RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 28 = 558 (2024): 49-56
DOI: https://doi.org/10.21857/9e31lhzl8m

NEW PARTITION IDENTITIES FOR ODD W ODD

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This note is dedicated to Marko Tadić at the occasion of his 70th birthday.

Abstract. In this note we conjecture Rogers-Ramanujan type colored partition identities for an array $\mathcal{N}_w^{\mathrm{odd}}$ with odd number of rows w such that the first and the last row consist of even positive integers. In a strange way this is different from the partition identities for the array $\mathcal{N}_w$ with odd number of rows w such that the first and the last row consist of odd positive integers—the partition identities conjectured by S. Capparelli, A. Meurman, A. Primc and the author and related to standard representations of the affine Lie algebra of type $C_\ell^{(1)}$ for $w = 2\ell + 1$. The conjecture is based on numerical evidence.

1. Introduction

We write a partition of positive integer n in terms of frequencies f_j—the number of occurrences of the part j in the partition

$$n = \sum_{j\in\mathbb{N}} f_j \cdot j = \underbrace{1 + \cdots + 1}_{f_1 \text{ times}} + \underbrace{2 + \cdots + 2}_{f_2 \text{ times}} + \dots . \tag{1.1}$$

It is clear that $f_j = 0$ for all but finitely many $j \in \mathbb{N}$ and that the partition (1.1) is determined by its sequence of frequencies $(f_i \mid i \in \mathbb{N})$.

The partition identities of Rogers (1894), Ramanujan (1913) and Schur (1917) for $k = 1$, and the partition identities of Gordon (1961) for $k \geq 2$, can be stated as:

Let $0 \leq a \leq k$. The number of partitions of n such that

$$f_j + f_{j+1} \leq k \quad \textit{for all } j \textit{ and} \tag{1.2}$$

$$f_1 \leq a \tag{1.3}$$

equals the number of partitions of n into parts $\not\equiv 0, \pm(a+1) \mod (2k+3)$. The conditions (1.2) on frequencies of two adjacent numbers are called *the difference conditions*, and the condition (1.3) on the frequency of number 1 is

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2020 *Mathematics Subject Classification.* 05A19, 17B67.
Key words and phrases. Colored partitions, Rogers-Ramanujan type identities.

called *the initial condition.* There are some other similar partition identities stating that the number of partitions of n satisfying certain difference & initial conditions is equal to the number of partitions of n with parts satisfying certain congruence conditions; these identities are often called the *classical* Rogers-Ramanujan type identities—see [1]. On the other side, some parts of representation theory of affine Kac-Moody Lie algebras lead to Rogers-Ramanujan type *colored* partition identities.

Let $N_1, \dots, N_r$, $r \geq 2$, be non-empty subsets of the set of positive integers $\mathbb{N}$ and let $\mathcal{N}$ be the multiset

$$\mathcal{N} = N_1 \cup \cdots \cup N_r. \tag{1.4}$$

If a positive integer a appears in several subsets N_i, then a appears in the multiset $\mathcal{N}$ several times. To see these elements in $\mathcal{N}$ as different, for each positive integer a we may "color" $a \in N_j$ with a "color" j by writting $a_j = (a, j) \in N_j \times \{j\}$, and then write (1.4) in terms of sets as

$$\mathcal{N} = (N_1 \times \{1\}) \cup \cdots \cup (N_r \times \{r\}) \subset \mathbb{N} \times \{1, \dots, r\}. \tag{1.5}$$

We say that elements in the multiset $\mathcal{N}$ appear in r colors. In this note a colored partition of positive integer n on the multiset $\mathcal{N}$ is

$$n = \sum_{a \in \mathcal{N}} f_a \cdot a. \tag{1.6}$$

It is clear that $f_a = 0$ for all but finitely many $a \in \mathcal{N}$ and that the partition (1.6) is determined by its "sequence" of frequencies $(f_a \mid a \in \mathcal{N})$.

EXAMPLE 1.1. Let $\mathcal{N} = N_1 \cup N_2$, where

$$N_1 = \{j \in \mathbb{N} \mid j \equiv 2, 8 \mod 10\}, N_2 = \{j \in \mathbb{N} \mid j \equiv 1, 2, 4, 5, 6, 8, 9 \mod 10\}.$$

Then parts a of colored partitions (1.6) for $\mathcal{N} = N_1 \cup N_2$ appear in two colors, 1 and 2: parts $\equiv 2, 8 \mod 10$ appear in both colors, and parts $\equiv 1, 4, 5, 6, 9 \mod 10$ appear only in color 2. Note that the generating function for colored partitions (1.6) is the infinite periodic product with modulus 10:

$$\prod_{j \equiv 1,2,2,4,5,6,8,8,9 \mod 10} (1 - q^j)^{-1}. \tag{1.7}$$

Lepowsky and Wilson gave in [4] a Lie theoretic interpretation of the classical Rogers-Ramanujan type partition identities in terms of characters of standard modules $L_{A_1^{(1)}}(\Lambda)$ for affine Kac-Moody Lie algebra of the type $A_1^{(1)}$. After their discovery it was expected that for each standard module $L_{\mathfrak{g}(A)}(\Lambda)$ for any affine Lie algebra $\mathfrak{g}(A)$ (cf. [3]) there is a Rogers-Ramanujan type partition identity, where $A_1^{(1)}$ is just "the smallest one" on the list of all affine Lie algebras:

$$A_1^{(1)}, A_\ell^{(1)}, B_\ell^{(1)}, C_\ell^{(1)}, D_\ell^{(1)}, \dots, G_2^{(1)}, A_2^{(2)}, A_{2\ell}^{(2)}, A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(2)}, D_4^{(3)}.$$

However, besides several sporadic results beyond $A_1^{(1)}$, so far this goal is not achieved.

In [2] Rogers-Ramanujan type partition identities are conjectured for all standard $C_\ell^{(1)}$-modules, stating that the number of colored partitions of n with parts satisfying certain congruence conditions is equal to the number of colored partitions (1.6) for a multiset $\mathcal{N} = \mathcal{N}_{2\ell+1}$ composed of ℓ copies of $\mathbb{N}$ and an additional copy of $(2\mathbb{N}+1)$, satisfying difference & initial conditions similar to (1.2)–(1.3) , but much more complicated. Moreover, in [2] another series of similar partition identities is conjectured for a multiset $\mathcal{N} = \mathcal{N}_{2\ell}$ composed of ℓ copies of $\mathbb{N}$, satisfying certain difference & initial conditions, but with no obvious connection to representation theory of affine Lie algebras.

In this note we conjecture yet another Rogers-Ramanujan type colored partition identities for a multiset $\mathcal{N} = \mathcal{N}_{2\ell-1}^{\mathrm{odd}}$, somewhat similar to the conjectured identities for standard $C_\ell^{(1)}$-modules, but again with no obvious connection to representation theory of affine Lie algebras.

2. Arrays with odd width w and even first row

Let $\mathcal{N} = \mathcal{N}_5^{\mathrm{odd}}$ be the colored array of natural numbers with 5 rows

$$
\begin{array}{cccccccccc}
 & 2_1 & & 4_1 & & 6_1 & & 8_1 & & \\
1_1 & & 3_1 & & 5_1 & & 7_1 & & & \\
 & 2_2 & & 4_2 & & 6_2 & & 8_2 & & \dots . \\
1_2 & & 3_2 & & 5_2 & & 7_2 & & & \\
 & 2_3 & & 4_3 & & 6_3 & & 8_3 & &
\end{array}
\tag{2.1}
$$

$\mathcal{N}$ is a multiset composed of 2 copies of $\mathbb{N}$ and an additional copy of $2\mathbb{N}$, but its elements are arranged in such a way that in the first row are even numbers and that numbers increase by one going to the right on any diagonal.

We consider colored partitions

$$
n = \sum_{a \in \mathcal{N}} f_a \cdot a, \tag{2.2}
$$

where f_a is the frequency of the part $a \in \mathcal{N}$ in the colored partition (2.2) of n. It is clear that $f_a = 0$ for all but finitely many $a \in \mathcal{N}$ and that the colored partition (2.2) is determined by its array $\mathcal{F}$ of frequencies

$$
\mathcal{F} = \begin{array}{cccccccccc}
 & f_{2_1} & & f_{4_1} & & f_{6_1} & & f_{8_1} & & \\
f_{1_1} & & f_{3_1} & & f_{5_1} & & f_{7_1} & & & \\
 & f_{2_2} & & f_{4_2} & & f_{6_2} & & f_{8_2} & & \dots . \\
f_{1_2} & & f_{3_2} & & f_{5_2} & & f_{7_2} & & & \\
 & f_{2_3} & & f_{4_3} & & f_{6_3} & & f_{8_3} & &
\end{array}
\tag{2.3}
$$

We say that two elements in the array $\mathcal{F}$ are *adjacent* if they are simultaneously on two adjacent rows and two adjacent diagonals. For example, f_{5_1} and f_{7_1} in the second row are adjacent to f_{6_1} in the first row and, just as well, adjacent to f_{6_2} in the third row. We say that the set[1] $\{a_1, a_2, a_3, \dots\}$ is a downward path $\mathcal{Z}$ in the array $\mathcal{F}$ if a_i is in the i-th row and if (a_i, a_{i+1}) is a pair of two adjacent elements for all i. For example, $\mathcal{Z} = \{f_{6_1}, f_{5_1}, f_{6_2}, f_{7_2}, f_{6_3}\}$ is a downward path in $\mathcal{F}$ and there are altogether 2^4 downward paths through f_{6_1} in the first row.

Let k be a positive integer. We say that the frequency array $\mathcal{F}$ *satisfies level k difference conditions* if

$$\sum_{m\in\mathcal{Z}} m \le k \quad \text{for all downward paths } \mathcal{Z} \text{ in } \mathcal{F}. \tag{2.4}$$

Note that the level k difference conditions for a frequency array $\mathcal{F}$ is similar to difference conditions (1.2) for a sequence of frequencies $(f_i \mid i \in \mathbb{N})$, but much more complicated.

Let $k_0, k_1, k_2, k_3 \in \mathbb{N}_0$, $k = k_0 + k_1 + k_2 + k_3 > 0$. We say that an array of frequencies $\mathcal{F}$ *is* $(k_0, k_1, k_2, k_3)^{odd}$*-admissible* if *the extended array of frequencies*

(2.5)

$$\mathcal{F}^{(k_0,k_1,k_2,k_3)\mathrm{odd}} = \begin{array}{cccccccccccc} & k_3 & & f_{2_1} & & f_{4_1} & & f_{6_1} & & f_{8_1} & & \\ k_2 & & f_{1_1} & & f_{3_1} & & f_{5_1} & & f_{7_1} & & & \\ & 0 & & f_{2_2} & & f_{4_2} & & f_{6_2} & & f_{8_2} & & \dots \\ k_1 & & f_{1_2} & & f_{3_2} & & f_{5_2} & & f_{7_2} & & & \\ & k_0 & & f_{2_3} & & f_{4_3} & & f_{6_3} & & f_{8_3} & & \end{array}$$

satisfies the level k difference conditions, that is

$$\sum_{m\in\mathcal{Z}} m \le k \quad \text{for all downward paths } \mathcal{Z} \text{ in } \mathcal{F}^{(k_0,k_1,k_2,k_3)\mathrm{odd}}. \tag{2.6}$$

Note the difference between (2.4) and (2.6): $(k_0, k_1, k_2, k_3)^{\mathrm{odd}}$-admissible frequency array $\mathcal{F}$ satisfies the level k difference conditions (2.4), but in addition to that there are new conditions on the frequencies at the beginning of the array, somewhat similar to initial condition (1.3), but much more complicated. For example, f_{1_1} in the second row must be $\le k_2$ because of (2.6) for the downward path $\mathcal{Z} = \{k_3, f_{1_1}, 0, k_1, k_0\}$.

We say that colored partitions (2.2) with $(k_0, k_1, k_2, k_3)^{\mathrm{odd}}$-admissible arrays of frequencies (2.3) are $(k_0, k_1, k_2, k_3)^{odd}$*-admissible colored partitions.*

Up till now we discussed the colored array of natural numbes $\mathcal{N} = \mathcal{N}_5^{\mathrm{odd}}$ with 5 rows, but all the notions can be extended to arrays $\mathcal{N}_w^{\mathrm{odd}}$ with $w = 2\ell - 1$ rows for $\ell = 2, 3, 4, \dots$ with initial "imaginary frequences" being (from the bottom row to the top row):

- $(k_0, k_1, k_2)^{\mathrm{odd}} = [k_0, k_1, k_2]$ for $\ell = 2$, $w = 3$,
- $(k_0, k_1, k_2, k_3)^{\mathrm{odd}} = [k_0, k_1, 0, k_2, k_3]$ for $\ell = 3$, $w = 5$,
- $(k_0, k_1, k_2, k_3, k_4)^{\mathrm{odd}} = [k_0, k_1, 0, k_2, 0, k_3, k_4]$ for $\ell = 4$, $w = 7$,

[1] or the sequence

- $\dots$
- $(k_0, k_1, \dots, k_{\ell-1}, k_\ell)^{\mathrm{odd}} = [k_0, k_1, 0, k_2, 0, k_3, \dots, 0, k_{\ell-1}, k_\ell]$ for $\ell \geq 5$.

We say that colored partitions (2.2) on $\mathcal{N} = \mathcal{N}_{2\ell-1}$ with $(k_0, k_1, \dots, k_\ell)^{\mathrm{odd}}$-admissible arrays of frequencies (2.3) are $(k_0, k_1, \dots, k_{\ell-1}, k_\ell)^{odd}$*-admissible colored partitions.*

Conjecture 2.1. *Let $\ell \geq 2$ and $k_0, k_1, \dots, k_\ell \in \mathbb{N}_0$, $k = k_0 + \cdots + k_\ell > 0$. Then the generating function for $(k_0, k_1, \dots, k_\ell)^{odd}$-admissible colored partitions can be expresed as an infinite periodic product with modulus $2\ell + 2k$.*

This conjecture is based on numerical evidence: we calculate[2] the number a_n of $(k_0, k_1, \dots, k_{\ell-1}, k_\ell)^{\mathrm{odd}}$-admissible colored partitions of n and then use Euler's factorization algorithm to write the generating function of partitions $\sum a_n q^n$ as an infinite periodic product—the Python code is available at
`https://github.com/mirkoprimc/odd_w_odd`
.

Bellow are listed some results, where

- "[1, 0, 0] product: [2, 3, 4] mod 6" means that the conjectured generating function for $(1, 0, 0)^{\mathrm{odd}}$-admissible colored partitions is
$$\prod_{j \equiv 2,3,4 \mod 6} \frac{1}{(1 - q^j)}$$
- and "[0, 1, 0] product: [1,2,-3,4,5] mod 6" means that the conjectured generating function for $(0, 1, 0)^{\mathrm{odd}}$-admissible colored partitions is
$$\frac{\prod_{j \equiv 3 \mod 6} (1 - q^j)}{\prod_{j \equiv 1,2,4,5 \mod 6} (1 - q^j)}.$$

```
[1, 0, 0] product: [2, 3, 4] mod 6

[0, 1, 0] product: [1, 2, -3, 4, 5] mod 6

[2, 0, 0] product: [2, 3, 4, 4, 5, 6] mod 8

[1, 1, 0] product: [1, 2, 4, 4, 6, 7] mod 8

[1, 0, 1] product: [2, 2, 3, 5, 6, 6] mod 8

[0, 2, 0] product: [1, 2, 2, 6, 6, 7] mod 8

[3, 0, 0] product: [2, 3, 4, 4, 5, 6, 6, 7, 8] mod 10
```

[2]by using a slightly modified code 21AAIC in [2] with built in option to choose even numbers in the top row (for p=0) or to choose odd numbers in the top row (for p=1);

```
[2, 1, 0] product: [1, 2, 4, 4, 5, 6, 6, 8, 9] mod 10

[2, 0, 1] product: [2, 2, 3, 4, 5, 6, 7, 8, 8] mod 10

[1, 2, 0] product: [1, 2, 2, 4, 5, 6, 8, 8, 9] mod 10

[1, 1, 1] product: [1, 2, 3, 4, 4, -5, 6, 6, 7, 8, 9] mod 10

[0, 3, 0] product: [1, 2, 2, 3, 4, -5, 6, 7, 8, 8, 9] mod 10

[4, 0, 0] product: [2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10] mod 12

[3, 1, 0] product: [1, 2, 4, 4, 5, 6, 6, 7, 8, 8, 10, 11] mod 12

[3, 0, 1] product: [2, 2, 3, 4, 5, 6, 6, 7, 8, 9, 10, 10] mod 12

[2, 2, 0] product: [1, 2, 2, 4, 5, 6, 6, 7, 8, 10, 10, 11] mod 12

[2, 1, 1] product: [1, 2, 3, 4, 4, 6, 6, 8, 8, 9, 10, 11] mod 12

[2, 0, 2] product: [2, 2, 3, 4, 4, 5, 7, 8, 8, 9, 10, 10] mod 12

[1, 3, 0] product: [1, 2, 2, 3, 4, 6, 6, 8, 9, 10, 10, 11] mod 12

[1, 2, 1] product: [1, 2, 2, 4, 4, 5, 7, 8, 8, 10, 10, 11] mod 12

[0, 4, 0] product: [1, 2, 2, 3, 4, 4, 8, 8, 9, 10, 10, 11] mod 12

[1, 0, 0, 0, 0] product: [2, 3, 4, 5, 6] mod 8

[0, 1, 0, 0, 0] product: [1, 2, 4, 6, 7] mod 8

[2, 0, 0, 0, 0] product: [2, 3, 4, 4, 5, 5, 6, 6, 7, 8] mod 10

[1, 1, 0, 0, 0] product: [1, 2, 3, 4, 4, 6, 6, 7, 8, 9] mod 10

[1, 0, 0, 1, 0] product: [1, 2, 2, 4, 5, 5, 6, 8, 8, 9] mod 10

[1, 0, 0, 0, 1] product: [2, 2, 3, 3, 4, 6, 7, 7, 8, 8] mod 10

[0, 2, 0, 0, 0] product: [1, 2, 2, 3, 4, 6, 7, 8, 8, 9] mod 10
```

```
[0, 1, 0, 1, 0] product: [1, 1, 2, 4, 4, 6, 6, 8, 9, 9] mod 10
```

REMARK 2.2. From the list above we see that we may expect Roger-Ramanujan type colored partition identities for most of $(k_0, k_1, \dots, k_\ell)^{\mathrm{odd}}$-admissible parameters—like the conjectured product formula (1.7) for the generating function of $(1,2,0)^{\mathrm{odd}}$-admissible colored partitions. It seems that for all the other parameters there is no infinite periodic product; the first possible case are parameters $[0,0,1,0,0]$ for $\ell = 3$ for which our code gives

```
/oddWodd/allWcasesProd
the first row parity p = 0, the highest_weight = [0, 0, 1, 0, 0]
N = 18
the exponents of the conjectured periodic product:
[2, 3, 3, 4, 4, -6, -6, -7, -7, 8, 9, 9, 9, 9, 10, 10, 10, 10,
10, 10, 10, 11, 11, -12, -12, -12, -12, -12, -12, -12, -13, -13,
-13, -13, -13, -13, -13, -13, -13, -13, -13, -13, -14, -14,
-14, -14, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15,
15, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,
16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16,
17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17, 17,
17, 17, 17, 17, 17, -18, -18, -18, -18, -18, -18, -18, -18,
-18, -18, -18, -18, -18, -18, -18, -18, -18, -18, -18, -18,
-18, -18, -18]
```

The list above encodes Euler's product for the first 19 terms in the generating function of $[0,0,1,0,0]$-admissible partitions (in the sense of [2]) for $p = 0$:

$$\frac{(1-q^6)^2(1-q^7)^2(1-q^{12})^7(1-q^{13})^{12}(1-q^{13})^{12}(1-q^{14})^4\dots}{(1-q^2)(1-q^3)^2(1-q^4)^2(1-q^8)(1-q^9)^4(1-q^{10})^7(1-q^{11})^2(1-q^{15})^{14}\dots}.$$

ACKNOWLEDGEMENTS.

I thank Marko Tadić for us being friends and the members of our Representation theory seminar in Zagreb for so many years.

I thank Stefano Capparelli, Arne Meurman and Andrej Primc for many of their ideas that are interwoven in this note.

This work is partially supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund—the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004).

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Novi kombinatorni identiteti za čudan w neparan

M. Primc

Sažetak. U ovom je članku iskazana slutnja o kombinatornim identitetima Rogers-Ramanujanovog tipa za obojene particije na aranžmanu prirodnih brojeva $\mathcal{N}_w^{\mathrm{odd}}$ s neparnim brojem redaka w tako da su prvi i zadnji redak aranžmana parni prirodni brojevi. To se na čudan način razlikuje od kombinatornih identiteta na aranžmanu $\mathcal{N}_w$ s neparnim brojem redaka w tako da su prvi i zadnji redak aranžmana neparni prirodni brojevi – kombinatornih identiteta koje su naslutili S. Capparelli, A. Meurman, A. Primc i autor vezano za standardne reprezentacije afine Liejeve algebre tipa $C_\ell^{(1)}$ za $w = 2\ell + 1$. Slutnja je zasnovana na numeričkoj provjeri.

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Received: 5.1.2023.
Revised: 24.1.2023.
Accepted: 14.2.2023.