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SPLITNESS OF THE VERONESEAN DUAL HYPEROVALS: A QUICK PROOF

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Dedicated to the memory of Zvonimir Janko

Abstract. Satoshi Yoshiara shows in [7] that the Veronesean dual hyperovals over $\mathbb{F}_2$ are of split type. So far there exists no published proof that a Veronesean dual hyperoval over any finite field of even characteristic is of split type. In this note we give a quick proof of this fact.

1. Introduction

Let $n \geq 2$. A set $\mathcal{A}$ of n-dimensional subspaces of a finite $\mathbb{F}_q$-vector space U is called a *dimensional dual arc of rank* n (we will use in the sequel the abbreviation DA), if:

(1) $\dim X_1 \cap X_2 = 1$ for every two $X_1, X_2 \in \mathcal{A}$.
(2) $\dim X_1 \cap X_2 \cap X_3 = 0$ for every three $X_1, X_2, X_3 \in \mathcal{A}$.
(3) The set $\mathcal{A}$ generates U.

The space $U = U(\mathcal{A})$ is called the *ambient space* of the DA. The axioms of a DA imply $|\mathcal{A}| \leq (q^n - 1)/(q - 1) + 1$. The DA of rank n over $\mathbb{F}_q$ is called a *dimensional dual hyperoval* (we use the abbreviation DHO) if $|\mathcal{A}| = (q^n - 1)/(q - 1) + 1$. The DA *splits over the subspace* Y, if $U = X \oplus Y$ for all $X \in \mathcal{A}$. We also say that Y is a *complement* of $\mathcal{A}$ and $\mathcal{A}$ *is splitting* or of *split type.*[1]

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[1]Often the terminology of projective geometry is used and a DA of rank n is called an $(n-1)$-dimensional dual arc. For background information on dimensional dual arcs and hyperovals we refer to Yoshiara's survey [5].

Let $\mathcal{S}$ and $\mathcal{T}$ be DAs over $\mathbb{F}_q$ of the same rank. A semilinear epimorphism $\phi : U(\mathcal{S}) \to U(\mathcal{T})$ is a *covering map* if $\mathcal{T} = \mathcal{S}\phi = \{X\phi \mid X \in \mathcal{S}\}$. One calls $\mathcal{T}$ a *quotient of* $\mathcal{S}$ and $\mathcal{S}$ a *cover of* $\mathcal{T}$.

Let q be a prime power. Let V be a n-dimensional $\mathbb{F}_q$-space, $n \geq 2$ and $U = S^2(V)$ the symmetric square of V. For $0 \neq e \in V$ define $X(e) = \{x \cdot e \mid x \in V\}$. Note, that $X(e) = X(\alpha e)$ for $0 \neq \alpha \in \mathbb{F}_q$. Then

$$\mathcal{VA}_n(q) = \{X(e) \mid 0 \neq e \in V\}$$

is a DA, the *Veronesean DA* over $\mathbb{F}_q$ of rank n (see [5, Section 5.2]). If q is a 2-power the set $X(\infty) = \{x^2 \mid x \in V\}$ is a subspace of $S^2(V)$ and

$$\mathcal{V}_n(q) = \{X(\infty)\} \cup \mathcal{VA}_n(q)$$

is even a DHO, the *Veronesean DHO* over $\mathbb{F}_q$ of rank n. We shall prove the following.

THEOREM 1.1. *For every integer $n \geq 2$ and every prime power q the Veronesean DA $\mathcal{VA}_n(q)$ is of split type. Let q be a 2-power. Then Veronesean DHO $\mathcal{V}_n(q)$ is of split type.*

2. The proof

We need the following observation.

LEMMA 2.1. *Let $\phi : U(\mathcal{S}) \to U(\mathcal{T})$ be covering map of the DA $\mathcal{S}$ onto the DA $\mathcal{T}$. Let $\mathcal{T}$ be of split type. Then $\mathcal{S}$ is of split type too.*

PROOF. Assume that $\mathcal{T}$ splits over the subspace Y of $U(\mathcal{T})$ and let Y' be the pre-image of Y with respect to ϕ. By definition of a covering map ϕ is injective on each $X \in \mathcal{S}$ and so on $X \cap Y'$. Since $(X \cap Y')\phi = X\phi \cap Y = 0$ we get $X \cap Y' = 0$, i.e. $\mathcal{S}$ splits over Y' as $U(\mathcal{S}) = X + Y'$ is the pre-image of $U(\mathcal{T}) = X\phi \oplus Y$ with respect to ϕ. ◻

With the help of this Lemma the splitness of the Veronesean DHOs can be deduced from previous work of Taniguchi [3] and Yoshiara [6]. Our proof of Theorem 1.1 includes a simplified, self-contained account of necessary material from [3] and [6]. We use the following lemma.

LEMMA 2.2. *Let $n \geq 2$, V be a $\mathbb{F}_q$-space of rank n and $\mathcal{S}$, $|\mathcal{S}| = (q^n - 1)/(q-1)$ a DA with ambient space $U = U(\mathcal{S})$. Let $\sigma : V \times V \to U$ be a symmetric, $\mathbb{F}_q$-bilinear mapping such that $\mathcal{S} = \{S(e) \mid 0 \neq e \in V\}$ where $S(e) = \{\sigma(x, e) \mid x \in V\}$. Then $\mathcal{S}$ is a quotient of $\mathcal{VA}_n(q)$.*

PROOF. By the universal property of $S^2(V)$ there exist an epimorphism $\phi : S^2(V) \to U$ with $(x \cdot y)\phi = \sigma(x, y)$. This shows $X(e)\phi = S(e)$, i.e. $\mathcal{VA}_n(q)\phi = \mathcal{S}$. ◻

Let q be a 2-power and $\mathcal{D}$ a DHO such that there exists $S_\infty \in \mathcal{D}$ such that $\mathcal{S} = \mathcal{D} - \{S_\infty\}$ satisfies the assumptions of Lemma 2.2. Then $\mathcal{D}$ is a quotient of $\mathcal{V}_n(q)$: for $0 \neq e \in V$ we see that $\mathbb{F}_q\sigma(e,e) - \{0\} = S(e) - \bigcup_{f \in V - \mathbb{F}_q e}(S(e) \cap S(f))$ (as $\mathbb{F}_q\sigma(e,f) \subseteq S(e) \cap S(f)$). This implies $S_\infty = \bigcup_{0 \neq e \in V} \mathbb{F}_q\sigma(e,e)$ and forces $X(\infty)\phi = S_\infty$. The claim follows.

Proof of Theorem 1.1. The following construction of a DA (a DHO) is a special case of Taniguchi's construction [3]. Let q be a prime power and γ be a generator of $\mathrm{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$. Define a set of n-spaces $\mathcal{S}$ over $\mathbb{F}_q$ in $U = U(\mathcal{S}) = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ by $\mathcal{S} = \{X(e) \mid 0 \neq e \in \mathbb{F}_{q^n}\}$ where for $0 \neq e \in \mathbb{F}_{q^n}$ one sets $S(e) = \{(xe, xe^\gamma + x^\gamma e) \mid x \in \mathbb{F}_{q^n}\}$. If q is a 2-power we set $S(\infty) = \{(x^2, 0) \mid x \in \mathbb{F}_{q^n}\}$ and $\mathcal{D} = \{S(\infty)\} \cup \mathcal{S}$. Then $\mathcal{S}$ is a DA and $\mathcal{D}$ even a DHO. Indeed, it is easy to see that $S(e) \cap S(f) = \mathbb{F}_q(ef, ef^\gamma + e^\gamma f)$ for $\mathbb{F}_q e \neq \mathbb{F}_q f$, $e \neq 0 \neq f$ and $S(\infty) \cap S(e) = \mathbb{F}_q(e^2, 0)$. Clearly, $\mathcal{S}$ as well as $\mathcal{D}$ split over $0 \times \mathbb{F}_{q^n}$. The mapping $\sigma : \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \to U$ defined by $\sigma(x,e) = (xe, xe^\gamma + x^\gamma e)$ is symmetric and bilinear. Then by Lemma 2.2 $\mathcal{S}$ is a quotient of $\mathcal{VA}_n(q)$ and the subsequent remark implies for q even that $\mathcal{D}$ is a quotient of $\mathcal{V}_n(q)$. This is observed already by Yoshiara in [6, Proposition 1]. Lemma 2.1 then forces that $\mathcal{VA}_n(q)$ and $\mathcal{V}_n(q)$ are of split type. □

Remark 2.3. (a) Our proof is a mere existence proof for complements. Yoshiara [7, Corollary 3] does more. He establishes a one-to-one correspondence between the complements of $\mathcal{V}_n(2)$ with classes of commutative pre-semifields of order 2^n.

(b) Lemma 2.1 becomes wrong if one interchanges the roles of $\mathcal{S}$ and $\mathcal{T}$: computer computations [2] used for [1] show that (the split DHO) $\mathcal{V}_5(2)$ has quotients which are not of split type.

(c) The splitness proof for the Taniguchi DHOs in [7] is not easy. Unfortunately Lemma 2.1 appears to be not helpful since the splitness question for the known quotients of the Taniguchi DHOs [4] seems to be difficult too.

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