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CZ-GROUPS WITH NONABELIAN NORMAL SUBGROUP OF ORDER p^4

Mario Osvin Pavčević and Kristijan Tabak
University of Zagreb and Rochester Institute of Technology, Croatia

Dedicated to the memory of Zvonimir Janko

Abstract. A p-group G with the property that its every nonabelian subgroup has a trivial centralizer (namely only its center) is called a CZ-group. In Berkovich's monograph (see [1]) the description of the structure of a CZ-group was posted as a research problem. Here we provide further progress on this topic based on results proved in [5]. In this paper we have described the structure of CZ-groups G that possess a nonabelian normal subgroup of order p^4 which is contained in the Frattini subgroup $\Phi(G)$. We manage to prove that such a group of order p^4 is unique and that the order of the entire group G is less than or equal to p^7, p being a prime. Additionally, all such groups G are shown to be of a class less than maximal.

1. Introduction

A p-group G is a group of order p^n, where p is prime. The conjugation of x by y is given by $x^y = y^{-1}xy$, where $x, y \in G$. If $x^y = x$, then x and y commute, i.e. $[x, y] = x^{-1}y^{-1}xy = 1$. Let $H \leq G$ be a subgroup of G. The centralizer of H in G is $C_G(H) = \{g \in G \mid h^g = h, \ \forall h \in H\}$. The center of G is given by $Z(G) = \{g \in G \mid x^g = x, \ \forall x \in G\}$. The center $Z(H)$ of a subgroup $H \leq G$ is defined in the same way.

A finite group G is called a *CZ-group* (this abbreviated form comes from the words centralizer and Zentrum) if $C_G(H) = Z(H)$ for all nontrivial $H \leq G$. The set of all CZ-groups that are at the same time p-groups will be denoted by CZ_p and sometimes we will call such a group a CZ_p-group. The question

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of determining the general structure of $G \in CZ_p$ was posted in [1] as one of the open problems in the theory of p-groups. More on p-groups can be found in [2] and [3]. The first results about groups $G \in CZ_p$ were published in [5], where it was shown that a minimal CZ_p-group has order at least p^5. Additionally, the structure of maximal abelian subgroups of a minimal CZ_p group has been described in that paper as well.

In this paper, we assume that the Frattini subgroup $\Phi(G)$, which is defined as the intersection of all maximal subgroups of G, contains a normal nonabelian subgroup of order p^4. A subgroup $H \leq G$ that is a normal subgroup of G will be sometimes called G-invariant (if we want to point out this fact, we will denoted it by $H \trianglelefteq G$). The existence of a normal subgroup of order p^4 in $\Phi(G)$ does not appear as a limitation, since we can always find normal subgroups in p-groups of any given order. What however appears o be a true assumption is that we, in addition assume, that this subgroup of order p^4 is nonabelian.

In the next result we will determine the lower bound for the order $|G|$ of $G \in CZ_p$.

LEMMA 1.1. *If $G \in CZ_p$, then $|G : Z(G)| \geq p^3$ and $|G| \geq p^5$.*

PROOF. Let us assume the opposite, so let $|G : Z(G)| \leq p^2$. Then immediately we get $|G : Z(G)| = p^2$, since otherwise G would be abelian. The factor group $G/Z(G)$ cannot be cyclic, otherwise G would be abelian again. Thus, $G/Z(G) \cong E_{p^2}$ (the elementary abelian group of order p^2). Since the Frattini subgroup is the smallest subgroup such that its factor group is elementary abelian, we get $\Phi(G) \leq Z(G)$. If $\Phi(G) < Z(G)$, there is some maximal subgroup M such that $Z(G) \nleq M$. Hence, M must be abelian, since otherwise, we would be able to find some $g \in Z(G) \setminus M$, leading further to $g \in C_G(M)$, which is a contradiction since $G \in CZ_p$. Therefore, $MZ(G) = G$ and G is abelian, which is a contradiction again. So, $|G : Z(G)| \geq p^3$ and $|G| \geq p^4$ (since $|Z(G)| \geq p$). If $|G| = p^4$, then $|Z(G)| = p$ and $Z(G) \leq \Phi(G)$. This implies that any maximal subgroup of G is minimally nonabelian, thus G is a minimal CZ group, from which follows that $|G| \geq p^5$ (as it was proved in [5]). This is a contradiction. Therefore, the only remaining option is $|G| \geq p^5$. □

LEMMA 1.2. *Let $G \in CZ_p$ and $M < G$, $M \in CZ_p$. Then $|G : Z(G)| \geq p^4$ and $|G| \geq p^5$.*

PROOF. Lemma 1.1 states that $|G : Z(G)| \geq p^3$. Let $M \in CZ_p$ and $M < G$. Then again by Lemma 1.1, $|M : Z(M)| \geq p^3$. It was proved in [5] that $Z(G) \leq Z(M)$. Thus $|G : Z(G)| > |M : Z(G)| \geq |M : Z(M)| \geq p^3$. Therefore, $|G : Z(G)| \geq p^4$ and $|G| \geq p^5$. □

The following statement establishes a connection between CZ_p-groups and the maximality of class.

THEOREM 1.3. *Let $G \in CZ_p$ and $B < G$ be a nonabelian group of order p^3. Then G is a group of maximal class.*

PROOF. Let $B < G$, where $|B| = p^3$ and B nonabelian. Then $C_G(B) < B$. Therefore, $Z(G) \leq Z(B)$. Clearly, $|Z(B)| = p$ and $Z(G) = Z(B)$.

It is known that if $H \in Syl_p(Aut(B))$ (a Sylow p-group), then $|H| = p^3$ and H is nonabelian. Therefore, $N_G(B)/C_G(B) \leq Aut(B)$ is a p-group. Also, $N_G(B) > B$ and $C_G(B) = Z(B)$. Therefore, $|N_G(B)/C_G(B)| \geq p^3$ since $|N_G(B) : C_G(B)| = |N_G(B) : B| \cdot |B : Z(B)| \geq p \cdot p^2 = p^3$. Thus, it is necessary that $N_G(B)/Z(G) \cong H \in Syl_p(Aut(B))$. Also, $N_G(B)/Z(G) < G/Z(G)$ (since $|Z(G)| = p$ and $|G| \geq p^5$ and H nonabelian of order p^3).

Obviously $C_G(N_G(B)/Z(G)) \leq N_G(B)/Z(G)$. Inductively, $G/Z(G)$ is of maximal class, where $|Z(G)| = p$. From here we deduce that G is of maximal class. □

2. CZ-GROUPS WITH NONABELIAN G-INVARIANT SUBGROUP $N \leq \Phi(G)$ OF ORDER p^4

Let us now we introduce the main assumption. We will assume further that G is a CZ_p group possessing a subgroup $N \leq \Phi(G)$ which is a nonabelian G-invariant subgroup of order p^4. The nilpotency class of a group G will be denoted by $cl(G)$. If the class is maximal, we will put $cl(G) = max$, otherwise $cl(G) < max$. If the group is generated by at least k elements, we shall say that it is a k-generated group and write $d(G) = k$.

We will make use of the following result. Its proof can be found in [1, Lemma 1.4.].

LEMMA 2.1. *Let G be a p-group for $p > 2$ and $N \trianglelefteq G$. If N has no abelian G-invariant subgroups of type (p, p), then N is cyclic.*

The structure of a p-subgroup N satisfying the properties mentioned above is partially described in the following result.

LEMMA 2.2. *Let $G \in CZ_p$ where $p > 2$ and $cl(G) < max$. Let $N \leq \Phi(G)$ be a G-invariant nonabelian group of order p^4. Then $\Phi(N) = Z(N) \cong E_{p^2}$ and N is a 2-generated group of exponent p^2.*

PROOF. Assume that $Z(N)$ is cyclic. Let $A \trianglelefteq G$ and $A \leq N$ of order p^2. Then $|N_G(A)/C_G(A)| = |G/C_G(A)| \leq |Aut(A)|_p = p$, where $|Aut(A)|_p$ is the maximal power of p that divides $|Aut(A)|$. Hence, $N \leq \Phi(G) \leq C_G(A)$ and $A \leq C_G(A)$. Thus, $A \leq Z(\Phi(G)) \cap N$ and $A \leq Z(N)$ (since $N \leq \Phi(G)$). Therefore, A is cyclic. Then, according to Lemma 2.1, N must be cyclic, which is a contradiction. Therefore, $Z(N)$ is not cyclic. If $d(Z(N)) \geq 3$, then $|Z(N)| \geq p^3$ and $|N : Z(N)| \leq p$. This would imply that N is abelian. Therefore, $d(Z(N)) = 2$ and $Z(N) \cong E_{p^2}$. Clearly, $d(N) \geq 2$. Assume that $d(N) = 4$. Then $N/\Phi(N) \cong E_{p^4}$ and $\Phi(N) = 1$. On the other hand, $\Phi(N) = N'\mho_1(N)$ and $N' = 1$. This is a contradiction. Thus, $d(N) \leq 3$.

If $d(N) = 3$, then $\Phi(N) \cong C_p$ and $1 < N' \leq \Phi(N)$. Thus, $\Phi(N) = N'$. Clearly, $N' \cap Z(N) > 1$ since $N' \trianglelefteq N$. Put $Z(N) = \langle x\rangle \times \langle y\rangle \cong C_p \times C_p$ such that $N' = \langle x\rangle$. Thus, there is $y \in Z(N) - \Phi(N)$.

Thus, y is a generator of N and the order of y is p. Then there is some maximal subgroup $M < N$ such that $y \notin M$. Therefore, $N = \langle M, y\rangle = M \times \langle y\rangle$. If $w^p = y$ for some w, then $y \in \mho_1(N) \leq \Phi(N)$. This is a contradiction since y is a generator. Because N is nonabelian, M must be nonabelian, otherwise $N = M \times \langle y\rangle$ would be abelian. Therefore, $M' > 1$ and $|M| = p^3$. Then, according to the Theorem 1.3, the class of the group G is maximal. This is a contradiction with our assumption. Therefore, $d(N) = 2$ and $N/\Phi(N) \cong E_{p^2}$ where $|\Phi(N)| = p^2$. If $Z(N) \nleq \Phi(N)$, then there is some maximal $M \lhd_p N$ such that $Z(N) \nleq M$ then $M' = 1$. Otherwise, by Theorem 1.3 we would get $cl(G) = max$. Thus, it is necessary that $Z(N) \leq \Phi(N)$. Since both groups have order p^2, we get $Z(N) = \Phi(N)$.

Let $exp(G) = p$. Then $|\mho_1(G)| = 1$ and $\Phi(N) = N'\mho_1(N) = N' \cong C_p \times C_p$. Since N has a maximal abelian subgroup, then $p^4 = |N| = p \cdot |N'| \cdot |Z(N)| = p \cdot p^2 \cdot p^2$. This is a contradiction. Thus, $exp(N) > p$.

If $exp(N) = p^3$, then $N \cong M_{p^4}$, where M_{p^4} is a minimal nonabelian group with a maximal cyclic subgroup. Then, there is some $w \in N$ of order p^3. Hence $\mho_1(N) = \langle w^p\rangle \cong C_{p^2}$ and $\mho_1(N) = \Phi(N) = Z(N)$ and $d(N) = 1$. Again, this is a contradiction. Thus, the only remaining option is $exp(N) = p^2$. □

The following result shows the uniqueness of the nonabelian G-invariant subgroup $N \leq \Phi(G)$, where $cl(G) < max,\ G \in CZ_p$ and $|N| = p^4$.

Lemma 2.3. *Let $G \in CZ_p,\ p > 2$ and $cl(G) < max$. Let $N \leq \Phi(G)$ be a G-invariant nonabelian subgroup of order p^4. Then N is uniquely determined by its generators and relations with $N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$.*

Proof. From Lemma 2.2 we have $Z(N) = \Phi(N) \cong E_{p^2}$. Also, $exp(N) = p^2$. If $M \lhd N$ is maximal, then $Z(N) < M$ and $M' = 1$. Thus, $|N| = p \cdot |N'||Z(N)|$. Hence, $|N'| = p$. We can put $Z(N) = \langle a\rangle \times \langle b\rangle$. We can assume $N' = \langle a\rangle$. There are $x, y \in N$ such that $x^p = a,\ y^p = b$. Otherwise, $a \notin \Phi(N) = N'\mho_1(N)$ and $b \notin \Phi(N)$. Now, take $[x, y] = a = x^{-1}y^{-1}xy = x^{-1}x^y = x^p$. This gives us $x^y = x^{1+p}$. □

Lemma 2.4. *Let the group N be defined as $N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$. Let $z^{-1} = x^p$. Then for all integers i, j, n the following relations hold: $y^j x = xy^j z^j$, $y^j x^i = x^i y^j z^{ij}$. Furthermore, $(x^i y^j)^n = x^{ni} y^{nj} z^{\binom{n}{2} ij}$ and the order $o(g) = p^2$, for all $g \in N - \Phi(N)$. The subgroup $\langle x^i y^{pj}\rangle \leq N$ is normal in N.*

Proof. Since $z^{-1} = x^p \in Z(N)$, it follows $x^y = xz^{-1}$ and $xyz = yx$. Then, $y^j x = y^{j-1}(yx) = y^{j-1}(xy)z = y^{j-2}(yx)yz = y^{j-2}(xy)yz^2 = y^{j-2}xy^2z^2 = \cdots = xy^j z^j$. We have $y^j x^i = y^j x x^{i-1} = xy^j x^{i-1} z^j =$

$xy^jxx^{i-2}z^j = xxy^jx^{i-2}z^{2j} = \cdots = x^iy^jz^{ij}$. We will use induction to prove the claim about $(x^iy^j)^n$. For $n = 1$ the claim is trivial. Assume that $(x^iy^j)^n = x^{ni}y^{nj}z^{\binom{n}{2}ij}$. Now we proceed with the induction step by computing

$$(x^iy^j)^{n+1} = x^{ni}y^{nj}z^{\binom{n}{2}ij}x^iy^j = x^{ni}y^{nj}x^iy^jz^{\binom{n}{2}ij} = x^{ni}x^iy^{nj}y^jz^{nij}z^{\binom{n}{2}ij}$$
$$= x^{(n+1)i}y^{(n+1)j}z^{\binom{n+1}{2}ij}.$$

Let $g \notin \Phi(N)$. Then $g = x^iy^j$, where either i or j is not divisible by p. Otherwise, $g \in \langle x^p, y^p\rangle = \Phi(N) = Z(N)$, (see Lemma 2.2). Since $z^p = 1$ and $p \mid \binom{p}{2}$, we obtain $(x^iy^j)^p = x^{pi}y^{pj}z^{\binom{p}{2}ij} = x^{pi}y^{pj}$. If $g^p = 1$, then $x^{pi} = y^{-pj} \in \langle x\rangle \cap \langle y\rangle$, which implies $x^{pi} = 1$ and $i \equiv 0 \pmod p$. In the other case $j \equiv 0 \pmod p$. This is a contradiction. Therefore $o(g) = p^2$.

Look now at x^iy^{pj}, where i and j are not divisible by p. Since $y^p \in Z(N)$, it follows $(x^iy^{pj})^x = x^iy^{pj}$. Let us assume that there is some integer n such that $(x^iy^{pj})^y = (x^iy^{pj})^n$. This would imply $\langle x^iy^{pj}\rangle \trianglelefteq N$. If such an n exists, this would imply $(xz^{-1})^iy^{pj} = (x^iy^{pj})^n$. Then, $x^iy^{pj}z^{-i} = x^{ni}y^{npj}$ and $y^{npj-pj} \in \langle x\rangle$. Thus $pj(n-1) \equiv 0 \pmod{p^2}$ and $n-1 \equiv 0 \pmod p$ since $j \not\equiv 0 \pmod p$. Let $n = 1 + mp$, for some integer m. Then $x^{i(1-n)} = z^i$ and $x^{-mpi} = z^i$. Therefore, $z^{mi} = z^i$. Take $m = 1$ and $n = 1 + p$. We conclude, such n exists and $\langle x^iy^{pj}\rangle \trianglelefteq N$. □

Lemma 2.5. *Let $G \in CZ_p$, $p > 2$ and $cl(G) < max$. Let $N \trianglelefteq G$ and $N \leq \Phi(G)$ be nonabelian of order p^4. Then $G/Z(N)$ is isomorphic to some subgroup of $Aut(N)$.*

Proof. Since $N_G(N) = G$ and $C_G(N) \leq N$, we get $C_G(N) = Z(N)$. Then by the N/C-theorem, $N_G(N)/C_G(N) \lesssim Aut(N)$. □

The following results is from [4, Theorem 12.2.2, page 178].

Theorem 2.6. *Let $|G| = p^n$ and $d(G) = d$. Then $|Aut(G)|$ divides $|Aut(E_{p^d}) \times \Phi(G)^d|$.*

The following result establishes an upper bound for the order of a group G with conditions we are studying here.

Theorem 2.7. *Let $G \in CZ_p$, $cl(G) < max$ and let $N \leq \Phi(G)$ be a normal nonabelian subgroup of G of order p^4. Then $|G| \leq p^7$.*

Proof. According to Lemma 2.2 and Lemma 2.3, $\Phi(N) = Z(N) \cong E_{p^2}$ and N is uniquely determined by $N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$. Applying Lemma 2.5 and Theorem 2.6, we have $G/Z(N) \lesssim Aut(N)$ and $|G/Z(N)|_p$ divides $|Aut(E_{p^2}) \times \Phi(N)^2|_p = |(p^2-1)(p^2-p)\cdot p^4|_p = p^5$. Since $|Z(N)| = p^2$, we get $|G| \leq p^7$. □

Corollary 2.8. *Let $G \in CZ_p$, $cl(G) < max$ and $N \leq \Phi(G)$ a normal nonabelian subgroup of order p^4. Then $|G| \in \{p^5, p^6, p^7\}$.*

We conclude this section with a technical result that we shall need.

Lemma 2.9. *Let $\langle x \rangle \cong C_{p^2}$, $p > 2$ and let $\varphi \in Aut(\langle x \rangle)$ be an automorphism of $\langle x \rangle$ of order p. Then, there is some $m \in \mathbb{N}$ such that $x^\varphi = x^{1+mp}$.*

3. The case $|G| = p^6$ and $cl(G) < max$

We shall continue with the same assumption that G possesses a nonabelian subgroup $N \leq \Phi(G)$ of order p^4. Additionally, we shall assume that G is **not of maximal class**. By Corollary 2.8, the order of G is at least p^5. If $|G| = p^5$, then $|G : \Phi(G)| = p$ and G is cyclic. Therefore, from this moment on, we can assume that $|G| \geq p^6$. If $|G| = p^6$, then $|G : \Phi(G)| = p^2$ and G is a 2-generated group.

We now prove additional results about the structure of the group N.

Lemma 3.1. *Let $N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p} \rangle$. Then $Z(N) = \Phi(N) = \langle x^p, y^p \rangle \cong E_{p^2}$ and $\langle y^{pi+1} \rangle \ntriangleleft N$, $i = 0, 1, \ldots, p-1$.*

Proof. From $(x^p)^y = (x^y)^p = (x^{1+p})^p = x^p$ and $[x, x^p] = 1$ we have $x^p \in Z(N)$. Furthermore, $x^{y^p} = x^{(1+p)^p} = x^p$ (since $(1+p)^p \equiv p \pmod{p^2}$). Therefore, $\langle x^p, y^p \rangle \leq Z(N)$. Since $|N : Z(N)| \geq p^2$ and $\langle x^p \rangle \cap \langle y^p \rangle = 1$, we have $Z(N) = \langle x^p, y^p \rangle$. Since N is 2-generated and $N/Z(N) \cong E_{p^2}$, it follows that $Z(N) = \Phi(N)$.

Now we shall prove the second claim. We firstly use the following: $(y^{pi+1})^x = (y^x)^{pi+1} = (yx^{-p})^{pi+1} = y^{pi+1}(x^{-p})^{pi+1} = y^{pi+1}x^{-p}$. If $\langle y^{pi+1} \rangle$ is N-invariant, then $y^{pi+1}x^{-p} \in \langle y^{pi+1} \rangle$. This implies $x^{-p} \in \langle y^{pi+1} \rangle \leq \langle y \rangle$ and $\langle x \rangle \cap \langle y \rangle > 1$, which is a contradiction. Therefore, $\langle y^{pi+1} \rangle \ntrianglelefteq N$. ∎

The following two results were proved in [1]. We shall present them here with slightly different proofs. We will use the following notation: if H is a normal subgroup of index p^i of G, then we shall write this as $H \lhd_{p^i} G$.

Theorem 3.2. *Let G be a p-group and let $K \trianglelefteq G$ contain a abelian maximal subgroup. Then K contains a maximal abelian subgroup that is G-invariant.*

Proof. If G is an abelian group, the claim is true. Let G be a nonabelian group, and let $A \lhd_p K \trianglelefteq G$, where A is an abelian subgroup. If $\{T \mid T \lhd_p K,\ T' = 1\} = \{A\}$, then $A^g \lhd_p K^g = K$ for all $g \in G$ (A^g is abelian as well). Therefore, $A^g = A$ for all $g \in G$. This implies that A is G-invariant.

Now assume that A_1 and A_2 are distinct maximal abelian subgroups of K. Then $A_i \lhd K$ and $A_1 A_2 = K$. Since $A_1 \cap A_2 \lhd_p A_i$, we have $A_1 \cap A_2 \leq C_K(A_1) \cap C_K(A_2)$. This implies $A_1 \cap A_2 \leq Z(K)$. Let K be a nonabelian group. Then $K/Z(K) \cong E_{p^2}$. There is a subgroup $C \leq K$ such that $C/Z(K) \cong C_p$. Then $C/Z(K) \lhd K/Z(K)$. There is a one-to-one map between $\{C/Z(K) \mid C/Z(K) \lhd K/Z(K)\}$ and $\{C/Z(K) \mid Z(K) \lhd_p C \lhd_p K\}$. Note that

$$|\{C/Z(K) \mid C/Z(K) \lhd_p K/Z(K)\}| = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_p = \frac{p^2-1}{p-1} = p+1.$$

This implies that K has $p+1$ abelian maximal subgroups. The group G acts via conjugation on $p+1$ maximal abelian subgroups of K. Orbits of this action have lengths $\equiv 0 \pmod p$. This implies that there is at least one fixed subgroup and that one is G-invariant. The proof is identical in the case when $K' = 1$. □

THEOREM 3.3. *Let $N \trianglelefteq G$ and $|N| > p^3$, where G is a p-group. Then there is some abelian $D < N$ of order p^3 such that $D \trianglelefteq G$.*

PROOF. There is a composition series that goes through each normal subgroup of G. It implies that there is a G-invariant subgroup $M < N$ of order p^4. Let $A < M$ be of order p^2. Then, A is abelian and $|M : A| = p^2$. By Theorem 3.2, there is a $B \trianglelefteq M$ of order p^2. Note that B is abelian as well. Since $|Aut(B)_p| = p$, it is necessary that $|N_M(B) : C_M(B)| \leq p$, where $N_M(B) = M$. If $M = C_M(B)$, then $B \leq Z(M)$. This implies that there is $g \in M - B$ such that $g^p \in B$ and $\langle B, g\rangle < M$ is an abelian group of order p^3.

If $C_M(B) \lhd_p M$, then $C_M(B)$ is abelian of order p^3. Thus, we can always find an abelian M-invariant subgroup of M the order of which is p^3. The claim follows from Theorem 3.2. □

PROPOSITION 3.4. *Let $G \in CZ_p$ be a 2-generated group of order p^6. Let $\Phi(G) = N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$ and $exp(G) \leq p^2$. Then $\mho_1(G) = \mho_1(N) = \langle x^p, y^p\rangle$ and $|G'| \geq p^3$.*

PROOF. From Lemma 3.1 the center of N is $Z(N) = \langle x^p, y^p\rangle = \Phi(N)$. Therefore, $\langle x^p, y^p\rangle \leq \mho_1(N) \leq \Phi(N) = \langle x^p, y^p\rangle$. This implies $\mho_1(N) = \langle x^p, y^p\rangle$. Thus, $\langle x^p, y^p\rangle \leq \mho_1(G)$. Since $exp(G) = p^2$, there is $x \in G$ and $o(g) = p^2$. Furthermore, $g^p \in \Phi(G) = N$ and $o(g^p) = p$. By Lemma 2.4, it follows that if $t \in N \setminus \Phi(N)$, then $o(t) = p^2$. Therefore, $g^p \in \Phi(N) = \langle x^p, y^p\rangle$. Thus, $\mho_1(G) \leq \langle x^p, y^p\rangle$ and finally $\mho_1(G) = \langle x^p, y^p\rangle$.

Since $\langle x^p\rangle = N' \leq G'$ and $\langle x^p\rangle \leq \mho_1(G)$, we have $|\mho_1(G) \cap G'| \geq p$ and

$$p \leq |\mho_1(G) \cap G'| = \frac{|\mho_1(G)||G'|}{|\mho_1(G)G'|} = \frac{p^2|G'|}{|\Phi(G)|} = \frac{|G'|}{p^2},$$

yielding $|G'| \geq p^3$. □

THEOREM 3.5. *Let $G \in CZ_p$ be of order p^6 and $\Phi(G) = N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$. Let $exp(G) = p^2$ and $|G'| = p^4$. Then $Z(G) = \langle x^p\rangle$ and $Z_2(G) = Z(N) = \langle x^p, y^p\rangle$.*

PROOF. By Proposition 3.4, $|G'| \geq p^3$. Since $|G : G'| \leq p^2$, the only options are $|G'| = p^3$ or $|G'| = p^4$. Let $|G'| = p^4$. Since $G' \leq \Phi(G) = \mho_1(G)G'$, we have $G' = N$. By Grünn's theorem (see [1]), we have $[G' : Z_2(G)] = 1$. Therefore, $[N, Z_2(G)] = 1$. Since $G \in CZ_p$, we have $Z_2(G) \leq C_G(N) = Z(N) = \langle x^p, y^p \rangle$ (see Lemma 3.1). This implies $Z_2(G)/Z_1(G) = Z(G/Z_1(G)) > 1$. Therefore, $Z_2(G) > Z_1(G) > 1$. Since $|Z_2(G)| = |\langle x^p, y^p \rangle| = p^2$, we have $|Z_1(G)| = |Z(G)| = p$. Note that $N' = \langle x^p \rangle$ is a characteristic subgroup of N, and N is a characteristic subgroup of G. It follows know that $N' \trianglelefteq G$ is of order p. Therefore, $|N' \cap Z(G)| > 1$ and $N' = Z(G) = \langle x^p \rangle$. □

THEOREM 3.6. *Let $G \in CZ_p$ be a group of order p^6. Let $\Phi(G) = N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p} \rangle$ and $exp(G) \leq p^2$. If $|G'| = p^4$, then G is a group of maximal class.*

PROOF. If we assume that $cl(G) < max$, then, by Theorem 3.3, there is a G-invariant, abelian subgroup $A \leq N$ of order p^3. This implies $Z(N) = \langle x^p, y^p \rangle \leq A$. Otherwise, N would be an abelian group. Note that $\mho_1(N) = Z(N) \leq A$. Since $\mho_1(N)$ is a characteristic subgroup of A, we have $\langle x^p, y^p \rangle \lhd G$. We know that $|G : \Phi(G)| = p^2$ and $G = \langle a, b \rangle$ for some $a, b \in G$. By Theorem 3.5, we have $Z(G) = \langle x^p \rangle$. This yields $[x^p, a] = [x^p, b] = 1$. If $(y^p)^a = (y^p)^b = y^p$, then $y^p \in Z(G) = \langle x^p \rangle$, which is a contradiction. Therefore, we have $(y^p)^a \neq y^p$. Since $\langle x^p, y^p \rangle \lhd G$, we have $(y^p)^a \in \langle x^p, y^p \rangle$. Also, $o(a^p) \leq p$ and $a^p \in N$. Therefore, $a^p \in \Omega_1(N) = \langle x^p, y^p \rangle$. It follows that $\langle x^p, y^p, a \rangle$ is a nonabelian group of order p^3 and by Theorem 1.3 we have $cl(G) = max$. This is the final contradiction which proves the theorem. □

THEOREM 3.7. *Let $G \in CZ_p$, $cl(G) < max$, $exp(G) \leq p^2$ and let $N \leq \Phi(G)$ be a nonabelian G-invariant subgroup of order p^4. Then $|G| = p^7$.*

PROOF. By Corollary 2.8, we have $p^5 \leq |G| \leq p^7$. Since $|\Phi(G)| \geq p^4$, it follows $|G| \geq p^6$ (since otherwise $d(G) = 1$). By Lemma 2.3, we know the structure of the group N.

Let $|G| = p^6$. By Proposition 3.4, we have $|G'| \geq p^3$. Since $|G| = p^6$ and $d(G) = 2$, it follows that $G' \leq \Phi(G)$ and $|G'| \leq p^4$. If $|G'| = p^4$, then by Theorem 3.6, the class of G would be maximal, contradicting the assumption. Hence $|G'| = p^3$. By Proposition 3.4, we have $\mho_1(G) = \mho_1(N) = Z(N) = \langle x^p, y^p \rangle = \Phi(N)$. Since $|G'| = p^3$, we have $G' \leq N = \Phi(G)$. On the other hand, G' is a maximal subgroup of N. Therefore $Z(N) = \mho_1(G) = \Phi(N) \leq G'$. This implies $\Phi(G) = \mho_1(G)G' = G' < N = \Phi(G)$. This is a contradiction. So, the only remaining possibility is $|G| = p^7$. □

4. The case $|G| = p^7$ and $cl(G) < max$

We shall continue with the same assumption that there is a nonabelian $N \leq \Phi(G)$ of order p^4. Additionally, we shall assume that G is **not of maximal class** and $|G| = p^7$, $exp(G) = p^2$. Note that $exp(G) \leq p^3$. We begin with the following result on the size of G'.

Lemma 4.1. *Let $G \in CZ_p$ be a group of order p^7 and $exp(G) = p^2$ where $N = \Phi(G) = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$. Then $\mho_1(G) = \mho_1(N) = \langle x^p, y^p\rangle$ and $|G'| \geq p^3$.*

Proof. Notice that $\mho_1(N) \leq \mho_1(G)$ and $exp(\mho_1(G)) = p$. By Lemma 3.1, we have $\mho_1(N) = \Phi(N) = \langle x^p, y^p\rangle$. We also have $\mho_1(G) \leq \Phi(G) = N$. Then $\mho_1(G) \leq \Phi(N) = \mho_1(N)$. This implies $\mho_1(G) = \mho_1(N) = \langle x^p, y^p\rangle$.

By Lemma 3.4, we have $\langle x^p\rangle = N' \leq G' \cap \mho_1(G)$. Therefore

$$p \leq |\mho_1(G) \cap G'| = \frac{|\mho_1(G)| \cdot |G'|}{|\mho_1(G) \cdot G'|} = \frac{p^2 \cdot |G'|}{|\Phi(G)|} = \frac{|G'|}{p^2}.$$

This yields $|G'| \geq p^3$. □

Theorem 4.2. *Let $G \in CZ_p$ be a group of order p^7 with $exp(G) = p^2$ where $N = \Phi(G) = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$. If $|G'| = p^4$, then $Z(G) = \langle x^p\rangle$ and $Z_2(G) = Z(N) = \langle x^p, y^p\rangle$.*

Proof. By Lemma 4.1, we have $|G'| \geq p^3$. Since $G' \leq \Phi(G) = N$, it follows $|G'| \leq p^4$. The rest of the proof follows the proof of Theorem 3.5. □

Now we shall present the main result.

Theorem 4.3. *Let $G \in CZ_p$ be of exponent p^2, where $cl(G) < max$. Let $N \leq \Phi(G)$ be a G-invariant nonabelian subgroup of order p^4. Then $|G| = p^7$ and $N = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p}\rangle$ is of index p in $\Phi(G)$.*

Proof. Assume that $N = \Phi(G)$ and $|G'| = p^4$. As in Theorem 3.5, we have $Z(G) = \langle x^p\rangle$ and $Z_2(G) = Z(N) = \langle x^p, y^p\rangle$. By Theorem 3.3, there is an abelian group $A \trianglelefteq G$ such that $A \leq N$ and $|A| = p^3$. Therefore $Z(N) \leq A$, since otherwise $AZ(N) = N$ and N would be an abelian group. By Lemma 4.1, we have $\mho_1(G) = \mho_1(N) = \langle x^p, y^p\rangle \leq A$. Note that $\mho_1(N) = Z(N)$. By Lemma 3.1, we have $Z(N) = \Phi(N) = \langle x^p, y^p\rangle \trianglelefteq G$ (since $\mho_1(G) = \langle x^p, y^p\rangle$ is a characteristic subgroup of G). From $G/\Phi(G) \cong E_{p^3}$, we have $G = \langle a, b, c\rangle$ for some generators $a, b, c \in G$. Since $x^p \in Z(G)$, it follows $[x^p, a] = [x^p, b] = [x^p, c]$.

If $(y^p)^a = (y^p)^b = (y^p)^c = y^p$, then $y^p \in Z(G)$. This is a contradiction. Thus, we may assume $(y^p)^a \neq y^p$. Furthermore, $o(a^p) \leq p$ (since $exp(G) = p^2$) and $a^p\mho_1(G) = \langle x^p, y^p\rangle$. This implies that $\langle x^p, y^p, a\rangle$ is a nonabelian group of order p^3 and by Theorem 1.3 the group G has maximal class. This is a contradiction. Therefore, by Lemma 4.1, we have $|G'| = p^3$. By Lemma 4.1, it follows that $\mho_1(G) = \mho_1(N) = Z(N) = \Phi(N) = \langle x^p, y^p \rangle$. From $|G'| = p^3$, we have $G' \leq N = G'\Phi(G)$. Since G' is maximal in N, it implies $\Phi(N) < G'$. Since $\mho_1(G) = \mho_1(N) = \Phi(N) < G'$, it follows that $\Phi(G) = \mho_1(G)G' \leq G' < N$. This yields now $\Phi(G) < N = \Phi(G)$, which is a contradiction. By Theorem 3.7, we have $|G| = p^7$. It follows $N < \Phi(G)$. The description of the group N is given by Lemma 2.3. Since $|G : \Phi(G)| \geq p^2$, we have $|\Phi(G) : N| = p$. ∎

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M. O. Pavčević
Department of applied mathematics
Faculty of Electrical Engineering and Computing
University of Zagreb
10000 Zagreb
Croatia
E-mail: mario.pavcevic@fer.hr

K. Tabak
Rochester Institute of Technology
Zagreb Campus
D.T. Gavrana 15, 10000 Zagreb
Croatia
E-mail: kxtcad@rit.edu

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