0\}| \text{ and } B[\Delta,\Gamma] = |\{T \in \Delta : T \subseteq K_0\}|,$$

where $T_0 \in \Delta$ and $K_0 \in \Gamma$ are any fixed representatives. Let A_{tk} and B_{tk} be the projections onto $\binom{X}{t}/G \times \binom{X}{k}/G$. Let $n_t = |\binom{X}{t}/G|$ be the number of orbits of t-subsets.

To construct a $\mathsf{Bus}(5,v,\lambda)$ with G as an automorphism group we let

$$M = \begin{bmatrix} A_{23} & \mathbb{O}_{24} & \mathbb{O}_{25} \\ \mathrm{I}_3 & A_{34} & \mathbb{O}_{35} \\ B_{34}^{\mathsf{T}} & \mathrm{I}_4 & A_{45} \end{bmatrix},$$

where I_t is the n_t by n_t identity matrix, and $\mathbb{O}_{tk}$ is the n_t by n_k matrix of zeros. A $\mathsf{Bus}(5,v,\lambda)$ with G as an automorphism group that may have repeated blocks is found when there is a non-negative valued solution U to $MU = \lambda J$, where $J = [\underbrace{1,1,\ldots,1}_{n_2+n_3+n_4}]^{\mathsf{T}}$.

Using a backtracking search we obtained the following $\mathsf{Bus}(5,v,\lambda)$s with $\lambda \in \{2,6\}$.

$\mathsf{Bus}(5,13,2)$.

Generators for automorphism group G of order 12.

```
(0,1,2,3,4,5,6,7,8,9,10,11)
```

Base blocks for a $\mathsf{Bus}(5,13,2)$ with automorphism group G

$\mathcal{B}_3$: $\{0,4,8\}$, $\{0,3,8\}$, $\{0,2,12\}$, $\{0,1,6\}$, $\{0,1,3\}$

$\mathcal{B}_4$: $\{0,4,8,12\}$, $\{0,3,7,12\}$, $\{0,2,6,8\}$, $\{0,2,4,9\}$, $\{0,2,4,7\}$, $\{0,1,10,12\}$, $\{0,1,7,12\}$, $\{0,1,5,6\}$, $\{0,1,4,10\}$, $\{0,1,3,9\}$, $\{0,1,2,9\}$, $\{0,1,2,5\}$

$\mathcal{B}_5$: $\{0,3,6,9,12\}$, $\{0,3,6,9,12\}$, $\{0,2,6,8,12\}$, $\{0,2,5,9,12\}$, $\{0,1,5,7,10\}$, $\{0,1,5,7,10\}$, $\{0,1,4,6,8\}$, $\{0,1,4,5,12\}$, $\{0,1,4,5,12\}$, $\{0,1,3,4,7\}$, $\{0,1,2,8,10\}$, $\{0,1,2,8,10\}$, $\{0,1,2,7,12\}$, $\{0,1,2,6,9\}$, $\{0,1,2,3,5\}$

$\mathsf{Bus}(5,21,6)$.

Generators for automorphism group G of order 336.

```
(0)(1,2,4,3,6,10,15,17)(5,8,14,16,20,18,13,19)(7,12,9,11)
(0,1,6)(2,3,7,17,10,11)(4,5,20,15,16,19)(8,9,18,14,12,13)
```

Base blocks for a $\mathsf{Bus}(5,21,6)$ with automorphism group G

$\mathcal{B}_3$: $\{0,5,9\}$, $\{0,1,19\}$, $\{0,1,2\}$

$\mathcal{B}_4$: $\{0,5,7,9\}$, $\{0,1,9,18\}$, $\{0,1,9,12\}$, $\{0,1,9,12\}$, $\{0,1,4,14\}$, $\{0,1,4,6\}$, $\{0,1,3,14\}$, $\{0,1,3,10\}$, $\{0,1,2,14\}$, $\{0,1,2,7\}$

$\mathcal{B}_5$: $\{0,5,7,9,20\}$, $\{0,1,9,12,13\}$, $\{0,1,6,8,9\}$, $\{0,1,4,13,14\}$, $\{0,1,4,8,20\}$, $\{0,1,4,8,20\}$, $\{0,1,4,6,15\}$, $\{0,1,3,8,14\}$, $\{0,1,2,18,19\}$, $\{0,1,2,8,16\}$, $\{0,1,2,6,13\}$, $\{0,1,2,6,13\}$, $\{0,1,2,5,12\}$, $\{0,1,2,4,9\}$, $\{0,1,2,3,15\}$, $\{0,1,2,3,4\}$

$\mathsf{Bus}(5,25,2)$. Let $f(X) = X^2 - X + 1$. Over $\mathbb{Z}_5$ the values of f are

X	0	1	2	3	4
$f(X)$	1	1	3	2	3

Hence $f(X)$ has no roots in $\mathbb{Z}_5$ and is irreducible, so $\mathbb{F}_{25} \approx \mathbb{Z}_5/(X^2 - X + 1)$ is the finite field of order 25. We identify the elements of $\mathbb{F}_{25}$ with the monomials $aX + b$, $a, b \in \mathbb{Z}_5$. Let $g = X + 1$. Place i in cell $[a, b]$ if $g^i = aX + b$ in an array

	0	1	2	3	4
0		0	18	6	12
1	20	1	3	17	16
2	14	11	19	10	21
3	2	9	22	7	23
4	8	4	5	15	13

.

Each cell (except [0,0]) is occupied by (a single) entry. Thus g generates $\mathbb{F}_{25}^*$ the multiplicative group of the field, and

$$\alpha : X \mapsto X + 1,$$
$$\beta : X \mapsto X + x,$$
$$\gamma : X \mapsto gX,$$

generates the affine group $AF(25) = \{Y \mapsto aY + b : a, b \in \mathbb{Z}_5, a \neq 0\}$.

Base blocks for a $\mathsf{Bus}(5, 25, 2)$ with automorphism group G

$\mathcal{B}_3$: $\{0, 1, g^{10}\}$

$\mathcal{B}_4$: $\{0, 1, g^{17}, g^7\}$, $\{0, 1, g^{20}, g\}$, $\{0, 1, g^{18}, g^{14}\}$

$\mathcal{B}_5$: $\{0, 1, g^{20}, g^{17}, g^{19}\}$, $\{0, 1, g^{18}, g^{10}, g^{23}\}$, $\{0, 1, g^{18}, g^{16}, g^9\}$, $\{0, 1, g^{18}, g^{16}, g^9\}$, $\{0, 1, g^{18}, g^{17}, g^{13}\}$, $\{0, 1, g^{18}, g^{17}, g^{13}\}$, $\{0, 1, g^{18}, g, g^{22}\}$, $\{0, 1, g^{18}, g^{20}, g^5\}$, $\{0, 1, g^{18}, g^{20}, g^5\}$, $\{0, 1, g^{18}, g^6, g^{12}\}$, $\{0, 1, g^{18}, g^6, g^{12}\}$

$\mathsf{Bus}(5, 27, 2)$.

Four generators for the automorphism group G of order 702.

```
(0,1,2)(3,4,5)(6,7,8)(9,10,11)(12,13,14)(15,16,17)
              (18,19,20)(21,22,23)(24,25,26)
(0,3,6)(1,4,7)(2,5,8)(9,12,15)(10,13,16)(11,14,17)
               (18,21,24)(19,22,25)(20,23,26)
(0,9,18)(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)
                (6,15,24)(7,16,25)(8,17,26)
(0)(1,6,9,8,12,26,16,14,20,7,15,17,11,2,3,18,4,24,13,
                 23,25,10,5,21,22,19)
```

Base blocks for a $\mathsf{Bus}(5, 27, 2)$ with automorphism group G

$\mathcal{B}_3$: $\{0, 1, 2\}$, $\{0, 1, 2\}$

$\mathcal{B}_4$: $\{0, 1, 3, 17\}$, $\{0, 1, 3, 17\}$

$\mathcal{B}_5$: $\{0, 1, 3, 15, 25\}$, $\{0, 1, 3, 15, 19\}$, $\{0, 1, 3, 12, 25\}$, $\{0, 1, 3, 12, 18\}$, $\{0, 1, 3, 10, 14\}$, $\{0, 1, 3, 7, 10\}$, $\{0, 1, 3, 5, 13\}$, $\{0, 1, 3, 4, 9\}$

These examples of $\mathsf{Bus}(5, v, 2)$s for $v \in \{13, 25, 27\}$ show that when $\lambda > 1$, Bussey systems with $k = 5$ can exist when v is not of the form $2^\ell - 1$. Is the current lack of such systems for $\lambda = 1$ simply because we have thus far not considered sufficiently large values of v?

6. Concluding Remarks

Our development of Steiner's problems and Bussey systems has focussed on combinatorial designs, and to a lesser extent on codes. We expect that other viewpoints can also be illuminating. Steiner's motivation was geometric, but we have not developed the connections with cubic and quartic curves here; instead we recommend an old but thorough monograph [53]. In another direction, because no block contains another, every $\mathsf{Bus}(k, v)$ is a special type of *Sperner family* [56] or *clutter*; see [31], for example.

We find it surprising that after 170 years, Steiner's problems remain almost unexplored when $k \geq 5$. Naturally the early confusion in attribution has had a lot to do with this. Nevertheless, Bussey's century-old Construction 2.1 remains the only general construction for all k. In this paper we have exploited connections with Steiner systems and with perfect binary one-error-correcting codes, together with extensive computation, to provide a wealth of $\mathsf{Bus}(5, 15)$s not arising from Bussey's construction, or even from perfect codes. Together with existence results for a higher-index analogue, these suggest that the solutions to Steiner's problem, which we call Bussey systems, admit a rich structure that warrants much further research.

References

[1] W. O. Alltop, *Extending t-designs*, J. Combinatorial Theory Ser. A **18** (1975), 177–186.
[2] I. Anderson, C. J. Colbourn, J. H. Dinitz, and T. S. Griggs, *Design theory: Antiquity to 1950*, in: Handbook of combinatorial designs, Chapman and Hall/CRC, Boca Raton, 2007, 11–22.
[3] J. E. F. Assmus Jr. and J. H. F. Mattson Jr., *On tactical configurations and error-correcting codes*, J. Combinatorial Theory **2** (1967), 243–257.
[4] W. W. R. Ball, Mathematical recreations and essays, The Macmillan Company, London, 1914.
[5] J. A. Barrau, *On the combinatory problem of Steiner*, Nederl. Akad. Wetensch. Proc. Ser. B **11** (1908), 352–360.
[6] S. Bays and E. de Weck, *Sur les systèmes de quadruples*, Comment. Math. Helv. **7** (1934), 222–241.
[7] R. A. Beezer, *Counting configurations in designs*, J. Combin. Theory Ser. A **96** (2001), 341–357.
[8] R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugenics **9** (1939), 353–399.
[9] G. Brunel, *Sur les deux systèmes de triades de treize éléments*, J. Math. Pures Appl. (5) **7** (1901) 305–330.
[10] W. H. Bussey, *On the tactical problem of Steiner*, Bull. Amer. Math. Soc. **16** (1909), 19–22.
[11] W. H. Bussey, *The tactical problem of Steiner*, Amer. Math. Monthly **21** (1914), 3–12.
[12] R. D. Carmichael, *Tactical configurations of rank two*, Amer. J. Math. **53** (1931), 217–240.

[13] L. G. Chouinard II, E. S. Kramer and D. L. Kreher, *Graphical t-wise balanced designs*, Discrete Math. **46** (1983), 227–240.
[14] C. J. Colbourn, *The configuration polytope of l-line configurations in Steiner triple systems*, Math. Slovaca **59** (2009), 77–108.
[15] C. J. Colbourn and J. H. Dinitz (eds.), Handbook of combinatorial designs, Chapman & Hall/CRC, Boca Raton, 2007.
[16] C. J. Colbourn, A. D. Forbes, M. J. Grannell, T. S. Griggs, P. Kaski, P. R. J. Östergård, D. A. Pike and O. Pottonen, *Properties of the Steiner triple systems of order 19*, Electron. J. Combin. **17** (2010), research paper 98.
[17] C. J. Colbourn and R. A. Mathon, *Steiner systems*, in: Handbook of combinatorial designs, Chapman and Hall/CRC, Boca Raton, 2007, 102–110.
[18] C. J. Colbourn and A. Rosa, Triple systems, The Clarendon Press, Oxford University Press, New York, 1999.
[19] F. N. Cole, L. D. Cummings, and H. S. White, *The complete enumeration of triad systems in 15 elements*, Proc. Natl. Acad. Sci. USA **3** (1917), 197–199.
[20] L. D. Cummings, *An undervalued Kirkman paper*, Bull. Amer. Math. Soc. **24** (1918), 336–339.
[21] V. De Pasquale, *Sui sistemi ternari di 13 elementi*, Rendiconti. Reale Istituto Lombardo di Science e Lettere, Serie II, **32** (1899), 213–221.
[22] I. Diener, E. Schmitt and H. L. de Vries, *All* 80 *Steiner triple systems on* 15 *elements are derived*, Discrete Math. **55** (1985), 13–19.
[23] C. Ehrhardt, *Tactics: in search of a long-term mathematical project (1844–1896)*, Historia Math. **42** (2015), 436–467.
[24] M. J. E. Golay, *Notes on digital coding*, Proc. I.R.E. **37** (1949), 657.
[25] M. J. Grannell, T. S. Griggs and R. A. Mathon, *Steiner systems $S(5,6,v)$ with $v = 72$ and* 84, Math. Comp. **67** (1998), 357–359, S1–S9.
[26] R. W. Hamming, *Error detecting and error correcting codes*, Bell System Tech. J. **29** (1950), 147–160.
[27] H. Hanani, *On quadruple systems*, Canadian J. Math. **12** (1960), 145–157.
[28] H. Hanani, *On the original Steiner systems*, Discrete Math. **51** (1984), 309–310.
[29] H. Hanani and J. Schönheim, *On Steiner systems*, Israel J. Math. **2** (1964), 139–142 and **4** (1966), 144.
[30] O. Heden, *A survey of perfect codes*, Adv. Math. Commun. **2** (2008), 223–247.
[31] S. Jukna, Extremal combinatorics, Springer, Heidelberg, 2011.
[32] P. Kaski and P. R. J. Östergård, *The Steiner triple systems of order 19*, Math. Comp. **73** (2004), 2075–2092.
[33] P. Kaski, P. R. J. Östergård and O. Pottonen, *The Steiner quadruple systems of order 16*, J. Combin. Theory Ser. A **113** (2006), 1764–1770.
[34] P. Kaski and O. Pottonen, `libexact` user's guide, version 1.0., Technical Report 2008-1, Helsinki Institute for Information Technology, 2008.
[35] P. Keevash, *The existence of designs*, 2014, arXiv:1401.3665.
[36] T. P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Mathematical Journal **2** (1847), 191–204.
[37] E. S. Kramer and D. M. Mesner, *t-designs on hypergraphs*, Discrete Math. **15** (1976), 263–296.
[38] K. A. Lauinger, D. L. Kreher, R. Rees and D. R. Stinson, *Computing transverse t-designs*, J. Combin. Math. Combin. Comput. **54** (2005), 33–56.
[39] B. D. McKay, `nauty` user's guide (version 1.5), Report TR-CS-90-02, Computer Science, Australian National University, 1990.
[40] N. S. Mendelsohn, *A theorem on Steiner systems*, Canadian J. Math. **22** (1970), 1010–1015.

[41] N. S. Mendelsohn and S. H. Y. Hung, *On the Steiner systems* $S(3, 4, 14)$ *and* $S(4, 5, 15)$, Utilitas Math. **1** (1972), 5–95.
[42] E. H. Moore, *Tactical Memoranda I–III*, Amer. J. Math. **18** (1896), 264–290.
[43] E. Netto, Lehrbuch der Combinatorik, Teubner, Leipzig, 1927.
[44] P. R. J. Östergård and O. Pottonen, *There exist Steiner triple systems of order 15 that do not occur in a perfect binary one-error-correcting code*, J. Combin. Des. **15** (2007), 465–468.
[45] P. R. J. Östergård and O. Pottonen, *There exists no Steiner system* $S(4, 5, 17)$, J. Combin. Theory Ser. A **115** (2008), 1570–1573.
[46] P. R. J. Östergård and O. Pottonen, *The perfect binary one-error-correcting codes of length 15: Part I. Classification*, IEEE Trans. Inform. Theory **55** (2009), 4657–4660.
[47] P. R. J. Östergård, O. Pottonen and K. T. Phelps, *The perfect binary one-error-correcting codes of length 15: Part II. Properties*, IEEE Trans. Inform. Theory **56** (2010), 2571–2582.
[48] K. T. Phelps, *A combinatorial construction of perfect codes*, SIAM J. Algebraic Discrete Methods **4** (1983), 398–403.
[49] K. T. Phelps, *A general product construction for error correcting codes*, SIAM J. Algebraic Discrete Methods **5** (1984), 224–228.
[50] M. Plotkin, *Binary codes with specified minimum distance*, IRE Trans. **IT-6** (1960), 445–450.
[51] J. Plücker, System der analytischen Geometrie, auf neue Betrachtungsweisen gegründet, und insbesondere eine ausfḧrliche Theorie der Curven dritter Ordnung enthalend, Duncker und Humboldt, Berlin, 1835.
[52] J. Plücker, Theorie der algebraischen Curven, gegründet auf eine meue Behandlungensweise der analytischen Geometrie, Marcus, Bonn, 1839.
[53] G. Salmon, A treatise on the higher plane curves, Hodges, Foster, and Figgis, Dublin, 1879.
[54] N. J. A. Sloane and D. S. Whitehead, *New family of single-error correcting codes*, IEEE Trans. Inform. Theory **IT-16** (1970), 717–719.
[55] F. I. Solov'eva, *Binary nongroup codes*, Metody Diskret. Analiz. **37** (1981), 65–76, 86.
[56] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Z. **27** (1928), 544–548.
[57] J. Steiner, *Combinatorische Aufgaben*, J. Reine Angew. Math. **45** (1853), 181–182.
[58] J. Steiner, *Allgemeine Eigenschaften der algebraischen*, Curven, J. Reine Angew. Math. **47** (1854), 1–6.
[59] J. Steiner, *Eigenschaften der Curven vierten Grades rücksichtlich ihrer Doppeltangenten*, J. Reine Angew. Math. **49** (1855), 265–272.
[60] D. R. Stinson and Y. J. Wei, *Some results on quadrilaterals in Steiner triple systems*, Discrete Math. **105** (1992), 207–219.
[61] J. J. Sylvester, *Elementary researches in the analysis of combinatorial aggregation*, Phil. Mag. (3) **24** (1844), 285–296.
[62] J. H. van Lint, *A survey of perfect codes*, Rocky Mountain J. Math. **5** (1975), 199–224.
[63] J. L. Vasil'ev, *On nongroup close-packed codes*, Problemy Kibernet. **8** (1962), 337–339, (in Russian). English translation in Probleme der Kybernetik **8** (1965), 375–378.
[64] O. Veblen and J. W. Young, Projective geometry, Ginn and Co., Boston, 1916.
[65] E. Witt, *Die 5-fach transitiven gruppen von Mathieu*, Abh. Math. Sem. Univ. Hamburg **12** (1937), 256–264.
[66] E. Witt, *Über Steinersche Systeme*, Abh. Math. Sem. Univ. Hamburg **12** (1937), 265–275.
[67] W. S. B. Woolhouse, Prize question 1733, Lady's and Gentleman's Diary, 1844.

C. J. Colbourn
Arizona State University
Tempe, Arizona
USA
E-mail: Charles.Colbourn@asu.edu

D. L. Kreher
Michigan Technological University
Houghton, Michigan
USA
E-mail: Kreher@mtu.edu

P. R. J. Östergård
Department of Information and Communications Engineering
Aalto University School of Electrical Engineering
00076 Aalto
Finland
E-mail: patric.ostergard@aalto.fi

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