GLASNIK MATEMATIČKI
Vol. 58(78)(2023), 247 – 257

BUSH-TYPE BUTSON HADAMARD MATRICES

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This paper is dedicated to the memory of Professor Zvonimir Janko

Abstract. Bush-type Butson Hadamard matrices are introduced. It is shown that a nonextendable set of mutually unbiased Butson Hadamard matrices is obtained by adding a specific Butson Hadamard matrix to a set of mutually unbiased Bush-type Butson Hadamard matrices. A class of symmetric Bush-type Butson Hadamard matrices over the group G of n-th roots of unity is introduced that is also valid over any subgroup of G. The case of Bush-type Butson Hadamard matrices of even order will be discussed.

1. Introduction

A Hadamard matrix, say H, is a square matrix of order n with entries from the set $\{-1, +1\}$ such that $HH^t = nI$. There is a great deal of interest in these matrices owing to their growing number of applications in fields as diverse as error-correcting codes (as used in the 1972 Mariner mission, for example) and modern CDMA cellphones (the Walsh transform). The interested reader may profitably consult sources such as Horadam [12] and Seberry [17], together with the references cited therein, for further discussion of the applications of these most useful objects.

In this paper, we will consider Hadamard matrices whose entries are taken from a larger set of values, namely, the roots of unity residing along the unit circle. Hadamard matrices whose entries are roots of unity are termed Butson Hadamard. Additionally, we will require the matrices studied here to be of

2020 *Mathematics Subject Classification.* 05B20.

Key words and phrases. Hadamard matrix, unbiased Hadamard matrix, Bush-type Hadamard matrix.

Bush-type, that is, they will have square order n^2 and be divided into n^2 blocks of order n which are either all ones or have row and column sum equal to 0. Butson Hadamard matrices were first studied by Butson [5,6] and Shrikhande [18], while Bush-type Hadamard matrices were first introduced by Bush [3,4]. For these and related structural constraints on Hadamard matrices, the reader may consult Colbourn and Dinitz [7], the standard reference of the field.

The remainder of this note is organized as follows. Sec. 2 recapitulates the necessary definitions and elementary results needed for the main constructions of this paper. Sec. 3 goes on to introduce the ω-circulant Bush-type Butson Hadamard matrices, a generalization of the negacirculant Bush-type Hadamard matrices first considered by Janko and Kharaghani [14].[1] The penultimate Sec. 4 of the main body of this work establishes the existence of families of unbiased Butson Hadamard matrices which are maximal in the sense that the set cannot be enlarged to a proper superset. Finally, the concluding Sec. 5 explores the use of generalized Hadamard matrices in the construction of symmetric Bush-type Hadamard matrices.

2. Preliminaries

A *Butson Hadamard matrix* is a square matrix, say H, of order n whose entries are from the m-th complex roots of unity such that $HH^* = I$. We denote this as $\mathrm{BH}(n, m)$.

Evidently, there is a Butson Hadamard matrix of every order n upon considering the matrix of the discrete Fourier transform, namely, $H = (\exp(2\pi n^{-1}\sqrt{-1}ij))_{i,j=0}^{n-1}$.

Example 2.1. Let $\xi = (1+\sqrt{-3})/2$. Following the construction intimated above, we obtain a $\mathrm{BH}(6, 6)$ given by

$$H_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \xi^3 & \xi^4 & \xi^5 \\ 1 & \xi^2 & \xi^4 & 1 & \xi^2 & \xi^4 \\ 1 & \xi^3 & 1 & \xi^3 & 1 & \xi^3 \\ 1 & \xi^4 & \xi^2 & 1 & \xi^4 & \xi^2 \\ 1 & \xi^5 & \xi^4 & \xi^3 & \xi^2 & \xi \end{pmatrix}.$$

A *Bush-type (Butson) Hadamard matrix*, say H, is a Butson Hadamard matrix of order n^2 over the m-th complex roots of unity which is subdivided into n^2 blocks $H_{11}, H_{12}, \ldots, H_{nn}$ of order n such that $JH_{ij} = H_{ij}J = \delta_{ij}nJ$.

[1] A structure similar to that of being negacirculant was also considered by Ionin and Kharaghani [13] in the construction of doubly regular asymmetric digraphs.

EXAMPLE 2.2. Continuing to let $\xi = (1+\sqrt{-3})/2$, we have a Bush-type $\mathrm{BH}(36,6)$ given by

$$\setcounter{MaxMatrixCols}{36}\begin{pmatrix}
1&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3&1&\xi^4&\xi^2&1&\xi^4&\xi^2&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi\\
1&1&1&1&1&1&\xi^5&1&\xi&\xi^2&\xi^3&\xi^4&\xi^4&1&\xi^2&\xi^4&1&\xi^2&\xi^3&1&\xi^3&1&\xi^3&1&\xi^2&1&\xi^4&\xi^2&1&\xi^4&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2\\
1&1&1&1&1&1&\xi^4&\xi^5&1&\xi&\xi^2&\xi^3&\xi^2&\xi^4&1&\xi^2&\xi^4&1&1&\xi^3&1&\xi^3&1&\xi^3&\xi^4&\xi^2&1&\xi^4&\xi^2&1&\xi^2&\xi&1&\xi^5&\xi^4&\xi^3\\
1&1&1&1&1&1&\xi^3&\xi^4&\xi^5&1&\xi&\xi^2&1&\xi^2&\xi^4&1&\xi^2&\xi^4&\xi^3&1&\xi^3&1&\xi^3&1&1&\xi^4&\xi^2&1&\xi^4&\xi^2&\xi^3&\xi^2&\xi&1&\xi^5&\xi^4\\
1&1&1&1&1&1&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^4&1&\xi^2&\xi^4&1&\xi^2&1&\xi^3&1&\xi^3&1&\xi^3&\xi^2&1&\xi^4&\xi^2&1&\xi^4&\xi^4&\xi^3&\xi^2&\xi&1&\xi^5\\
1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3&1&\xi^4&\xi^2&1&\xi^4&\xi^2&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1\\
\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&1&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3&1&\xi^4&\xi^2&1&\xi^4&\xi^2\\
\xi^2&\xi&1&\xi^5&\xi^4&\xi^3&1&1&1&1&1&1&\xi^5&1&\xi&\xi^2&\xi^3&\xi^4&\xi^4&1&\xi^2&\xi^4&1&\xi^2&\xi^3&1&\xi^3&1&\xi^3&1&\xi^2&1&\xi^4&\xi^2&1&\xi^4\\
\xi^3&\xi^2&\xi&1&\xi^5&\xi^4&1&1&1&1&1&1&\xi^4&\xi^5&1&\xi&\xi^2&\xi^3&\xi^2&\xi^4&1&\xi^2&\xi^4&1&1&\xi^3&1&\xi^3&1&\xi^3&\xi^4&\xi^2&1&\xi^4&\xi^2&1\\
\xi^4&\xi^3&\xi^2&\xi&1&\xi^5&1&1&1&1&1&1&\xi^3&\xi^4&\xi^5&1&\xi&\xi^2&1&\xi^2&\xi^4&1&\xi^2&\xi^4&\xi^3&1&\xi^3&1&\xi^3&1&1&\xi^4&\xi^2&1&\xi^4&\xi^2\\
\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&1&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^4&1&\xi^2&\xi^4&1&\xi^2&1&\xi^3&1&\xi^3&1&\xi^3&\xi^2&1&\xi^4&\xi^2&1&\xi^4\\
1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3&1&\xi^4&\xi^2&1&\xi^4&\xi^2&1\\
\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&1&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3\\
\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^2&\xi&1&\xi^5&\xi^4&\xi^3&1&1&1&1&1&1&\xi^5&1&\xi&\xi^2&\xi^3&\xi^4&\xi^4&1&\xi^2&\xi^4&1&\xi^2&\xi^3&1&\xi^3&1&\xi^3&1\\
\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^3&\xi^2&\xi&1&\xi^5&\xi^4&1&1&1&1&1&1&\xi^4&\xi^5&1&\xi&\xi^2&\xi^3&\xi^2&\xi^4&1&\xi^2&\xi^4&1&1&\xi^3&1&\xi^3&1&\xi^3\\
\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi^4&\xi^3&\xi^2&\xi&1&\xi^5&1&1&1&1&1&1&\xi^3&\xi^4&\xi^5&1&\xi&\xi^2&1&\xi^2&\xi^4&1&\xi^2&\xi^4&\xi^3&1&\xi^3&1&\xi^3&1\\
\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&1&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^4&1&\xi^2&\xi^4&1&\xi^2&1&\xi^3&1&\xi^3&1&\xi^3\\
\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1&\xi^3&1&\xi^3&1&\xi^3&1\\
\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&1&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4\\
\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^2&\xi&1&\xi^5&\xi^4&\xi^3&1&1&1&1&1&1&\xi^5&1&\xi&\xi^2&\xi^3&\xi^4&\xi^4&1&\xi^2&\xi^4&1&\xi^2\\
\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^3&\xi^2&\xi&1&\xi^5&\xi^4&1&1&1&1&1&1&\xi^4&\xi^5&1&\xi&\xi^2&\xi^3&\xi^2&\xi^4&1&\xi^2&\xi^4&1\\
\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi^4&\xi^3&\xi^2&\xi&1&\xi^5&1&1&1&1&1&1&\xi^3&\xi^4&\xi^5&1&\xi&\xi^2&1&\xi^2&\xi^4&1&\xi^2&\xi^4\\
\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&1&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^4&1&\xi^2&\xi^4&1&\xi^2\\
\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi^2&\xi^4&1&\xi^2&\xi^4&1\\
\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&1&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5\\
\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^2&\xi&1&\xi^5&\xi^4&\xi^3&1&1&1&1&1&1&\xi^5&1&\xi&\xi^2&\xi^3&\xi^4\\
\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^3&\xi^2&\xi&1&\xi^5&\xi^4&1&1&1&1&1&1&\xi^4&\xi^5&1&\xi&\xi^2&\xi^3\\
\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi^4&\xi^3&\xi^2&\xi&1&\xi^5&1&1&1&1&1&1&\xi^3&\xi^4&\xi^5&1&\xi&\xi^2\\
\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&1&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi\\
\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&1\\
\xi&\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&1&1&1&1&1&1\\
1&\xi&\xi^2&\xi^3&\xi^4&\xi^5&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^2&\xi&1&\xi^5&\xi^4&\xi^3&1&1&1&1&1&1\\
\xi^5&1&\xi&\xi^2&\xi^3&\xi^4&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^3&\xi^2&\xi&1&\xi^5&\xi^4&1&1&1&1&1&1\\
\xi^4&\xi^5&1&\xi&\xi^2&\xi^3&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi^4&\xi^3&\xi^2&\xi&1&\xi^5&1&1&1&1&1&1\\
\xi^3&\xi^4&\xi^5&1&\xi&\xi^2&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi^3&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1&1\\
\xi^2&\xi^3&\xi^4&\xi^5&1&\xi&\xi^3&\xi^5&\xi&\xi^3&\xi^5&\xi&\xi^4&\xi&\xi^4&\xi&\xi^4&\xi&\xi^5&\xi^3&\xi&\xi^5&\xi^3&\xi&1&\xi^5&\xi^4&\xi^3&\xi^2&\xi&1&1&1&1&1&1
\end{pmatrix}.$$

The Bush-type Butson Hadamard matrix of the previous example has an additional structure that we will attend to in the next section, namely, it is block ω-circulant (in this case, $\omega = \xi$).

Two $\mathrm{BH}(n^2, m)$s H_1 and H_2 are *unbiased* if $n^{-1}H_1H_2^*$ is also a $\mathrm{BH}(n^2, m)$. A collection $\{H_1, \ldots, H_s\}$ of $\mathrm{BH}(n^2, m)$s is mutually unbiased in the event that each pair $\{H_1, H_2\}$ is unbiased.

Example 2.3. It can be checked that the following is a pair of unbiased Bush-type BH$(16, 2)$s.

$$\begin{pmatrix}
1&1&1&1&1&1&-&-&1&-&1&-&1&-&-&1\\
1&1&1&1&1&1&-&-&-&1&-&1&-&1&1&-\\
1&1&1&1&-&-&1&1&1&-&1&-&-&1&1&-\\
1&1&1&1&-&-&1&1&-&1&-&1&1&-&-&1\\
1&1&-&-&1&1&1&1&1&-&-&1&1&-&1&-\\
1&1&-&-&1&1&1&1&-&1&1&-&-&1&-&1\\
-&-&1&1&1&1&1&1&-&1&1&-&1&-&1&-\\
-&-&1&1&1&1&1&1&1&-&-&1&-&1&-&1\\
1&-&1&-&1&-&-&1&1&1&1&1&1&1&-&-\\
-&1&-&1&-&1&1&-&1&1&1&1&1&1&-&-\\
1&-&1&-&-&1&1&-&1&1&1&1&-&-&1&1\\
-&1&-&1&1&-&-&1&1&1&1&1&-&-&1&1\\
1&-&-&1&1&-&1&-&1&1&-&-&1&1&1&1\\
-&1&1&-&-&1&-&1&1&1&-&-&1&1&1&1\\
-&1&1&-&1&-&1&-&-&-&1&1&1&1&1&1\\
1&-&-&1&-&1&-&1&-&-&1&1&1&1&1&1
\end{pmatrix}, \quad
\begin{pmatrix}
1&1&1&1&1&-&1&-&1&-&-&1&1&1&-&-\\
1&1&1&1&-&1&-&1&-&1&1&-&1&1&-&-\\
1&1&1&1&1&-&1&-&-&1&1&-&-&-&1&1\\
1&1&1&1&-&1&-&1&1&-&-&1&-&-&1&1\\
1&-&1&-&1&1&1&1&1&1&-&-&1&-&-&1\\
-&1&-&1&1&1&1&1&1&1&-&-&-&1&1&-\\
1&-&1&-&1&1&1&1&-&-&1&1&-&1&1&-\\
-&1&-&1&1&1&1&1&-&-&1&1&1&-&-&1\\
1&-&-&1&1&1&-&-&1&1&1&1&1&-&1&-\\
-&1&1&-&1&1&-&-&1&1&1&1&-&1&-&1\\
-&1&1&-&-&-&1&1&1&1&1&1&1&-&1&-\\
1&-&-&1&-&-&1&1&1&1&1&1&-&1&-&1\\
1&1&-&-&1&-&-&1&1&-&1&-&1&1&1&1\\
1&1&-&-&-&1&1&-&-&1&-&1&1&1&1&1\\
-&-&1&1&-&1&1&-&1&-&1&-&1&1&1&1\\
-&-&1&1&1&-&-&1&-&1&-&1&1&1&1&1
\end{pmatrix}.$$

Finally, a Butson Hadamard matrix is said to be *normalized* if the first row and column consist entirely of ones.

3. ω-Circulant Bush-Type Matrices

The construction of real symmetric Bush-type Hadamard matrices was initiated by Bush [10] when he constructed symmetric Bush-type Hadamard matrices of order $4n^2$ from a projective plane of order $2n$. The expectation was to show, for example, there is no symmetric Bush-type Hadamard matrix of order 100 and thus no projective plane of order 10. Wallis [20] used mutually orthogonal latin squares in a circuitous route via design graphs, constructing many symmetric Bush-type Hadamard matrices of order $16n^2$. Further, it was shown there that if a Hadamard matrix of order n exists, then a Bush-type Hadamard matrix of order n^2 exists by an application of affine resolvable designs.

Best and Kharaghani [1] and Holzmann et al. [11] simplified the construction of Bush-type Hadamard matrices considerably by using the auxiliary matrices corresponding to a Hadamard matrix. Furthermore, appealing to the so-called mutually suitable latin squares, one may construct sets of mutually unbiased Hadamard matrices. We define these objects now.

Given a normalized BH(n, m), label the rows consecutively as $r_0, \ldots, r_{n-1}$. Then the *auxiliary matrices* of the matrix are the projection matrices $c_i = r_i^* r_i$ $(i = 0, \ldots, n-1)$ corresponding to each row. In [15] Kharaghani showed the following result.

Lemma 3.1. *The auxiliary matrices $c_0, \ldots, c_{n-1}$ of a BH(n, m) satisfy*

(1) $c_i^* = c_i$,
(2) $c_i c_j = \delta_{ij} n c_i$, *and*
(3) $\sum_i c_i = nI$.

Recall that a *latin square* of side n is an $n \times n$ matrix whose rows and columns are permutations of an n-set. Clearly, if there is a symmetric latin square of side n with constant diagonal over the auxiliary matrices of a $\mathrm{BH}(n,m)$, then there is a Hermitian Bush-type $\mathrm{BH}(n^2,m)$. We can, in fact, impose additional structure on the matrix.

THEOREM 3.2. *Let $c_0,\dots,c_{n-1}$ be the auxiliary matrices of a $BH(n,m)$, and let L be any latin square of side n. Then $(\xi_{ij}c_{L_{ij}})_{i,j=0}^{n-1}$, where ξ_{ij} is any complex m'-th root of unity, is a Butson Hadamard matrix of order n^2 over the M-th roots of unity with $M = lcm(m,m')$.*

PROOF OF THEOREM 3.2. Let $R_0,\dots,R_{n-1}$ be the block rows of the constructed matrix. Then, for $i \neq j$, we have $R_iR_j^* = \sum_k \xi_{ik}\xi_{jk}^{-1}c_{L_{ik}}c_{L_{jk}} = O$ and

$$R_iR_i^* = \sum_k \xi_{ik}\xi_{ik}^{-1}c_{L_{ik}}c_{L_{ik}} = n\sum_k c_{ik} = n^2 I.$$

This concludes the proof. ∎

Let A be a matrix over the complex m-th roots of unity, and let ω be a primitive m-th root. If A has first row $(a_0,\dots,a_{n-1})$, then A is *ω-circulant* in the event that $A_{ij} = a_{(j-i)\pmod n}$ if $j \geqq i$, and $A_{ij} = \omega a_{(j-i)\pmod n}$ if $j < i$. A is block ω-circulant if each a_i is a matrix (or block). Note that towards simplicity, we will usually abstain from including the descriptor *block*. We then have the following result.

COROLLARY 3.3. *If there is a $BH(n,m)$, then there is an ω-circulant Bush-type $BH(n^2,m)$, where ω is a primitive m-th root of unity.*

PROOF OF COROLLARY 3.3. $L = \mathrm{circ}(0,\dots,n-1)$ is a latin square with constant diagonal. Then $(\xi_{ij}c_{L_{ij}})$, where $c_0,\dots,c_{n-1}$ are the auxiliary matrices of the $\mathrm{BH}(n,m)$, and where $\xi_{ij} = 1$ if $j \geqq i$ and $\xi_{ij} = \omega$ if $j < i$, is the required matrix. ∎

Two latin squares of the same side are *orthogonal* in the event that in the superimposition of one square over the other, every ordered pair of the alphabet appears precisely once. A collection of latin squares of the same side is mutually orthogonal if every pair of squares is orthogonal.

Two latin squares of the same side are *suitable* in the event that the superimposition of a row of one square over any row of the other contains precisely one ordered pair in which the abscissa and ordinate coincide. A collection of latin squares of the same side are mutually suitable if every pair is suitable.

Evidently, orthogonality and suitability of latin squares are equivalent concepts (see Holzmann et al. [11]).

Proposition 3.4. *Given a latin square L, form the matrix whose (k, j)-th entry is i if and only if the (i, j)-th entry of L is k. This defines a bijection between sets of mutually orthogonal and mutually suitable latin squares of the same side.*

Corollary 3.5. *For every $n > 2$ with $n \neq 6$, there exists a pair of suitable latin squares.*

Corollary 3.6. *If q is a prime power, there is a complete set of mutually suitable latin squares of side q consisting of $q-1$ matrices.*

It is a straightforward exercise to construct the squares of the previous corollary directly. Indeed, take $\mathbf{F} = \{x_0 = 0, x_1, \ldots, x_{q-1}\}$ to be the Galois field of q elements. For each $s \neq 0$, define L_{x_s} by $L_{x_{s_{ij}}} = x_s(x_i - x_j)$. Then $\{L_{x_s} : s = 1, \ldots, q-1\}$ is a collection of mutually suitable latin squares.

If L_1 and L_2 are two suitable latin squares, then we define their product $L_1 \circ L_2$ by taking the (i, j)-th entry to be the point of agreement between the i-th row of L_1 and the j-th row of L_2. Clearly, $L_1 \circ L_2$ is again a latin square.

Using Lemma 3.1 and Theorem 3.2, we have the following theorem.[2]

Theorem 3.7. *For $n > 2$ with $n \neq 6$, if there is a BH(n, m), then there are at least two mutually unbiased Bush-type Hadamard matrices of order n^2. Furthermore, if n is a prime power, then there are $n - 1$ mutually unbiased ω-circulant Bush-type Hadamard matrices.*

Remark 3.8. In general, it is known that there are at most n mutually unbiased Hadamard matrices of order n. The reader may consult the comprehensive reference Durt et al. [9] for this and closely related topics. Here we have constructed a collection of unbiased Hadamard matrices of a particular block form that can never meet this optimal bound.

4. New Sets of Unbiased Bush-Type Butson Hadamard Matrices

In the previous section, it was shown that given a BH(n, m), we can construct families of mutually unbiased Bush-type Butson Hadamard matrices using the mutually suitable latin squares. In the cases of 2-nd and 4-th roots of unity, it is shown by Holzmann et al. [11] and Best and Kharaghani [1] that we may add another matrix not of Bush-type which is unbiased with each of the previous matrices. We can apply this result to the general Butson matrices as follows.

Theorem 4.1. *If there is a BH(n, m), and if there are ℓ mutually suitable latin squares of side n, then there are $\ell + 1$ mutually unbiased BH(n^2, m)s.*

[2]Special versions of this theorem were given in the aforementioned articles. We include it here in its full generality.

Proof of Theorem 4.1. Let $L_1, \ldots, L_\ell$ be the mutually suitable latin squares over the auxiliary matrices of a $\mathrm{BH}(n, m)$. Then there are mutually unbiased $\mathrm{BH}(n^2, m)$s.

If $r_0, \ldots, r_{n-1}$ are the rows of the assumed $\mathrm{BH}(n, m)$, then form the matrix K by $K_{ij} = r_j^* r_i$. Since $(r_u^* r_u)(r_j^* r_i)^* = \delta_{ui} n r_u^* r_j$, it follows that K is unbiased with each of $L_1, \ldots, L_\ell$. □

We now show that the set of unbiased matrices constructed in the previous theorem is maximal, in the sense that the set cannot be enlarged to a proper superset, in the case that $n = m$ is a prime power and $\ell = n - 1$. Indeed, let $q = p^n$ be any prime power, and let $Q = \{x \in \mathbf{C} : x^q = 1\}$. If S is any multiset consisting of elements of Q, then the main theorem of Lam and Leung [16] shows that $\sum_{\alpha \in S} \alpha = 0$ only if $|S| \equiv 0 \pmod{p}$. Thus we have the following result.

Lemma 4.2. *If S and T are any multisets over elements of Q, then $\sum_{\alpha \in S} \alpha = \sum_{\beta \in T} \beta$ only if $|S|^2 \equiv |T|^2 \pmod{p}$.*

Proof of Lemma 4.2. Let $s = |S|$ and $t = |T|$. Take $R = s \cdot Q$, that is, R is the multiset containing s copies of every element of Q; then $\sum_{r \in R} r = 0$. Now, remove from R the elements of S (taking into account multiplicities), and add a further s copies of the elements of T. Call this new multiset R'. Since $\sum_{\alpha \in S} \alpha = \sum_{\beta \in T} \beta$, it follows that $\sum_{r' \in R'} r' = 0$ so that $|R'| \equiv 0 \pmod{p}$. However, we also have that $|R'| = sq - s^2 + st$, whereupon $s^2 \equiv st \pmod{p}$. Swapping S and T, we find that $t^2 \equiv st \pmod{p}$, and the result follows. □

For a given prime power, let $L_1, \ldots, L_{q-1}$, and K be the mutually unbiased $\mathrm{BH}(q^2, q)$s obtained from Theorem 4.1 using the discrete Fourier transform matrix of order q. Choose the first row from each matrix, and label them $\ell_1, \ldots, \ell_{q-1}$, and k (note that $k = \mathbf{1}$).

We have that $|\langle x, y \rangle| = q$, for any distinct $x, y \in \{\ell_1, \ldots, \ell_{q-1}, k\}$. Assume that we can add a further vector v over Q such that $|\langle x, v \rangle| = q$, for every $x \in \{\ell_1, \ldots, \ell_{q-1}, k\}$. We write $v = (r_1, \ldots, r_q)$, where each $r_i = (y_{i1}, \ldots, y_{iq})$ has length q. We can then assume the following situation corresponding to the rows $\ell_0, \ldots, \ell_{q-1}$, and k

$$
\begin{array}{ccc|cccc|c|cccc}
 & r_1 & & & r_2 & & & \cdots & & r_q & & \\
1 & \ldots & 1 & 1 & * & \ldots & * & \cdots & 1 & * & \ldots & * \\
1 & \ldots & 1 & 1 & * & \ldots & * & \cdots & 1 & * & \ldots & * \\
 & \vdots & & & \vdots & & & & & \vdots & & \\
1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \cdots & 1 & 1 & \ldots & 1
\end{array}
$$

where the bottom row corresponds to k and the first to v. First taking the inner product of v with respect to each of $\ell_1, \ldots, \ell_{q-1}, k$, and then adding, we obtain $qz_1 + \cdots qz_q$ where each $z_i \in Q$. Next, multiplying with respect to the compartments and adding, we obtain $qy_{11} + \cdots qy_{1q} + qy_{21} + qy_{31} + \cdots + qy_{q1}$ where each $y_{ij} \in Q$. We then have that

$$y_{11} + \cdots y_{1q} + y_{21} + y_{31} + \cdots + y_{q1} = z_1 + \cdots + z_q.$$

The left summand contains $2q - 1$ q-th roots of unity, while the right summand contains q q-th roots of unity. As $(2q-1)^2 \not\equiv q^2 \pmod p$, we have a contradiction. Therefore the following result holds.

THEOREM 4.3. *For every prime power q there is a nonextendable set of q mutually unbiased $BH(q^2, q)$s.*

5. Application of Generalized Hadamard Matrices

Inspired by a result of Verheiden [19], it will be shown that Bush-type Butson Hadamard matrices are also constructible from generalized Hadamard matrices. A *generalized Hadamard matrix* is a square matrix $H = [H_{ij}]$ of order n over an additive group G such that the multisets $S_{ij} = \{H_{ik} - H_{jk} : 0 \leqq k < n\}$, for $0 \leqq i < j < n$, each contain λ copies of every group element in G. We then write H is a $\mathrm{GH}(G, \lambda)$.[3]

Drake [8] showed that for a prime power q there is a symmetric $\mathrm{GH}(G, 1)$ over any elementary abelian group G of order q. It is also known that from a $\mathrm{GH}(G, 1)$ a *group divisible design* $\mathrm{GDD}(q^2, q, q, q, 0, 1)$ is constructible by simply representing the group elements by their linear permutation representation (see Beth et al. [2] for the necessary definitions). Such a GDD consists of q^2 disjoint permutation matrix blocks of size q. Moreover, continuing to let H be our group matrix, one easily sees that

$$HH^t = H^tH = qI_{q^2} - I_q \otimes J_q + J_q \otimes J_q.$$

For the prime power q, let ξ be a primitive q-th root of unity, and let

$$D = \mathrm{diag}(I_q, \xi I_q, \xi^2 I_q, \cdots, \xi^{q-1} I_q),$$

be a block-diagonal matrix of order q^2. We then have the following.

THEOREM 5.1. *Let H be a $GH(G, 1)$ over an abelian group G of prime power order q viewed as the $(0, 1)$ incidence matrix of a $GDD(q^2, q, q, q, 0, 1)$. Then $K = HDH^t + I_q \otimes J_q$ is a symmetric Bush-type $BH(q^2, q)$.*

PROOF OF THEOREM 5.1. From $HH^t = H^tH = qI_{q^2} - I_q \otimes J_q + J_q \otimes J_q$ it follows that

$$HD(H^tH)D^*H^t = qI_{q^2}HH^t - H(I_q \otimes J_q)H^t + HD(J_q \otimes J_q)(HD)^*.$$

Noting that $H(I_q \otimes J_q)H^t = J_q \otimes J_q$ and $(HDH^t)(J \otimes J) = 0$, it follows that

$$KK^* = (HDH^t + I_q \otimes J_q)(HDH^t + I_q \otimes J_q)^* = q^2 I_{q^2}.$$

[3]These matrices are also referred to as difference matrices in the literature.

Since

$$(HDH^t + I_q \otimes J_q)^t = HDH^t + I_q \otimes J_q,$$

K is symmetric. It can be seen that the off-diagonal blocks have zero row and column sum. ☐

Remark 5.2. For a prime p, Winterhof [21] has shown that a $\mathrm{BH}(n, p)$ over a group G of the p-th roots of unity is also a $\mathrm{GH}(G, p)$. The $\mathrm{BH}(36, 6)$ of Example 2 is not a $\mathrm{GH}(G, 6)$ over the group G of the 6-th roots of unity.

The construction method for Theorem 5.1 has the extra property that:

Corollary 5.3. *Theorem 5.1 is valid for any subgroup of the group of q-th roots of unity and the matrix K is a $GH(G, q)$ over the group G of the q-th roots of unity.*

Example 5.4. Applying Theorem 5.1 to the group C_4 of the 4-th roots of unity, and for a normalized $\mathrm{GH}(C_4, 1)$, we obtain a Bush-type $\mathrm{GH}(C_4, 4)$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & - & i & j & 1 & i & j & - & 1 & j & - & i \\
1 & 1 & 1 & 1 & - & 1 & j & i & i & 1 & - & j & j & 1 & i & - \\
1 & 1 & 1 & 1 & i & j & 1 & - & j & - & 1 & i & - & i & 1 & j \\
1 & 1 & 1 & 1 & j & i & - & 1 & - & j & i & 1 & i & - & j & 1 \\
1 & - & i & j & 1 & 1 & 1 & 1 & 1 & j & - & i & 1 & i & j & - \\
- & 1 & j & i & 1 & 1 & 1 & 1 & j & 1 & i & - & i & 1 & - & j \\
i & j & 1 & - & 1 & 1 & 1 & 1 & - & i & 1 & j & j & - & 1 & i \\
j & i & - & 1 & 1 & 1 & 1 & 1 & i & - & j & 1 & - & j & i & 1 \\
1 & i & j & - & 1 & j & - & i & 1 & 1 & 1 & 1 & 1 & - & i & j \\
i & 1 & - & j & j & 1 & i & - & 1 & 1 & 1 & 1 & - & 1 & j & i \\
j & - & 1 & i & - & i & 1 & j & 1 & 1 & 1 & 1 & i & j & 1 & - \\
- & j & i & 1 & i & - & j & 1 & 1 & 1 & 1 & 1 & j & i & - & 1 \\
1 & j & - & i & 1 & i & j & - & 1 & - & i & j & 1 & 1 & 1 & 1 \\
j & 1 & i & - & i & 1 & - & j & - & 1 & j & i & 1 & 1 & 1 & 1 \\
- & i & 1 & j & j & - & 1 & i & i & j & 1 & - & 1 & 1 & 1 & 1 \\
i & - & j & 1 & - & j & i & 1 & j & i & - & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

For the subgroup $\{1, -1\}$, we obtain a Bush-type $\mathrm{BH}(16, 2)$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
- & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
- & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

COROLLARY 5.5. *Applying Theorem 5.1 to* $p = 2n$ *and the subgroup* $\{1, -1\}$*, implies the existence of a symmetric Bush-type Hadamard matrix of order* $4n^2$.

No symmetric Bush-type Hadamard matrix of order $4n^2$, n odd, is known. We suspect that there is no symmetric Bush-type $\mathrm{GH}(4n^2, 2n)$, n odd.

ACKNOWLEDGEMENTS.

Thanks to the anonymous referee for the constructive comments. Hadi Kharaghani is supported by an NSERC Discovery Grant. Caleb Van't Land and Vlad Zaitsev were supported by Alberta Innovates' SRS Program.

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Received: 25.11.2022.
Revised: 7.6.2023.