RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 28 = 558 (2024): 57-70
DOI: https://doi.org/10.21857/90836c2xgy

ON HIGHER ORDER WEIERSTRASS POINTS ON $X_0(N)$

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For Marko Tadić, with our admiration and appreciation

Abstract. Let Γ be the Fuchsian group of the first kind. For an even integer $m \geq 4$, we describe the space $H^{m/2}(\mathfrak{R}_\Gamma)$ of $m/2$–holomorphic differentials in terms of a subspace $S_m^H(\Gamma)$ of the space of (holomorphic) cuspidal modular forms $S_m(\Gamma)$. This generalizes classical isomorphism $S_2(\Gamma) \simeq H^1(\mathfrak{R}_\Gamma)$. We study the properties of $S_m^H(\Gamma)$. As an application, we describe the algorithm implemented in SAGE for testing if a cusp at ∞ for non–hyperelliptic $X_0(N)$ is a $\frac{m}{2}$–Weierstrass point.

1. Introduction

Let Γ be the Fuchsian group of the first kind [4, Section 1.7, page 28]. Examples of such groups are the important Hecke congruence groups $\Gamma_0(N)$, $N \geq 1$. Let $\mathbb{H}$ be the complex upper half-plane. The quotient $\Gamma\backslash\mathbb{H}$ can be compactified by adding a finite number of Γ-orbits of points in $\mathbb{R}\cup\{\infty\}$ called cusps of Γ. In this way we obtain a compact Riemann surface which will be denoted by $\mathfrak{R}_\Gamma$. We write $g(\Gamma)$ for the genus of $\mathfrak{R}_\Gamma$. For $\Gamma = \Gamma_0(N)$, we let $X_0(N) = \mathfrak{R}_\Gamma$. For $x \in \mathbb{H}$ or $x \in \mathbb{R}\cup\{\infty\}$ a cusp for Γ, let $\mathfrak{a}_x$ be the Γ–orbit of x i.e., the corresponding point in $\mathfrak{R}_\Gamma$. For $l \geq 1$, let $H^l(\mathfrak{R}_\Gamma)$ be the space of all holomorphic differentials on $\mathfrak{R}_\Gamma$ (see [2], or Section 2) in this paper).

Let $m \geq 2$ be an even integer. Let $S_m(\Gamma)$ be the space of (holomorphic) cusp forms of weight m. It is well–known that $S_2(\Gamma)$ is naturally isomorphic to the vector space $H^1(\mathfrak{R}_\Gamma)$ (see [4, Theorem 2.3.2]). This is employed on many instances in studying various properties of modular curves (see for example [15, Chapter 6]). In this paper we study the generalization of this concept to the holomorphic differentials of higher order.

For an even integer $m \geq 4$, the space $S_m(\Gamma)$ is usually too big to be isomorphic to $H^{m/2}(\mathfrak{R}_\Gamma)$ due to presence of cusps and elliptic points. So, in

2020 *Mathematics Subject Classification.* 11F11.
Key words and phrases. Wronskians, holomorphic differentials, cuspidal modular forms.

general, $H^{m/2}(\mathfrak{R}_\Gamma)$ corresponds to a subspace $S_m^H(\Gamma)$ of $S_m(\Gamma)$ (see Lemma 3.1). We study the space $S_m^H(\Gamma)$ in detail in Section 3 (see Lemma 3.2).

We recall in Section 2 (see Definition 2.1) that $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is a $m/2$-Weierstrass point if there exists a non–zero holomorphic differential $\omega \in H^{m/2}(\mathfrak{R}_\Gamma)$ such that $\nu_{\mathfrak{a}}(\omega) \geq \dim H^{m/2}(\mathfrak{R}_\Gamma)$. When $m = 2$ we speak about classical Weierstrass points. So, 1-Weierstrass points are simply Weierstrass points. Weierstrass points on modular curves are very-well studied (see for example [15, Chapter 6], [12], [13], [16], [17], [18], [1]). Higher–order Weierstrass points has not been not studied much (see for example [12], [14]).

The case $m \geq 4$ is more complex. It is studied in Section 4. We recall that $\mathfrak{R}_\Gamma$ is hyperelliptic if $g(\Gamma) \geq 2$, and there is a degree two map onto $\mathbb{P}^1$. Under the assumptions that $\mathfrak{R}_\Gamma$ is not hyperelliptic and that $\mathfrak{a}_\infty$ is a cusp for Γ, we develop a criterion that $\mathfrak{a}_\infty$ is a $\frac{m}{2}$–Weierstrass point for $\mathfrak{R}_\Gamma$ (see Theorem 4.5). The Section 4 is mainly devoted to the proof of Theorem 4.5. For $\Gamma = \Gamma_0(N)$, $X_0(N)$ is not hyperelliptic for most of values of N (see below), and Theorem 4.5 contains an algorithm for testing that $\mathfrak{a}_\infty$ is a $\frac{m}{2}$–Weierstrass point. We illustrate this by several examples (see the end of Section 4).

Let $\Gamma = \Gamma_0(N)$, $N \geq 1$. We recall that $g(\Gamma_0(N)) \geq 2$ unless

$$\begin{cases} N \in \{1-10, 12, 13, 16, 18, 25\} & \text{when } g(\Gamma_0(N)) = 0, \text{ and} \\ N \in \{11, 14, 15, 17, 19-21, 24, 27, 32, 36, 49\} & \text{when } g(\Gamma_0(N)) = 1. \end{cases}$$

Let $g(\Gamma_0(N)) \geq 2$. Then, we remark that Ogg [13] has determined all $X_0(N)$ which are hyperelliptic curves. In view of Ogg's paper, we see that $X_0(N)$ is **not hyperelliptic** for $N \in \{34, 38, 42, 43, 44, 45, 51-58, 60-70\}$ or $N \geq 72$. This implies $g(\Gamma_0(N)) \geq 3$.

In this paper we continue our earlier approach in studying various aspects of modular curves ([3], [7], [8], [9], [10]). This paper contains large parts of previous manuscript [11]. The rest of manuscript [11], related to cups forms constructed out of Wronskians, would be extended and published separately since does not fit here.

We would like to thank I. Kodrnja for her help with the SAGE system. Also, we would like to thank M. Kazalicki and F. Najman for some useful discussions about modular forms and curves in general.

2. Preliminaries I: Holomorphic Differentials and m–Weierstrass Points

The goal of the present section is to recall necessary facts about holomorphic differentials and m–Weierstrass points on a general compact Riemann surface, phrased in terms of $\mathfrak{R}_\Gamma$ where Γ is a Fuchsian group of the first kind. We let $D^m(\mathfrak{R}_\Gamma)$ (resp., $H^m(\mathfrak{R}_\Gamma)$)be the space of meromorphic (resp., holomorphic) differential of degree m on $\mathfrak{R}_\Gamma$ for each $m \in \mathbb{Z}$. We recall that

$D^0(\mathfrak{R}_\Gamma) = \mathbb{C}(\mathfrak{R}_\Gamma)$, and $D^m(\mathfrak{R}_\Gamma) \neq 0$ for all other $m \in \mathbb{Z}$. In fact, if we fix a non–zero $\omega \in D^1(\mathfrak{R}_\Gamma)$, then $D^m(\mathfrak{R}_\Gamma) = \mathbb{C}(\mathfrak{R}_\Gamma)\,\omega^n$. We have the following:

$$\deg(\mathrm{div}(\omega)) = 2m(g(\Gamma)-1), \quad \omega \in D^m(\mathfrak{R}_\Gamma),\ \omega \neq 0. \tag{2.1}$$

We shall be interested in the case $m \geq 1$, and in holomorphic differentials. We recall [2, Proposition III.5.2] that

$$\dim H^m(\mathfrak{R}_\Gamma) = \begin{cases} 0 & \text{if } m \geq 1, g(\Gamma) = 0; \\ g(\Gamma) & \text{if } m = 1,\ g(\Gamma) \geq 1; \\ g(\Gamma) & \text{if } m \geq 2,\ g(\Gamma) = 1; \\ (2m-1)(g(\Gamma)-1) & \text{if } m \geq 2,\ g(\Gamma) \geq 2. \end{cases} \tag{2.2}$$

This follows easily from Riemann-Roch theorem. Recall that the canonical class K is simply the divisor of any non–zero meromorphic form ω on $\mathfrak{R}_\Gamma$. Different choices of a ω differ by a divisor of a non–zero function $f \in \mathbb{C}(\mathfrak{R}_\Gamma)$

$$\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega).$$

Different choices of ω have the same degree since $\deg(\mathrm{div}(f)) = 0$.

For a divisor $\mathfrak{a}$, we let

$$L(\mathfrak{a}) = \{f \in \mathbb{C}(\mathfrak{R}_\Gamma);\quad f = 0 \text{ or } \mathrm{div}(f) + \mathfrak{a} \geq 0\}.$$

We have the following three facts:

(1) for $\mathfrak{a} = 0$, we have $L(\mathfrak{a}) = \mathbb{C}$;
(2) if $\deg(\mathfrak{a}) < 0$, then $L(\mathfrak{a}) = 0$;
(3) the Riemann-Roch theorem: $\dim L(\mathfrak{a}) = \deg(\mathfrak{a}) - g(\Gamma) + 1 + \dim L(K - \mathfrak{a})$.

Now, it is obvious that $f\omega^m \in H^m(\mathfrak{R}_\Gamma)$ if and only if

$$\mathrm{div}(f\omega^m) = \mathrm{div}(f) + m\mathrm{div}(\omega) = \mathrm{div}(f) + mK \geq 0.$$

Equivalently, $f \in L(mK)$. Thus, we have that $\dim H^m(\mathfrak{R}_\Gamma) = \dim L(mK)$. Finally, by the Riemann-Roch theorem, we have the following:

$$\begin{aligned} \dim L(mK) &= \deg(mK) - g(\Gamma) + 1 + \dim L((1-m)K) \\ &= (2m-1)(g(\Gamma)-1) + \dim L((1-m)K). \end{aligned}$$

Now, if $g(\Gamma) \geq 2$, then $\deg(K) = 2(g(\Gamma)-1) > 0$, and the claim easily follows from (1) and (2) above. Next, assume that $g(\Gamma) = 1$. If $\omega \in \dim H^1(\mathfrak{R}_\Gamma)$ s non–zero, then it has a degree zero. Thus, it has no zeroes. This means that $\omega H^{l-1}(\mathfrak{R}_\Gamma) = H^l(\mathfrak{R}_\Gamma)$ for all $l \in \mathbb{Z}$. But since obviously $H^0(\mathfrak{R}_\Gamma)$ consists of constants only, we obtain the claim. Finally, the case $g(\Gamma) = 0$ is obvious from (2) since the degree of mK is $2m(g(\Gamma)-1) < 0$ for all $m \geq 1$.

Assume that $g(\Gamma) \geq 1$ and $m \geq 1$. Then, $\dim H^m(\mathfrak{R}_\Gamma) \neq 0$. Let $t = \dim H^m(\mathfrak{R}_\Gamma)$. We fix the basis $\omega_1, \ldots, \omega_t$ of $H^m(\mathfrak{R}_\Gamma)$. Let z be any local coordinate on $\mathfrak{R}_\Gamma$. Then, locally there exists unique holomorphic functions

$\varphi_1, \ldots, \varphi_t$ such that $\omega_i = \varphi_i (dz)^m$, for all i. Then, again locally, we can consider the Wronskian W_z defined by

$$W_z(\omega_1, \ldots, \omega_t) \stackrel{def}{=} \begin{vmatrix} \varphi_1(z) & \cdots & \varphi_t(z) \\ \frac{d\varphi_1(z)}{dz} & \cdots & \frac{d\varphi_t(z)}{dz} \\ & \cdots & \\ \frac{d^{t-1}\varphi_1(z)}{dz^{k-1}} & \cdots & \frac{d^{t-1}\varphi_t(z)}{dz^{t-1}} \end{vmatrix}. \tag{2.3}$$

As proved in [2, Proposition III.5.10], collection of all

$$W_z(\omega_1, \ldots, \omega_t)(dz)^{\frac{t}{2}(2m-1+t)}, \tag{2.4}$$

defines a non–zero holomorphic differential form

$$W(\omega_1, \ldots, \omega_t) \in H^{\frac{t}{2}(2m-1+t)}(\mathfrak{R}_\Gamma).$$

We call this form the Wronskian of the basis $\omega_1, \ldots, \omega_t$. It is obvious that a different choice of a basis of $H^m(\mathfrak{R}_\Gamma)$ results in a Wronskian which differs from $W(\omega_1, \ldots, \omega_t)$ by a multiplication by a non–zero complex number. Also, the degree is given by

$$\deg(\operatorname{div}(W(\omega_1, \ldots, \omega_t))) = t(2m-1+t)(g(\Gamma)-1). \tag{2.5}$$

Following [2, III.5.9], we make the following definition:

Definition 2.1. *Let $m \geq 1$ be an integer. We say that $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is a m-Weierstrass point if there exists a non–zero $\omega \in H^m(\mathfrak{R}_\Gamma)$ such that*

$$\nu_{\mathfrak{a}}(\omega) \geq \dim H^m(\mathfrak{R}_\Gamma).$$

Equivalently [2, Proposition III.5.10] , if

$$\nu_{\mathfrak{a}}(W(\omega_1, \ldots, \omega_t)) \geq 1.$$

When $m = 1$ we speak about classical Weierstrass points. So, 1-Weierstrass points are simply Weierstrass points.

3. Preliminaries II: Interpretation in Terms of Modular Forms

In this section we give interpretation of results of Section 2 in terms of modular forms. Again, Γ stand for a Fuschsian group of the first kind. Let $m \geq 2$ be an even integer. We consider the space $\mathcal{A}_m(\Gamma)$ be the space of all meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that $f(\gamma.z) = j(\gamma, z)^m f(z)$ ($z \in \mathbb{H}$, $\gamma \in \Gamma$) which are meromorphic at every cusp for Γ. By [4, Theorem 2.3.1], there exists isomorphism of vector spaces $\mathcal{A}_m(\Gamma) \longrightarrow D^{m/2}(\mathfrak{R}_\Gamma)$, denoted by $f \longmapsto \omega_f$ such that the following holds (see [4, Theorem 2.3.3]):

$$\begin{aligned}
&\nu_{\mathfrak{a}_\xi}(f) = \nu_{\mathfrak{a}_\xi}(\omega_f) + \frac{m}{2}\left(1 - \frac{1}{e_{\mathfrak{a}_\xi}}\right) \quad \text{if } \xi \in \mathbb{H},\\
&\nu_{\mathfrak{a}}(f) = \nu_{\mathfrak{a}}(\omega_f) + \frac{m}{2} \quad \text{for } \Gamma\text{-cusp } \mathfrak{a},\\
&\operatorname{div}(f) = \operatorname{div}(\omega_f) + \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \frac{m}{2}\left(1 - \frac{1}{e_{\mathfrak{a}}}\right)\mathfrak{a},
\end{aligned} \tag{3.1}$$

where $1/e_{\mathfrak{a}} = 0$ if $\mathfrak{a}$ is a cusp, and $\mathfrak{a}_\xi \in \mathfrak{R}_\Gamma$ is the projection of ξ to $\mathfrak{R}_\Gamma$. Let $f \in M_m(\Gamma)$ be a modular form. Using (3.1), we obtain

$$\operatorname{div}(\omega_f) = \mathfrak{c}'_f - \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right]\mathfrak{a} - \frac{m}{2}\sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma,\\ cusp}} \mathfrak{b}, \tag{3.2}$$

where as usual in our previous papers (see for example [9, Lemma 2.2]), $\mathfrak{c}'_f$ is an effective integral divisor which satisfies

$$\operatorname{div}(f) = \mathfrak{c}'_f + \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \left(\frac{m}{2}(1 - 1/e_{\mathfrak{a}}) - \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right]\right)\mathfrak{a}. \tag{3.3}$$

This shows that ω_f is holomorphic everywhere except maybe at cusps and elliptic points. Moreover, if $f \in S_m(\Gamma)$ (a space of cuspidal modular forms), then

$$\operatorname{div}(\omega_f) = \mathfrak{c}_f - \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right]\mathfrak{a} - \left(\frac{m}{2} - 1\right)\sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma,\\ cusp}} \mathfrak{b}, \tag{3.4}$$

Next, we determine all $f \in M_m(\Gamma)$ such that $\omega_f \in H^{m/2}(\mathfrak{R}_\Gamma)$. From (3.2) we see that such f must belong to $S_m(\Gamma)$, and from (3.4)

$$\mathfrak{c}_f \geq \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right]\mathfrak{a} + \left(\frac{m}{2} - 1\right)\sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma,\\ cusp}} \mathfrak{b}, \tag{3.5}$$

where the integral divisor c_f is defined by (see [9, Lemma 2.2])

$$\mathfrak{c}_f \overset{def}{=} \mathfrak{c}'_f - \sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma,\\ cusp}} \mathfrak{b}.$$

Now, above considerations immediately imply the following result:

LEMMA 3.1. *We define the subspace of $S_m(\Gamma)$ by*

$$S_m^H(\Gamma) = \{f \in S_m(\Gamma);\quad f = 0 \text{ or } f \text{ satisfies (3.5)}\}.$$

Then, the map $f \longmapsto \omega_f$ is an isomorphism of $S_m^H(\Gamma)$ onto $H^{m/2}(\mathfrak{R}_\Gamma)$.

We remark that when $m=2$, (3.5) and reduces to obvious $\mathfrak{c}_f \geq 0$. Hence, $S_2^H(\Gamma) = S_2(\Gamma)$ recovering the standard isomorphism of $S_2(\Gamma)$ and $H^1(\mathfrak{R}_\Gamma)$ (see [4, Theorem 2.3.2]). We continue by collecting a few properties of spaces $S_m^H(\Gamma)$.

LEMMA 3.2. *Assume that $m, n \geq 2$ are even integers. Let Γ be a Fuchsian group of the first kind. Then, we have the following:*

(i) $S_2^H(\Gamma) = S_2(\Gamma)$.
(ii) $S_m^H(\Gamma)$ *is isomorphic to* $H^{m/2}(\mathfrak{R}_\Gamma)$.
(iii) $S_m^H(\Gamma) = \{0\}$ *if* $g(\Gamma) = 0$.
(iv) *Assume that* $g(\Gamma) = 1$. *Let us write* $S_2(\Gamma) = \mathbb{C} \cdot f$, *for some non–zero cuspidal form* f. *Then, we have* $S_m^H(\Gamma) = \mathbb{C} \cdot f^{m/2}$.
(v) $\dim S_m^H(\Gamma) = (m-1)(g(\Gamma)-1)$ *if* $g(\Gamma) \geq 2$ *and* $m \geq 4$.
(vi) $S_m^H(\Gamma) \cdot S_n^H(\Gamma) \subset S_{m+n}^H(\Gamma)$.
(vii) *There are no* $m/2$*–Weierstrass points on* $\mathfrak{R}_\Gamma$ *for* $g(\Gamma) \in \{0, 1\}$.
(viii) *Assume that* $g(\Gamma) \geq 2$, *and* $\mathfrak{a}_\infty$ *is a* Γ*-cusp. Then,* $\mathfrak{a}_\infty$ *is a* $\frac{m}{2}$*–Weierstrass point if and only if there exists* $f \in S_m^H(\Gamma)$, $f \neq 0$, *such that*

$$\mathfrak{c}'_f(\mathfrak{a}_\infty) \geq \begin{cases} \frac{m}{2} + g(\Gamma) & \text{if } m = 2; \\ \frac{m}{2} + (m-1)(g(\Gamma)-1) & \text{if } m \geq 4. \end{cases}$$

(ix) *Assume that* $g(\Gamma) \geq 2$, *and* $\mathfrak{a}_\infty$ *is a* Γ*-cusp. Then, there exists a basis* $f_1, \dots f_t$ *of* $S_m^H(\Gamma)$ *such that their* q*–expansions are of the form*

$$f_u = a_u q^{i_u} + \text{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$\frac{m}{2} \leq i_1 < i_2 < \cdots < i_t \leq \frac{m}{2} + m\,(g(\Gamma) - 1)\,,$$

and

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

(x) *Assume that* $g(\Gamma) \geq 2$, *and* $\mathfrak{a}_\infty$ *is a* Γ*-cusp. Then,* $\mathfrak{a}_\infty$ *is* **not** *a* $\frac{m}{2}$*–Weierstrass point if and only if there exists a basis* $f_1, \dots f_t$ *of* $S_m^H(\Gamma)$ *such that their* q*–expansions are of the form*

$$f_u = a_u q^{u+m/2-1} + \text{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

(xi) *Assume that* $g(\Gamma) \geq 1$. *Let us fix a basis* $f_1, \dots, f_t$ *of* $S_m^H(\Gamma)$, *and let* $\omega_1, \dots, \omega_t$ *be the corresponding basis of* $H^{m/2}(\mathfrak{R}_\Gamma)$. *As in Section 2, we construct holomorphic differential* $W(\omega_1, \dots, \omega_t) \in H^{\frac{t}{2}(m-1+t)}(\mathfrak{R}_\Gamma)$. *We also construct the Wronskian* $W(f_1, \dots, f_t) \in S_{t(m+t-1)}(\Gamma)$ *(see Lemma 3.3 below). Then, we have the following equality* $\omega_{W(f_1,\dots,f_t)} = W(\omega_1, \dots, \omega_t)$. *In particular, we obtain the following:* $W(f_1, \dots, f_t) \in$

$S^H_{t(m+t-1)}(\Gamma)$. *Moreover, assume that* $\mathfrak{a}_\infty$ *is a* Γ*-cusp. Then,* $\mathfrak{a}_\infty$ *is a* $\frac{m}{2}$*–Weierstrass point if and only if*

$$\mathfrak{c}_{W(f_1,\dots,f_t)}(\mathfrak{a}_\infty) \geq \frac{t}{2}(m-1+t).$$

Proof. (i) and (ii) follow from above discussion. Next, using above discussion and (2.2) we obtain

$$\dim S^H_m(\Gamma) = \dim H^{m/2}(\mathfrak{R}_\Gamma) = \begin{cases} 0 & \text{if } \ m \geq 2, g(\Gamma) = 0; \\ g(\Gamma) & \text{if } \ m = 2, \ g(\Gamma) \geq 1; \\ g(\Gamma) & \text{if } \ m \geq 4, \ g(\Gamma) = 1; \\ (m-1)(g(\Gamma)-1) & \text{if } \ m \geq 4, \ g(\Gamma) \geq 2. \end{cases}$$

This immediately implies (iii) and (v). Next, assume that $g(\Gamma) = 1$. Then, we see that $\dim S^H_m(\Gamma) \leq 1$ for all even integers $m \geq 4$. It is well known that $f^{m/2} \in S_m(\Gamma)$. Next, (3.5) for $m = 2$, and [9, Lemma 2.2] imply $\mathrm{div}(\omega_f) = \mathfrak{c}_f = 0$. Using [4, Theorem 2.3.2], we obtain $\omega_{f^{m/2}} = \omega_f^{m/2}$. Hence,

$$\mathrm{div}(\omega_{f^{m/2}}) = \frac{m}{2}\mathrm{div}(\omega_f) = 0.$$

Then, applying (3.4) with $f^{m/2}$ in place of f, we obtain

$$\mathfrak{c}_{f^{m/2}} = \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma, \\ elliptic}} \Big[\frac{m}{2}(1-1/e_\mathfrak{a})\Big]\,\mathfrak{a} + \Big(\frac{m}{2}-1\Big) \sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b}.$$

This shows that $f^{m/2} \in S^H_m(\Gamma)$ proving (iv). (vi) follows from [4, Theorem 2.3.1]. (vii) follows immediately form the discussion in Section 2, and it is well–known. (viii) is a reinterpretation of Definition 2.1. The details are left to the reader as an easy exercise. Now, we prove (ix) and (x). The case of $g(\Gamma) = 1$ are obvious since we have $S_2(\Gamma) = \mathbb{C} \cdot f$ where

$$\mathfrak{c}'_f = \mathfrak{a}_\infty + \sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma, \ cusp \\ \mathfrak{b}\neq\mathfrak{a}_\infty}} \mathfrak{b}.$$

Next, we prove (ix) and (x) in the case $g(\Gamma) \geq 2$. Let $f \in S^H_m(\Gamma)$, $f \neq 0$. Then, by the definition of $S^H_m(\Gamma)$, we obtain

$$\mathfrak{c}'_f(\mathfrak{a}_\infty) = 1 + \mathfrak{c}_f(\mathfrak{a}_\infty) \geq 1 + \Big(\frac{m}{2}-1\Big) = \frac{m}{2}. \tag{3.6}$$

On the other hand, again by the definition of $S^H_m(\Gamma)$ (see (3.5)) and the fact that $\mathfrak{c}'_f \geq 0$, we obtain

$$\deg(\mathfrak{c}'_f) = \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \mathfrak{c}'_f(\mathfrak{a}) \geq \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \mathfrak{c}'_f(\mathfrak{a}) + \sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma,\ cusp\\ \mathfrak{b}\neq\mathfrak{a}_\infty}} \mathfrak{c}'_f(\mathfrak{b}) + \mathfrak{c}'_f(\mathfrak{a}_\infty) \geq$$
$$\sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}} \left[\frac{m}{2}(1-1/e_\mathfrak{a})\right] + \frac{m}{2}(t-1) + \mathfrak{c}'_f(\mathfrak{a}_\infty)$$

where t is the number of nonequivalent Γ–cusps. The degree $\deg(\mathfrak{c}'_f)$ is given by

$$\begin{aligned}\deg(\mathfrak{c}'_f) &= \dim M_m(\Gamma) + g(\Gamma) - 1\\ &= \begin{cases} 2(g(\Gamma)-1)+t & \text{if } \ m=2;\\ m(g(\Gamma)-1)+\frac{m}{2}t+\sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma,\\ elliptic}}\left[\frac{m}{2}(1-1/e_\mathfrak{a})\right] & \text{if } \ m\geq 4.\end{cases}\end{aligned}$$

Combining with the previous inequality, we obtain

$$\mathfrak{c}'_f(\mathfrak{a}_\infty) \leq \frac{m}{2} + m(g(\Gamma)-1) \ \text{ if } \ m\geq 2.$$

Having in mind (3.6), the rest of (ix) has a standard argument (see for example [6, Lemma 4.3]). Finally, (x) follows (viii) and (ix). The last claim (xi) follows easily if we note that $\omega_{W(f_1,\ldots,f_t)} = W(\omega_1,\ldots,\omega_t)$ is equality of two meromorphic differentials. So, it is enough to pick a non–elliptic point $w\in\mathbb{H}$ and check the equality in a small neighborhood of w in $\mathbb{H}$. But this local identity is obvious, and the claim (xi) follows. □

We end this section by recalling a generalization of the usual notion of the Wronskian of modular forms [18], ([15], 6.3.1), ([6], the proof of Theorem 4-5), and ([8], Lemma 4-1).

LEMMA 3.3. *Let $m\geq 1$. Then, for any sequence $f_1,\ldots,f_k\in M_m(\Gamma)$, the Wronskian*

$$W(f_1,\ldots,f_k)(z) \stackrel{def}{=} \begin{vmatrix} f_1(z) & \cdots & f_k(z)\\ \frac{df_1(z)}{dz} & \cdots & \frac{df_k(z)}{dz}\\ & \cdots & \\ \frac{d^{k-1}f_1(z)}{dz^{k-1}} & \cdots & \frac{d^{k-1}f_k(z)}{dz^{k-1}}\end{vmatrix}$$

is a cuspidal modular form in $S_{k(m+k-1)}(\Gamma)$ if $k\geq 2$. If $f_1,\ldots,f_k$ are linearly independent, then $W(f_1,\ldots,f_k)\neq 0$.

4. An Algorithm For $X_0(N)$

In this section we describe the algorithm for testing $\mathfrak{a}_\infty$ to be a $\frac{m}{2}$–Weierstrass point on $X_0(N)$ for all even integers $m = 2, 4, 6, \dots$ assuming that $X_0(N)$ is not hyperelliptic (see Introduction).

We begin with the following remark. The criterion in Lemma 3.2 (x) is a quite good criterion to check whether or not $\mathfrak{a}_\infty$ is a Weierstrass points (the case $m = 2$) using computer systems such as SAGE since we need just to list the basis. This case is well-known (see [15, Definition 6.1]). This criterion has been used in practical computations in combination with SAGE in [9] for $\Gamma = \Gamma_0(N)$.

But, Lemma 3.2 (x) is not good when $m \geq 4$ in most interesting cases. For example, when Γ has elliptic points and for m large enough, there is a basis of the space of cuspidal forms $S_m(\Gamma)$ which contains properly normalized cusp forms having leading terms $q^{m/2}, \dots, q^{m/2+m(g(\Gamma)-1)}$. This follows from the following two lemmas.

First, we recall [8, Lemma 2.9] which is well-known in a slightly different notation ([16], [17]):

Lemma 4.1. *Let $m \geq 4$ be an even integer such that $\dim S_m(\Gamma) \geq g(\Gamma)+1$. Then, for all $1 \leq i \leq t_m - g$, there exists $f_i \in S_m(\Gamma)$ such that $\mathfrak{c}'_{f_i}(\mathfrak{a}_\infty) = i$.*

The second lemma is even more elementary, and it follows from the explicit formula for the dimension of $S_m(\Gamma)$. The details are left to the reader as an exercise.

Lemma 4.2. *Assume that Γ has elliptic points. (For example, $\Gamma = \Gamma_0(N)$.) Then, for a sufficiently large even integer m, we have $\frac{m}{2} + m(g(\Gamma) - 1) \leq \dim S_m(\Gamma) - g(\Gamma)$.*

Thus, under above assumptions, in view of Lemma 3.2 (ix), the listing of basis of $S_m(\Gamma_0(N))$ in SAGE does not give any information about basis of $S_m^H(H)$.

Now, we explain the algorithm for testing that $\mathfrak{a}_\infty$ is a $\frac{m}{2}$–Weierstrass point for $m \geq 4$. In what follows we assume that $g(\Gamma) \geq 2$ (see Lemma 3.2 (vii)). We start with the following lemma.

Lemma 4.3. *Let $m \geq 4$ be an even integer. Let us select a basis $f_0, \dots, f_{g-1}$, $g = g(\Gamma)$, of $S_2(\Gamma)$. Then, all of $\binom{g+\frac{m}{2}-1}{\frac{m}{2}}$ monomials $f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}$, $\alpha_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}$, belong to $S_m^H(\Gamma)$. We denote by $S_{m,2}^H(\Gamma)$ this subspace of $S_m^H(\Gamma)$.*

Proof. This follows from Lemma 3.2 (vi) since $S_2(\Gamma) = S_2^H(\Gamma)$ (see Lemma 3.2 (i)). ∎

The next lemma is crucial for the algorithm.

LEMMA 4.4. *Let $m \geq 4$ be an even integer. Assume that $\mathfrak{R}_\Gamma$ is not hyperelliptic. Then, we have $S^H_{m,2}(\Gamma) = S^H_m(\Gamma)$.*

PROOF. We use notation of Section 2 freely. The reader should review Lemma 3.2. Let $F \in S_2(\Gamma)$, $F \neq 0$. We define a holomorphic differential form $\omega \in H(\mathfrak{R}_\Gamma)$ by $\omega = \omega_F$. Define a canonical class K by $K = \operatorname{div}(\omega)$. We prove the following:

$$L\left(\frac{m}{2}K\right) = \left\{ \frac{f}{F^{m/2}};\quad f \in S^H_m(\Gamma) \right\}. \tag{4.1}$$

The case $m = 2$ is of course well–known. By the Riemann-Roch theorem and standard results recalled in Section 2 we have

$$\begin{aligned} \dim L\left(\frac{m}{2}K\right) &= \deg\left(\frac{m}{2}K\right) - g(\Gamma) + 1 + \dim L\left(\left(1 - \frac{m}{2}\right)K\right) \\ &= (m-1)(g(\Gamma)-1) + \begin{cases} 1 & \text{if } \ m = 2; \\ 0 & \text{if } \ m \geq 4. \end{cases} \end{aligned}$$

Next, we recall that $S_2(\Gamma) = S^H_2(\Gamma)$ (see Lemma 3.2 (i)). Then, Lemma 3.2 (vi) we obtain $F^{\frac{m}{2}} \in S^H_m(\Gamma)$. Therefore, $f/F^{\frac{m}{2}} \in \mathbb{C}(\mathfrak{R}_\Gamma)$ for all $f \in S^H_m(\Gamma)$.

By the correspondence described in (3.1) we have

$$\begin{aligned} \operatorname{div}(F) = \operatorname{div}(\omega_F) + \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma}\left(1 - \frac{1}{e_\mathfrak{a}}\right)\mathfrak{a} &= K + \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma}\left(1 - \frac{1}{e_\mathfrak{a}}\right)\mathfrak{a} \\ &= K + \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma, \\ elliptic}} (1 - 1/e_\mathfrak{a})\mathfrak{a} + \sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b}. \end{aligned}$$

Thus, for $f \in S^H_m(\Gamma)$, we have the following:

$$\begin{aligned} \operatorname{div}\left(\frac{f}{F^{\frac{m}{2}}}\right) + \frac{m}{2}K &= \operatorname{div}(f) - \frac{m}{2}\operatorname{div}(F) + \frac{m}{2}K \\ &= \operatorname{div}(f) - \frac{m}{2}\sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma, \\ elliptic}} (1 - 1/e_\mathfrak{a})\mathfrak{a} - \frac{m}{2}\sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b} \end{aligned}$$

Next, using [9, Lemma 2.2] (see (3.3) in this paper), the right–hand side becomes

$$\mathfrak{c}'_f - \sum_{\substack{\mathfrak{a}\in\mathfrak{R}_\Gamma, \\ elliptic}} \left[\frac{m}{2}(1 - 1/e_\mathfrak{a})\right]\mathfrak{a} - \frac{m}{2}\sum_{\substack{\mathfrak{b}\in\mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b} \geq 0$$

by the definition of $S^H_m(\Gamma)$. Hence, $f/F^{\frac{m}{2}} \in L\left(\frac{m}{2}K\right)$. Now, comparing the dimensions in (4.1), we obtain their equality. This proves (4.1).

Next, let W be any finite dimensional $\mathbb{C}$–vector space. Let $\mathrm{Symm}^k(W)$ be symmetric tensors of degree $k \geq 1$. Then, by Max Noether theorem ([5], Chapter VII, Corollary 3.27) the multiplication induces a surjective map $\mathrm{Symm}^k(L(K)) \to L\left(\frac{m}{2}K\right)$. The lemma follows. □

Now, after all of these preparations, we come to the main result of the paper. It gives a good criterion for **testing** that $\mathfrak{a}_\infty$ is a $\frac{m}{2}$–Weierstrass point for $m \geq 4$. We give examples of explicit computations below.

THEOREM 4.5. *Let $m \geq 4$ be an even integer. Assume that $\mathfrak{R}_\Gamma$ is not hyperelliptic. Assume that $\mathfrak{a}_\infty$ is a cusp for Γ. Let us select a basis $f_0, \ldots, f_{g-1}$, $g = g(\Gamma)$, of $S_2(\Gamma)$ (listed by their q–expansions using SAGE system if $\Gamma = \Gamma_0(N)$). Compute q–expansions of all monomials*

$$f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \ \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}.$$

Then, $\mathfrak{a}_\infty$ is **not** *a $\frac{m}{2}$–Weierstrass point if and only if there exist $\mathbb{C}$–linear combinations of such monomials, say $F_1, \ldots F_t$, $t = (m-1)(g-1)$, such that their q–expansions are of the form*

$$F_u = a_u q^{u+m/2-1} + \text{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

PROOF. We combine Lemmas 4.4 and 3.2 (x). □

We make the method of Theorem 4.5 more explicit as follows for $\Gamma = \Gamma_0(N)$. First, the number of monomials is

$$\binom{g + m/2 - 1}{m/2}.$$

Then, by selecting the first $m/2 + m \cdot (g-1)$ terms from q–expansions of the monomials, we can create the matrix of size

$$\binom{g + m/2 - 1}{m/2} \times \left(\frac{m}{2} + m \cdot (g-1)\right).$$

Then, we perform suitable integral Gaussian elimination method to transform the matrix into row echelon form. The procedure is as follows. We successively sort and transform the row matrices to cancel the leading row coefficients with the same number of leading zeros as their predecessor. We use the *Quicksort algorithm* for sorting. We obtain the transformed matrix and the transformation matrix. The non-null rows of the transformed matrix give the q-expansions of the basis elements of $S_m^H(\Gamma)$, and the corresponding rows

of the transformation matrix give the corresponding linear combinations of monomials. Using this method, we check the following from Theorem 4.5.

Let $m = 4$. Then, for $X_0(34)$ the basis of $S_4^H(\Gamma_0(34))$ is given by

$$\begin{aligned}
f_0^2 &= q^2 - 4q^5 - 4q^6 + 12q^8 + 12q^9 - 2q^{10} \\
f_0f_1 &= q^3 - q^5 - 2q^6 - 2q^7 + 2q^8 + 5q^9 + 2q^{10} \\
f_0f_2 &= q^4 - 2q^5 - q^6 - q^7 + 6q^8 + 6q^9 + 2q^{10} \\
-f_1^2 + f_0f_2 &= -2q^5 + q^6 - q^7 + 5q^8 + 6q^9 + 4q^{10} \\
-f_1^2 + f_0f_2 + 2f_1f_2 &= -3q^6 - 5q^7 + 11q^8 + 16q^9 + 2q^{10} \\
-f_1^2 + f_0f_2 + 2f_1f_2 + 3f_2^2 &= -17q^7 + 17q^8 + 34q^9 + 17q^{10}
\end{aligned}$$

Their first exponents are $\frac{m}{2} = 2, 3, 4, 5, 6, \frac{m}{2} + (m-1)(g-1) - 1 = 7$ which shows that $\mathfrak{a}_\infty$ is not 2–Weierstrass point for $X_0(34)$.

For $X_0(55)$, the basis of $S_4^H(\Gamma_0(55))$ is given by

$$\begin{aligned}
f_0^2 &= q^2 - 2q^8 + \cdots \\
f_0f_1 &= q^3 - 2q^7 + \cdots \\
f_0f_2 &= q^4 - 2q^7 + \cdots \\
f_0f_3 &= q^5 - 2q^7 + \cdots \\
f_0f_4 &= q^6 - 2q^{11} + \cdots \\
-f_1f_2 + f_0f_3 &= -2q^7 + q^8 + \cdots \\
-f_1f_2 + f_0f_3 + 2f_2f_3 &= q^8 + 2q^9 + \cdots \\
-f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 &= 2q^9 - q^{10} + \cdots \\
-f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 - 2f_3f_4 &= -q^{10} + 11q^{12} + \cdots \\
-f_1f_2 + f_0f_3 + 2f_2f_3 - f_3^2 - 2f_3f_4 + f_4^2 &= 11q^{12} - 11q^{13} + \cdots \\
-f_1f_2 - f_2^2 + f_0f_3 + 2f_2f_3 - f_3^2 + f_0f_4 - 6f_3f_4 - f_4^2 &= -22q^{13} + 44q^{15} + \cdots \\
-f_2^2 + f_3^2 + f_0f_4 - f_2f_4 - 4f_3f_4 + 2f_4^2 &= -22q^{14} + 22q^{15} + \cdots
\end{aligned}$$

The last exponent is $14 > \frac{m}{2} + (m-1)(g-1) - 1 = 13$. So, $\mathfrak{a}_\infty$ is a 2–Weierstrass point for $X_0(55)$.

We end the section with the following remark. When $\mathfrak{R}_\Gamma$ is hyperelliptic, $S_{m,2}^H(\Gamma)$ could be a proper subspace of $S_m^H(\Gamma)$. For example, assume that $g(\Gamma) = 2$. Let f_0, f_1 be a basis of $S_2(\Gamma)$. Then, for any even integer $m \geq 4$,

$f_0^u f_1^{\frac{m}{2}-u}$, $0 \le u \le m$ is a basis of $S^H_{m,2}(\Gamma)$. Therefore,

$$\dim S^H_m(\Gamma) = (m-1)(g(\Gamma)-1) = m-1 \ge \frac{m}{2}+1 = \dim S^H_{m,2}(\Gamma), \quad \text{for } m \ge 4.$$

Thus, $S^H_{4,2}(\Gamma) = S^H_4(\Gamma)$ while $S^H_{m,2}(\Gamma)$ is a proper subset of $S^H_m(\Gamma)$ for $m \ge 6$.

Acknowledgements.

The author acknowledges Croatian Science Foundation grant IP-2018-01-3628.

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O Weierstrassovim točkama višeg reda na $X_0(N)$

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SAŽETAK. Neka je Γ Fuchsova grupa prve vrste. Za paran cijeli broj $m \geq 4$ opisujemo prostor $H^{m/2}(\mathfrak{R}_\Gamma)$ svih $m/2$–holomorfnih diferencijala u terminima potprostora $S_m^H(\Gamma)$ prostora (holomorfnih) kuspidalnih modularnih formi $S_m(\Gamma)$. To generalizira klasični izomorfizam $S_2(\Gamma) \simeq H^1(\mathfrak{R}_\Gamma)$. Proučavamo svojstva prostora $S_m^H(\Gamma)$. Kao primjenu, dajemo algoritam implementiran u SAGE-u koji testira je li kusp u ∞ za nehipereliptički $X_0(N)$ $\frac{m}{2}$–Weierstrassova točka.

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Received: 7.1.2023.
Accepted: 14.3.2023.