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# JIANG'S CONJECTURE AND FIBERS OF THE BARBASCH-VOGAN DUALITY

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*To Marko Tadić on the occasion of his 70th Birthday*

Abstract. The well-known Shahidi's conjecture says that tempered $L$-packets have generic members. As a natural generalization of Shahidi's conjecture to non-tempered local Arthur packets, Jiang's conjecture characterizes the relation between the structure of local Arthur parameters and the upper bound of wavefront sets of representations in local Arthur packets. One of the main ingredients in Jiang's conjecture is the Barbasch-Vogan duality. In this paper, first we briefly survey the recent progress on Jiang's conjecture, then towards the general case of Jiang's conjecture, we explicitly describe the fibers of the Barbasch-Vogan duality for classical groups.

## 1. Introduction

Let $F$ be a non-Archimedean local field. Let $\mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}$ be quasi-split classical groups, where $\alpha$ is a square class in $F$, and let $G_n = \mathrm{G}_n(F)$. Here, we identify a square class with the corresponding quadratic character of the Weil group $W_F$ via the local class field theory. Their Langlands dual groups are

$$\widehat{\mathrm{G}}_n(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{SO}_{2n}(\mathbb{C}).$$

Let ${}^L\mathrm{G}_n$ be the $L$-group of $G_n$,

$${}^L\mathrm{G}_n = \begin{cases} \widehat{\mathrm{G}}_n(\mathbb{C}) & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \\ \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_F & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}. \end{cases}$$

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In his fundamental work [2], Arthur introduced the local Arthur packets which are finite sets of representations of $G_n$, parameterized by local Arthur parameters. Local Arthur parameters are defined as a direct sum of irreducible representations

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{G}_n$$

$$\psi = \bigoplus_{i=1}^{r} \phi_i \otimes S_{m_i} \otimes S_{n_i}, \tag{1.1}$$

satisfying the following conditions:

(1) $\phi_i(W_F)$ is bounded and consists of semi-simple elements, and $\dim(\phi_i) = k_i$;

(2) the restrictions of $\psi$ to the two copies of $\mathrm{SL}_2(\mathbb{C})$ are analytic, $S_k$ is the $k$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, and

$$\sum_{i=1}^{r} k_i m_i n_i = N = N_n := \begin{cases} 2n+1 & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ 2n & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}. \end{cases}$$

Assuming the Ramanujan conjecture, Arthur ([2]) showed that these local Arthur packets characterize the local components of square-integrable automorphic representations. For $1 \leq i \leq r$, let $a_i = k_i m_i$, $b_i = n_i$. Let

$$\underline{p}(\psi) = [b_1^{a_1}, b_2^{a_2}, \ldots, b_r^{a_r}]$$

be a partition of $N$, where without loss of generality, we assume that $b_1 \geq b_2 \geq \cdots \geq b_r$. A local Arthur parameter $\psi$ is called tempered or generic if for all $1 \leq i \leq r$, $b_i = 1$. Given a local Arthur parameter $\psi$ as in (1.1), the local Arthur packet is denoted by $\widetilde{\Pi}_\psi$. An irreducible admissible representation $\pi$ of $G_n$ is called of Arthur type if it lies in a local Arthur packet.

Given an irreducible representation $\pi$ of $G_n$, one important invariant is a set $\mathfrak{n}(\pi)$ which is defined to be all the $F$-rational nilpotent orbits $\mathcal{O}$ in the Lie algebra $\mathfrak{g}_n$ of $G_n$ such that the coefficient $c_{\mathcal{O}}(\pi)$ in the Harish-Chandra-Howe local expansion of the character $\Theta(\pi)$ of $\pi$ is nonzero (see [9] and [21]). Let $\mathfrak{n}^m(\pi)$ be the subset of $\mathfrak{n}(\pi)$ consisting of maximal nilpotent orbits, under the dominant order of nilpotent orbits. Let $\overline{\mathfrak{n}}(\pi)$ and $\overline{\mathfrak{n}}^m(\pi)$ be the sets of corresponding nilpotent orbits over $\overline{F}$. Then $\overline{\mathfrak{n}}^m(\pi)$ is called the wavefront set of $\pi$. Note that nilpotent orbits $\mathcal{O}$ of $G_n$ are parametrized by data $(\underline{p}, \underline{q})$, where $\underline{p}$ is partition of $2n$ (or $2n+1$ when $\mathrm{G}_n = \mathrm{SO}_{2n+1}$) and $\underline{q}$ is certain non-degenerate quadratic form ([26, Section I.6]). Let $\mathfrak{p}(\pi)$ be the set of partitions corresponding to $\mathfrak{n}(\pi)$. Under the dominant order of partitions, let $\mathfrak{p}^m(\pi)$ be the maximal elements in $\mathfrak{p}(\pi)$. Then $\mathfrak{p}^m(\pi)$ can be identified exactly with the set $\overline{\mathfrak{n}}^m(\pi)$, except for very even orbits of even special orthogonal groups in which case it could be one-to-two (for those very even partitions, see §2.1), is also called the wavefront set of $\pi$.

For tempered $L$-packets, Shahidi has the following conjecture in general.

CONJECTURE 1.1 (Shahidi's conjecture). *For any quasi-split reductive group $G$, tempered $L$-packets have generic members.*

Conjecture 1.1 can be enhanced as follows.

CONJECTURE 1.2 (Enhanced Shahidi's conjecture). *For any quasi-split reductive group $G$, local Arthur packets are tempered if and only if they have generic members.*

Jiang's conjecture is a natural generalization of Shahidi's conjectures above from tempered local Arthur packets to non-tempered ones, on the characterization of the set $\mathfrak{p}^m(\pi)$ for $\pi$ in local Arthur packets. Note that for a generic representation $\pi$, the set $\mathfrak{p}^m(\pi)$ contains only regular nilpotent orbits. The global version of this conjecture can be found in [14, Conjecture 4.2].

One of the main ingredients in Jiang's conjecture is the Barbasch-Vogan duality map $d_{BV}$, from nilpotent orbits in $\widehat{\mathrm{G}}(\mathbb{C})$ to those in $\mathrm{G}(\mathbb{C})$, see Definition 2.5 for details in the cases of $\mathrm{G} = \mathrm{G}_n$. Note that nilpotent orbits in $\mathrm{G}(\mathbb{C})$ are naturally identified with those in $\mathrm{G}(\overline{F})$ (see [3, 20, 23]).

CONJECTURE 1.3 (Jiang's conjecture). *Given any local Arthur parameter $\psi$ of $G_n$ as in* (1.1)*, the followings hold.*

1. *For any $\pi \in \widetilde{\Pi}_\psi$, any partition $\underline{p}$ in $\mathfrak{p}^m(\pi)$ has the property that*
$$\underline{p} \leq d_{BV}(\underline{p}(\psi)).$$
2. *There exists $\pi \in \widetilde{\Pi}_\psi$, such that $d_{BV}(\underline{p}(\psi)) \in \mathfrak{p}^m(\pi)$.*

There has been many recent progress towards Conjecture 1.3. In [19], the first named author and the third named author studied Jiang's conjecture adapting the matching method of endoscopic transfer in [22] and the work of [12, 13, 15, 17] to construct a particular element in each local Arthur packet. We obtain results assuming a conjecture as follows.

Let $\theta$ be the standard outer automorphism of $\mathrm{G}(N) = \mathrm{GL}(N)$: $g \mapsto {}^t g^{-1}$ and let $\tilde{\theta}(N) = Int(\tilde{J}) \circ \theta : g \mapsto \tilde{J}\theta(g)\tilde{J}^{-1}$, where

$$\tilde{J} = \tilde{J}(N) = \begin{pmatrix} & 0 & & 1 \\ & & -1 & \\ & \cdots & & \\ (-1)^{N-1} & & & 0 \end{pmatrix}.$$

Let $\pi_\psi$ be the representation of $\mathrm{GL}_N(F)$ corresponding to $\phi_\psi$ via local Langlands correspondence, which is unitary and self-dual, and let $\widetilde{\pi}_\psi$ be its canonical extension to the bitorsor $\widetilde{\mathrm{GL}}_N(F) = \mathrm{GL}_N(F) \rtimes \widetilde{\theta}(N)$.

Taking the character expansion for the representation $\widetilde{\pi}_\psi$ of the bitorsor $\widetilde{\mathrm{GL}}_N(F)$ at

$$\theta_{\widehat{\mathrm{G}}_n} = s_{\widehat{\mathrm{G}}_n} \rtimes \widetilde{\theta}(N) \in \widetilde{\mathrm{GL}}_N(F), \tag{1.2}$$

(see [4], also see [16, Theorem 3.2] and [25, Theorems 4.20, 4.23]), where

$$s_{\widehat{\mathrm{G}}_n} = \begin{cases} I_N, & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \\ \begin{pmatrix} I_n & & \\ & 1 & \\ & & -I_n \end{pmatrix}, & \text{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \\ \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}, & \text{when } \mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}, \end{cases}$$

we can define the sets $\mathfrak{n}^m(\widetilde{\pi}_\psi)$ and $\mathfrak{p}^m(\widetilde{\pi}_\psi)$ similarly. Note that when $\mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}$, the connected component of the stabilizer of $\theta_{\widehat{\mathrm{G}}_n}$ in $\widetilde{\mathrm{GL}}_N(F)$ is $\widehat{\mathrm{G}}_n(F)$ and $\mathfrak{n}^m(\widetilde{\pi}_\psi)$ consists of $F$-rational nilpotent orbits in the Lie algebra of $\widehat{\mathrm{G}}_n(F)$. When $\mathrm{G}_n = \mathrm{Sp}_{2n}$, the connected component of the stabilizer of $\theta_{\widehat{\mathrm{G}}_n}$ in $\widetilde{\mathrm{GL}}_{2n+1}(F)$ is $\mathrm{G}_n(F) \times \mathrm{SO}_1$ and $\mathfrak{n}^m(\widetilde{\pi}_\psi)$ consists of $F$-rational nilpotent orbits in the Lie algebra of $\mathrm{G}_n(F)$. Then we have the following conjecture regarding the set $\mathfrak{p}^m(\widetilde{\pi}_\psi)$.

Conjecture 1.4. *For any $\underline{p} \in \mathfrak{p}^m(\widetilde{\pi}_\psi)$,*

$$\underline{p} \leq \begin{cases} \{(\underline{p}(\psi)^*)_{\widehat{\mathrm{G}}_n}\}, & \textit{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}, \\ \{((\underline{p}(\psi)^*)^-)_{\mathrm{G}_n}\}, & \textit{when } \mathrm{G}_n = \mathrm{Sp}_{2n}, \end{cases}$$

*where $(\underline{p}(\psi)^*)_{\widehat{\mathrm{G}}_n}$ is the $\widehat{\mathrm{G}}_n$-collapse of the partition $\underline{p}(\psi)^*$ (transpose of $\underline{p}(\psi)$), which is the largest $\widehat{\mathrm{G}}_n$-partition smaller than or equal to $\underline{p}(\psi)^*$, $(\underline{p}(\psi)^*)^-$ is decreasing the smallest part of $\underline{p}(\psi)^*$ by 1 and $((\underline{p}(\psi)^*)^-)_{\mathrm{G}_n}$ is the $\mathrm{G}_n$-collapse of $(\underline{p}(\psi)^*)^-$.*

We also believe that the following stronger conjecture holds.

Conjecture 1.5.

$$\mathfrak{p}^m(\widetilde{\pi}_\psi) = \begin{cases} \{(\underline{p}(\psi)^*)_{\widehat{\mathrm{G}}_n}\}, & \textit{when } \mathrm{G}_n = \mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}^{\alpha}, \\ \{((\underline{p}(\psi)^*)^-)_{\mathrm{G}_n}\}, & \textit{when } \mathrm{G}_n = \mathrm{Sp}_{2n}. \end{cases}$$

Conjectures 1.4 and 1.5 are inspired by the result of Konno in [16, Theorem 4.1], where certain cases of these conjectures are confirmed. For more comments on Conjectures 1.4 and 1.5, please see [19, Remark 1.4]. The main results in [19] towards Jiang's conjecture 1.3 can be summarized in the following theorem.

Theorem 1.6 ([19, Theorem 1.9]). *Let $\psi$ be a local Arthur parameter as in (1.1), with $\underline{p}(\psi) = [b_1^{a_1}, b_2^{a_2}, \ldots, b_r^{a_r}]$ and $b_1 \geq b_2 \geq \cdots \geq b_r$. Assume that Conjecture 1.4 is true. Then we have the followings.*

1. *Conjecture 1.3 Part (1) is partially valid, i.e., for any partition $\underline{p} > d_{BV}(\underline{p}(\psi))$ and any $\pi \in \widetilde{\Pi}_\psi$, $\underline{p} \notin \mathfrak{p}^m(\pi)$.*

*2. Conjecture 1.2 is true.*

*3. Let*

$$\underline{p}_1 = \left[\lfloor \frac{b_1}{2} \rfloor^{a_1}, \lfloor \frac{b_2}{2} \rfloor^{a_2}, \ldots, \lfloor \frac{b_r}{2} \rfloor^{a_r}\right]^*,$$

*and* $n_0 = \lfloor \frac{\sum_{b_i \text{ odd}} a_i}{2} \rfloor$. *Then Conjecture 1.3 Part (2) holds for the following cases.*

*(a) When* $\mathrm{G}_n = \mathrm{Sp}_{2n}$*, and*

$$([\underline{p}_1, \underline{p}_1, (2n_0)]^*)_{\mathrm{Sp}_{2n}} = ([b_1^{a_1}, \ldots, b_r^{a_r}]^-)_{\mathrm{Sp}_{2n}}. \tag{1.3}$$

*In particular, if*

*(i)* $a_r = b_r = 1$ *and* $b_i$ *are all even for* $1 \le i \le r-1$*,*

*(ii) or,* $b_i$ *are all odd,*

*then* (1.3) *holds and thus Conjecture 1.3 Part (2) is valid.*

*(b) When* $\mathrm{G}_n = \mathrm{SO}_{2n+1}$*, and*

$$([\underline{p}_1, \underline{p}_1, (2n_0+1)]^*)_{\mathrm{SO}_{2n+1}} = ([b_1^{a_1}, \ldots, b_r^{a_r}]^+)_{\mathrm{SO}_{2n+1}}. \tag{1.4}$$

*In particular, if*

*(i)* $b_1$ *is even and* $a_1 = 1$*, and* $b_i$ *are all odd for* $2 \le i \le r$*,*

*(ii) or,* $b_i$ *are all even,*

*then* (1.4) *holds and thus Conjecture 1.3 Part (2) is valid.*

*(c) When* $\mathrm{G}_n = \mathrm{SO}_{2n}^{\alpha}$*, and*

$$[\underline{p}_1, \underline{p}_1, (2n_0-1), 1]^{\mathrm{SO}_{2n}} = ([b_1^{a_1}, \ldots, b_r^{a_r}]^*)_{\mathrm{SO}_{2n}}. \tag{1.5}$$

*If all* $b_i$ *are of the same parity, then* (1.5) *holds and thus Conjecture 1.3 Part (2) is valid. Here given any partition* $\underline{q}$ *of* $\mathrm{SO}_{2n}$*,* $\underline{q}^{\mathrm{SO}_{2n}}$ *is the* $\mathrm{SO}_{2n}$*-expansion of* $\underline{q}$*, which is the smallest special* $\mathrm{SO}_{2n}$*-partition bigger than or equal to* $\underline{q}$*.*

Conjecture 1.4 plays an important role in all parts of Theorem 1.6, in the matching process of endoscopic transfer as in [22]. In Conjecture 1.3 Part (1), what is missing is that for any partition $\underline{p}$ which is not related to $d_{BV}(\underline{p}(\psi))$ and any $\pi \in \widetilde{\Pi}_\psi$, $\underline{p} \notin \mathfrak{p}^m(\pi)$. Unfortunately, our current method can not rule out these partitions, due to the expectation that wavefront sets of representations may not be singleton (see the example provided in [24]). For more discussion on Conjectures 1.3, 1.4, and Theorem 1.6, we refer to [19, Remark 1.10] and the discussions afterwards.

As another approach towards Jiang 's conjecture, joint with Hazeltine ([11]), we proved the following very interesting reduction on Part (1) of Conjecture 1.3, by an inductive process on the $L$-data for the Aubert-Zelevinsky dual of representations in local Arthur packets.

Theorem 1.7 ([11, Theorem 1.6]). *The following statements are equivalent.*

1. *Part (1) of Conjecture 1.3 holds for any local Arthur parameter.*
2. *Part (1) of Conjecture 1.3 holds for any anti-tempered local Arthur parameter, i.e., the dual of a tempered local Arthur parameter.*

We remark that recent work of Ciubotaru-Mason-Brown-Okada ([6–8]) and Waldspurger ([27]), combining the closure relation result of [10], imply that Jiang's conjecture 1.3 holds for any local Arthur parameter $\psi$ of $\mathrm{Sp}_{2n}(F)$ or split $\mathrm{SO}_{2n+1}(F)$ which is trivial on $W_F$. For more discussion, see [11].

As we can see above, one of the main ingredients in Jiang's conjecture 1.3 is the Barbasch-Vogan duality. Towards the general case of Jiang's conjecture, in this paper, we explicitly describe the fibers of the Barbasch-Vogan duality for classical groups in §3 as follows.

THEOREM 1.8 (Theorem 3.4). *Let $(X, X') \in \{(B,C),(C,B),(D,D)\}$ and $\underline{\mathfrak{p}}$ be a partition of type $X'$ in the image of the Barbasch-Vogan duality. Write $d_{BV}(\underline{\mathfrak{p}}) =: \underline{p} = [p_1^{m_1}, \ldots, p_r^{m_r}]$, and define a subset $I \subseteq \{1, \ldots, r\}$ type by type as follows. (We set $p_{r+1} = 0$ and $m_{r+1} = 1$.)*

*(i) When $X = B$,*

$$I := \left\{ 1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is odd, and } m_i+\sum_{j=1}^{i-1} m_j p_j \text{ is odd.}} \right\}.$$

*(ii) When $X = C$,*

$$I := \left\{ 1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is even, and } m_i+\sum_{j=1}^{i-1} m_j (p_j+1) \text{ is even.}} \right\}.$$

*(iii) When $X = D$,*

$$I := \left\{ 1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is odd, and } m_i+\sum_{j=1}^{i-1} m_j p_j \text{ is even.}} \right\}.$$

*For any subset $J \subseteq I$, we define $\underline{p}_J$ from $\underline{p}$ by reducing the multiplicity of $p_j$ and $p_j - 2$ by 1 and increasing the multiplicity of $p_j - 1$ by 2 for each $j \in J$.*

*Then the following map is a bijection of partially ordered sets*

$$\begin{aligned} (2^I, \geq) &\longrightarrow (d_{BV}^{-1}(\underline{\mathfrak{p}}), \geq) \\ J &\longmapsto \underline{p}_J, \end{aligned}$$

*where $(2^I, \geq)$ is the power set of $I$ with the partial ordering defined by $J_1 \geq J_2$ if $J_1 \subseteq J_2$.*

The result in this paper facilitates the understanding of the structure of the local Arthur packets and is expected to play an important role towards the general case of Jiang's conjecture and many other problems related to local Arthur packets. As an example, in [18], the first and second named authors have applied the description of fibers of Barbasch-Vogan duality to prove the weak local Arthur packets conjecture proposed by Ciubotaru-Mason-Brown-Okada ([7, Conjecture 3.1.2]).

The following is the structure of this paper. In §2, we give some preliminaries on partitions and nilpotent orbits of $G_n(\mathbb{C})$ and the Barbasch-Vogan duality. In §3, we describe the fibers of the Barbasch-Vogan duality and prove our main result (Theorem 1.8).

## 2. Preliminaries

In this section, we give some preliminaries on partitions and nilpotent orbits of $G_n(\mathbb{C})$ and the Barbasch-Vogan duality.

2.1. *Partitions and nilpotent orbits of $G_n(\mathbb{C})$.* In this subsection, we recall the basic notation for partitions and the correspondence between nilpotent orbits of $\mathfrak{g}_n(\mathbb{C})$ and partitions, following [5].

First, we denote the set of partitions of $n$ by $\mathcal{P}(n)$. We express a partition $\underline{p} \in \mathcal{P}(n)$ in one of the following forms:

(i) $\underline{p} = [p_1, \ldots, p_N]$, such that $p_i$'s are non-increasing non-negative integers and $\sum_{i=1}^N p_i = n$. We denote the the *length* of $\underline{p}$ by $l(\underline{p}) = |\{1 \leq i \leq N \mid p_i > 0\}|$.

(ii) $\underline{p} = [p_1^{r_1}, \ldots, p_N^{r_N}]$, such that $p_i$'s are decreasing non-negatives integers and $\sum_{i=1}^N r_i p_i = n$. We assume $r_i > 0$ unless specified.

Also, we denote $|\underline{p}| = n$ if $\underline{p} \in \mathcal{P}(n)$.

Next, we recall the definitions for partitions of type $B$, $C$ and $D$.

Definition 2.1. *For $\epsilon \in \{\pm 1\}$, we define*

$$\mathcal{P}_\epsilon(n) = \{[p_1^{r_1}, \ldots, p_N^{r_N}] \in \mathcal{P}(n) \mid r_i \text{ is even for all } p_i \text{ with } (-1)^{p_i} = \epsilon\}.$$

*Then we say*

1. *$\underline{p} \in \mathcal{P}(n)$ is of type $B$ if $n$ is odd and $\underline{p} \in \mathcal{P}_1(n)$.*
2. *$\underline{p} \in \mathcal{P}(n)$ is of type $C$ if $n$ is even and $\underline{p} \in \mathcal{P}_{-1}(n)$.*
3. *$\underline{p} \in \mathcal{P}(n)$ is of type $D$ if $n$ is even and $\underline{p} \in \mathcal{P}_1(n)$.*

*We denote $\mathcal{P}_X(n)$ the set of partitions of $n$ of type $X$.*

Denote the set of nilpotent orbits of $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{C})$ by $\mathcal{N}_B(2n+1)$, $\mathcal{N}_C(2n)$ and $\mathcal{N}_D(2n)$ respectively. Also, we denote

$$\mathcal{N}_B = \bigcup_{n \geq 0} \mathcal{N}_B(2n+1), \ \mathcal{N}_C = \bigcup_{n \geq 0} \mathcal{N}_C(2n), \ \mathcal{N}_D = \bigcup_{n \geq 0} \mathcal{N}_D(2n).$$

For $(X, N) \in \{(B, 2n+1), (C, 2n), (D, 2n)\}$, there is a surjection

$$\begin{aligned} \mathcal{N}_X(N) &\longrightarrow \mathcal{P}_X(N), \\ \mathcal{O} &\longmapsto \underline{p}_{\mathcal{O}}. \end{aligned}$$

The fiber of $\underline{p} = [p_1^{m_1}, \ldots, p_r^{m_r}] \in \mathcal{P}_X(N)$ under this map is a singleton, which we denote by $\{\mathcal{O}_{\underline{p}}\}$, except when $\underline{p}$ is "very even"; i.e., $\underline{p}$ is of type $D$ and $p_i$'s are all even. When $\underline{p}$ is very even, the fiber consists of two nilpotent orbits, which we denote by $\mathcal{O}^I_{\underline{p}}$ and $\mathcal{O}^{II}_{\underline{p}}$.

The surjection $\mathcal{O} \mapsto \underline{p}_{\mathcal{O}}$ carries the closure ordering on $\mathcal{N}_X(N)$ to the dominance ordering on $\mathcal{P}_X(N)$ in the sense that $\mathcal{O} > \mathcal{O}'$ if and only if $\underline{p}_{\mathcal{O}} > \underline{p}_{\mathcal{O}'}$. Note that when $\underline{p}$ is very even, $\mathcal{O}^I_{\underline{p}}$ and $\mathcal{O}^{II}_{\underline{p}}$ are not comparable.

2.2. *Barbasch-Vogan duality.* In this subsection, following [1, 3, 20, 23], we introduce several operations on the set of partitions, and then use them to describe the definition of the Barbasch-Vogan duality on the level of partitions and nilpotent orbits.

First, we need the following operations to construct or decompose partitions.

Definition 2.2. *Suppose $\underline{p} \in \mathcal{P}(n_1)$ and $\underline{q} \in \mathcal{P}(n_2)$.*

*(i) Write $\underline{p} = [p_1^{r_1}, \dots, p_N^{r_N}]$ and $\underline{q} = [p_1^{s_1}, \dots, p_N^{s_N}]$, where we allow $r_i = 0$ or $s_i = 0$. Then we define*

$$\underline{p} \sqcup \underline{q} = [p_1^{r_1+s_1}, \dots, p_N^{r_N+s_N}] \in \mathcal{P}(n_1 + n_2).$$

*(ii) Write $\underline{p} = [p_1, \dots, p_N]$, we define*

$$\underline{p}^+ = [p_1 + 1, p_2, \dots, p_N] \in \mathcal{P}(n_1 + 1),$$
$$\underline{p}^- = [p_1, \dots, p_{N-1}, p_N - 1] \in \mathcal{P}(n_1 - 1).$$

The following notation is useful in the computation.

Definition 2.3. *For $\underline{p} = [p_1, \dots, p_N] \in \mathcal{P}(n)$ and $b \in \mathbb{Z}$, we define*

$$\underline{p}_{>b} = [p_1, \dots, p_i]$$

*where $i = \max\{1 \le j \le N \mid p_j > b\}$. We define $\underline{p}_{\bullet b}$ similarly for $\bullet \in \{=, <, \le, \ge\}$ so that $\underline{p} = \underline{p}_{>b} \sqcup \underline{p}_{\le b} = \underline{p}_{>b} \sqcup \underline{p}_{=b} \sqcup \underline{p}_{<b}$, etc.*

We recall the definition of transpose (or conjugation) of partitions.

Definition 2.4. *For $\underline{p} = [p_1, \dots, p_N] \in \mathcal{P}(n)$, we define $\underline{p}^* = [p_1^*, \dots, p_{N'}^*] \in \mathcal{P}(n)$ by*

$$p_i^* = |\{j \mid p_j \ge i\}|.$$

It is easy to see that for any two partitions $\underline{p}$, $\underline{q}$ of $n$, $(\underline{p} \sqcup \underline{q})^* = \underline{p}^* + \underline{q}^*$.

Next, we recall the definition of collapse. Let $n$ be a positive integer and let $X = B$ if $n$ is odd and $X \in \{C, D\}$ if $n$ is even. For any $\underline{p} \in \mathcal{P}(n)$, there exists a unique maximal partition $\underline{p}_X \in \mathcal{P}(n)$ of type $X$ such that $\underline{p}_X \le \underline{p}$. We denote $\underline{p}_X$ the $X$-*collapse* of $\underline{p}$.

The following lemma, which is a special case of [1, Lemma 3.1], gives an inductive way to compute collapse. Note that $\underline{p}_{>x}$ is always superior than $(\underline{p}_{\le x})^+$ in the notation there. Following the notation in [1], we often omit the parentheses between the superscript and subscript. For example, we shall write ${\underline{p}_{>x,D}}^{+}{}_{B}{}^{-*}$ instead of $(((((\underline{p}_{>x})_D)^+)_B)^-)^*$.

Lemma 2.1. *Let $x$ be a positive integer and $\underline{p}$ be a partition. Then for $X \in \{B, C, D\}$, the $X$-collapse (if defined) of $\underline{p}$ is given by the following table.*

| | $l(\underline{p}_{>x})$ *even* | | $l(\underline{p}_{>x})$ *odd* | |
|---|---|---|---|---|
| | $\lvert\underline{p}_{>x}\rvert$ *even* | $\lvert\underline{p}_{>x}\rvert$ *odd* | $\lvert\underline{p}_{>x}\rvert$ *even* | $\lvert\underline{p}_{>x}\rvert$ *odd* |
| $\underline{p}_B$: | $\underline{p}_{>x,D} \sqcup \underline{p}_{\leq x,B}$ | ${\underline{p}_{>x}}^{-}{}_{D} \sqcup {\underline{p}_{\leq x}}^{+}{}_{B}$ | ${\underline{p}_{>x}}^{-}{}_{B} \sqcup {\underline{p}_{\leq x}}^{+}{}_{D}$ | $\underline{p}_{>x,B} \sqcup \underline{p}_{\leq x,D}$ |
| $\underline{p}_C$: | $\underline{p}_{>x,C} \sqcup \underline{p}_{\leq x,C}$ | ${\underline{p}_{>x}}^{-}{}_{C} \sqcup {\underline{p}_{\leq x}}^{+}{}_{C}$ | $\underline{p}_{>x,C} \sqcup \underline{p}_{\leq x,C}$ | ${\underline{p}_{>x}}^{-}{}_{C} \sqcup {\underline{p}_{\leq x}}^{+}{}_{C}$ |
| $\underline{p}_D$: | $\underline{p}_{>x,D} \sqcup \underline{p}_{\leq x,D}$ | ${\underline{p}_{>x}}^{-}{}_{D} \sqcup {\underline{p}_{\leq x}}^{+}{}_{D}$ | ${\underline{p}_{>x}}^{-}{}_{B} \sqcup {\underline{p}_{\leq x}}^{+}{}_{B}$ | $\underline{p}_{>x,B} \sqcup \underline{p}_{\leq x,B}$ |

Finally, we recall the definition of Barbasch-Vogan duality maps for partitions of type $X$.

Definition 2.5. *(i) For $\underline{p} \in \mathcal{P}_B(2n+1)$, we define $d_{BV}(\underline{p}) := {\underline{p}^{-}}_{C}{}^{*}$, which is in $\mathcal{P}_C(2n)$.*

*(ii) For $\underline{p} \in \mathcal{P}_C(2n)$, we define $d_{BV}(\underline{p}) := {\underline{p}^{+}}_{B}{}^{*}$, which is in $\mathcal{P}_B(2n+1)$.*

*(iii) For $\underline{p} \in \mathcal{P}_D(2n)$, we define $d_{BV}(\underline{p}) := {\underline{p}^{*}}_{D}$, which is in $\mathcal{P}_D(2n)$.*

The Barbasch-Vogan duality map can be extended to the level of nilpotent orbits. If $\underline{p} \in \mathcal{P}_D(2n)$ is very even, then define

$$d_{BV}(\mathcal{O}^I_{\underline{p}}) = \begin{cases} \mathcal{O}^I_{\underline{p}} & \text{if } n \text{ is even,} \\ \mathcal{O}^{II}_{\underline{p}} & \text{if } n \text{ is odd.} \end{cases}$$

Otherwise, define $d_{BV}(\mathcal{O}_{\underline{p}}) = \mathcal{O}_{d_{BV}(\underline{p})}$. See [5, Corollary 6.3.5]. Also see Proposition 3.7 (a) below for the well-definedness, i.e., $d_{BV}(\underline{p})$ is very even only if $\underline{p}$ is also very even. A nilpotent orbit or a partition is *special* if it is in the image of the Barbasch-Vogan duality map. Sometimes, to make things more clear, we may use $d_{BV,(X,X')}$ to denote the Barbasch-Vogan duality map sending partitions of type $X$ to partitions of type $X'$.

## 3. Fibers of the Barbasch-Vogan duality

Let $(X, X') \in \{(B, C), (C, B), (D, D)\}$. In this section, we study the structure of the sets of partitions

$$d_{BV}^{-1}(\underline{\mathfrak{p}}) := \{\underline{p} \in \mathcal{P}_X \mid d_{BV}(\underline{p}) = \underline{\mathfrak{p}}\},$$

for a special partition $\underline{\mathfrak{p}} \in \mathcal{P}_{X'}$. We give a explicit description of $d_{BV}^{-1}(\underline{\mathfrak{p}})$ in Theorem 3.4 and relate it with

$$d_{BV}^{-1}(\mathcal{O}') := \{\mathcal{O} \in \mathcal{N}_X \mid d_{BV}(\mathcal{O}) = \mathcal{O}'\}$$

for any special $\mathcal{O}' \in \mathcal{N}_{X'}$ in Proposition 3.7.

For convenience, we introduce the following notation. Recall that when we write a partition $\underline{p}$ as $[p_1, \ldots, p_r]$, we require that $p_i$ is non-increasing. Set $p_t = 0$ for any $t > r$ throughout this section.

Definition 3.1. *Let $n$ be a positive integer and $\underline{p} = [p_1, \ldots, p_r], \underline{q} = [q_1, \ldots, q_s]$ be two arbitrary partitions in $\mathcal{P}(n)$.*

1. *If $\underline{p} \neq \underline{q}$, we define $x(\underline{p}, \underline{q})$ to be the unique index such that $p_i = q_i$ for $1 \leq i < x$ and $p_x \neq q_x$. If $\underline{p} = \underline{q}$, then we set $x(\underline{p}, \underline{q}) = l(\underline{p}) + 1$. Occasionally, we write $x = x(\underline{p}, \underline{q})$ if there is no confusion.*
2. *We say $\underline{p} \succcurlyeq \underline{q}$ if $p_{x(\underline{p},\underline{q})} \geq q_{x(\underline{p},\underline{q})}$. This gives a total order on $\mathcal{P}(n)$.*

First, we study the relation between two partitions $\underline{p} = [p_1, \ldots, p_r], \underline{q} = [q_1, \ldots, q_s]$ such that $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$. Assuming $\underline{p} \succcurlyeq \underline{q}$ and denote $x = x(\underline{p}, \underline{q})$ for short, in the following lemma, we show that $\underline{p}_{\geq p_x - 1} \sqcup [p_x - 2]$ and $\underline{q}_{\geq p_x - 1}$ are closely related. Moreover, we construct partition $\underline{q}^\sharp \geq \underline{q}$ (resp. $\underline{p}^\flat \leq \underline{p}$) in the same fiber of Barbasch-Vogan duality map, which is "closer" to $\underline{p}$ (resp. $\underline{q}$), in the sense that if $\underline{p} \neq \underline{q}$, then

$$x(\underline{p}, \underline{q}^\sharp) > x(\underline{p}, \underline{q}), \ \ x(\underline{p}^\flat, \underline{q}) > x(\underline{p}, \underline{q}).$$

Lemma 3.1. *Suppose $\underline{p} = [p_1, \ldots, p_r], \underline{q} = [q_1, \ldots, q_s]$ are two partitions in $\mathcal{P}_X(n)$ where $X \in \{B, C, D\}$ such that*

*(i)* $\underline{p} \succcurlyeq \underline{q}$ *and* $\underline{p} \neq \underline{q}$.
*(ii)* $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$.

*We denote $x = x(\underline{p}, \underline{q})$ for short and define*

$$y := \min(\{1 \leq i \leq r \mid p_i < p_x - 1\} \sqcup \{r + 1\}).$$

*Then the followings hold. (We set $p_\alpha = 0 = q_\beta$ for $\alpha > r$ and $\beta > s$.)*

*(a)* $p_y = p_x - 2$*, which is odd if $X \in \{B, D\}$ and even if $X = C$.*
*(b)* $q_x = p_x - 1$*, $q_z = p_z$ for $x < z < y$, $q_y = p_y + 1$, and $q_{y+1} \leq p_y$.*
*(c) Consider partitions*

$$\underline{q}^\sharp := [p_1, \ldots, p_y, q_{y+1}, \ldots, q_s] \geq \underline{q},$$
$$\underline{p}^\flat := [q_1, \ldots, q_y, p_{y+1}, \ldots, p_r] \leq \underline{p}.$$

*The partitions $\underline{q}^\sharp$ and $\underline{p}^\flat$ are of type $X$, and*

$$d_{BV}(\underline{p}) = d_{BV}(\underline{p}^\flat) = d_{BV}(\underline{q}^\sharp) = d_{BV}(\underline{q}).$$

Proof. We prove the lemma for type $B, C$ and $D$ separately. In the computation of collapse of partitions below, we frequently apply Lemma 2.1, while sometimes the detail is omitted in the argument.

**Type B**: For a partition $\underline{\mathfrak{p}}$ of type $B$, its Barbasch-Vogan dual is given by

$$d_{BV}(\underline{\mathfrak{p}}) = {\underline{\mathfrak{p}}^{-}}_{C}{}^{*}.$$

Since taking transpose of partitions is a bijection, Assumption (ii) is equivalent to

$$\underline{p}^{-}{}_{C} = \underline{q}^{-}{}_{C}. \tag{3.1}$$

First, we deal with the case that $|\underline{p}_{>p_x}|$ is even. By Lemma 2.1, we may replace $\underline{p}$ and $\underline{q}$ with $\underline{p}_{\leq p_x}$ and $\underline{q}_{\leq p_x}$ respectively, and assume $p_x = p_1$. After the replacement, (3.1) still holds, and $\underline{p}$ and $\underline{q}$ are still of type $B$. Note that $p_1 > 1$ by Assumption (i).

Since $\underline{q}^{-}{}_{C}$ contains at most $x-1$ copies of $p_1$, in order that (3.1) holds, $p_1$ and $x$ must be both odd and the multiplicity of $p_1$ in $\underline{p}$ must be exactly $x$. Then we may write

$$\underline{p} = [p_1^x, (p_1-1)^{y-x-1}] \sqcup [p_y, \dots, p_r].$$

Note that $p_y < p_1 - 1$ by the definition of $y$, and hence $[p_y, \dots, p_r] = \underline{p}_{<p_1-1}$. Since $\underline{p}$ is of type $B$ and $p_1 - 1$ is even, $y - x + 1$, the multiplicity of $p_1 - 1$ in $\underline{p}$, must be even. Under these parity conditions, we have

$$\underline{p}^{-}{}_{C} = [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup [p'_y, p'_{y+1}, \dots],$$

where $0 \leq p'_y \leq p_y + 1 \leq p_1 - 1$ and $0 \leq p'_{y+1} \leq p_y < p_1 - 1$. Now we write

$$\underline{q} = [p_1^{x-1}, (p_1-1)^{2l}] \sqcup \underline{q}_{<p_1-1},$$

for some non-negative integer $l$. Since $|\underline{q}|$ is odd, we have $|\underline{q}_{<p_1-1}| \geq 1$ and hence

$$\underline{q}^{-} = [p_1^{x-1}, (p_1-1)^{2l}] \sqcup (\underline{q}_{<p_1-1})^{-}.$$

Applying Lemma 2.1 again,

$$\underline{q}^{-}{}_{C} = [p_1^{x-1}, (p_1-1)^{2l}] \sqcup (\underline{q}_{<p_1-1})^{-}{}_{C}.$$

Note that $(\underline{q}_{<p_1-1})^{-} \geq (\underline{q}_{<p_1-1})^{-}{}_{C}$, and hence any piece of $(\underline{q}_{<p_1-1})^{-}{}_{C}$ is smaller than $p_1 - 1$. Therefore, comparing the multiplicity of $p_1 - 1$ in both sides of (3.1), we must have $2l = y - x + 1$ and $p'_y = p_y + 1 = p_1 - 1$. This verifies both Parts (a) and (b).

For Part (c), note that $[q_{y+1}, \dots, q_s] = \underline{q}_{<p_1-1}$. We have

$$\begin{aligned}
(\underline{q}^{\sharp})^{-}{}_{C} &= ([p_1^x, (p_1-1)^{y-x-1}] \sqcup [p_1 - 2, q_{y+1}, \dots, q_s]^{-})_{C} \\
&= ([p_1^x, (p_1-1)^{y-x-1}])^{-}{}_{C} \sqcup ([p_1 - 1] \sqcup (\underline{q}_{<p_1-1})^{-})_{C} \\
&= [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup [p_1 - 1] \sqcup (\underline{q}_{<p_1-1})^{-}{}_{C} \\
&= \underline{q}^{-}{}_{C}.
\end{aligned}$$

Similarly, since $[p_y, p_{y+1}, \dots, p_r] = \underline{p}_{\le p_1-2}$, we have

$$\begin{aligned}
{\underline{p}^-}_C &= ([p_1^x, (p_1-1)^{y-x-1}] \sqcup (\underline{p}_{\le p_1-2})^-)_C \\
&= ([p_1^x, (p_1-1)^{y-x-2}, p_1-2])_C \sqcup {(\underline{p}_{\le p_1-2})^{+-}}_C \\
&= [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup ([p_1-1] \sqcup [p_{y+1}, \dots, p_r]^-)_C \\
&= [p_1^{x-1}, (p_1-1)^{y-x+1}] \sqcup {([p_{y+1}, \dots, p_r])^-}_C \\
&= {([p_1^{x-1}, (p_1-1)^{y-x+1}] \sqcup [p_{y+1}, \dots, p_r])^-}_C \\
&= {(\underline{p}^\flat)^-}_C.
\end{aligned}$$

This completes the proof of the case that $|\underline{p}_{>p_x}|$ is even.

Next, we deal with the case that $|\underline{p}_{>p_x}|$ is odd. Denote $\widetilde{\underline{p}} := \underline{p}_{\le p_x}$ and $\widetilde{\underline{q}} := \underline{q}_{\le p_x}$, which are of type $D$. Lemma 2.1 implies

$$\widetilde{\underline{p}}^{+-}{}_C = \widetilde{\underline{q}}^{+-}{}_C. \tag{3.2}$$

Rewrite

$$\widetilde{\underline{p}} = [p_1^x, p_{x+1}, \dots],\ \widetilde{\underline{q}} = [p_1^{x-1}] \sqcup \widetilde{\underline{q}}_{<p_1},$$

where $p_{x+1} \le p_1$, and denote

$$\ell_p := l((\widetilde{\underline{p}}^{+-}{}_C)_{\ge p_1}),\ \ell_q := l((\widetilde{\underline{q}}^{+-}{}_C)_{\ge p_1})$$

for short. Note that (3.2) implies that $\ell_p = \ell_q$.

If $p_1$ is even, then $\ell_q \le x$ and $\ell_p \ge x$, where $\ell_p \ge x$ only if $p_{x+1} < p_1$. Thus $\ell_p = \ell_q$ implies that $p_{x+1} < p_1$ must hold, but then one of $\widetilde{\underline{p}}$ and $\widetilde{\underline{q}}$ is not of type $D$, a contradiction. Therefore, $p_1$ must be odd. Then $\ell_q \le x-1$ and $\ell_p \ge x-1$, where $\ell_p = x-1$ only if $p_{x+1} < p_1$ and $x-1$ is odd. Thus we may rewrite

$$\begin{aligned}
\widetilde{\underline{p}} &= [p_1^x, (p_1-1)^{y-x-1}] \sqcup [p_y, \dots, p_r], \\
\widetilde{\underline{q}} &= [p_1^{x-1}, (p_1-1)^{2l}] \sqcup \widetilde{\underline{q}}_{<p_1-1}.
\end{aligned}$$

Note that since $|\widetilde{\underline{q}}|$ is even, $|\widetilde{\underline{q}}_{<p_1-1}| \ge 1$, and hence

$$\widetilde{\underline{q}}^{+-} = [p_1+1, p_1^{x-2}, (p_1-1)^{2l}] \sqcup (\widetilde{\underline{q}}_{<p_1-1})^-.$$

Therefore,

$$\begin{aligned}
\widetilde{\underline{p}}^{+-}{}_C &= [p_1+1, p_1^{x-2}, (p_1-1)^{y-x}] \sqcup [p'_y, p'_{y+1}, \dots], \\
\widetilde{\underline{q}}^{+-}{}_C &= [p_1+1, p_1^{x-2}, (p_1-1)^{2l}] \sqcup {(\widetilde{\underline{q}}_{<p_1-1})^-}_C,
\end{aligned}$$

where $0 \le p'_y \le p_y + 1 \le p_1 - 1$ and $p'_{y+1} \le p_y < p_1 - 1$. As $\widetilde{\underline{p}}$ is of type $D$, $y - x$ is odd. Therefore, comparing the multiplicity of $p_1 - 1$ in both sides of (3.2), we must have $2l = y - x + 1$ and $p'_y = p_y = p_1 - 1$. This verifies both Parts (a) and (b).

For Part (c), we consider

$$\begin{aligned}\widetilde{\underline{q}}^{\sharp} := (\underline{q}^{\sharp})_{\le p_x} =& [p_1^x, (p_1-1)^{y-x-1}] \sqcup [p_y, q_{y+1}, \dots, q_s]\\ =& [p_1^x, (p_1-1)^{y-x-1}, p_1-2] \sqcup \widetilde{\underline{q}}_{<p_1-1},\end{aligned}$$

and

$$\widetilde{\underline{p}}^{\flat} := (\underline{p}^{\flat})_{\le p_x} = [p_1^{x-1}, (p_1-1)^{y-x+1}] \sqcup [p_{y+1}, \dots, p_r].$$

Then by Lemma 2.1, it suffices to show that $(\widetilde{\underline{q}}^{\sharp})^{+-}{}_C = \widetilde{\underline{q}}^{+-}{}_C$ and $(\widetilde{\underline{p}}^{\flat})^{+-}{}_C = \widetilde{\underline{p}}^{+-}{}_C$. Indeed,

$$\begin{aligned}(\widetilde{\underline{q}}^{\sharp})^{+-}{}_C &= ([p_1+1, p_1^{x-1}, (p_1-1)^{y-x-1}])^{-}{}_C \sqcup ([p_1-1] \sqcup (\widetilde{\underline{q}}_{<p_1-1})^{-})_C\\ &= [p_1+1, p_1^{x-2}, (p_1-1)^{y-x}] \sqcup [p_1-1] \sqcup (\widetilde{\underline{q}}_{<p_1-1})^{-}{}_C\\ &= \widetilde{\underline{q}}^{+-}{}_C.\end{aligned}$$

Similarly, since $[p_y, p_{y+1}, \dots, p_r] = \widetilde{\underline{p}}_{\le p_1-2}$, we have

$$\begin{aligned}\widetilde{\underline{p}}^{+-}{}_C &= ([p_1+1, p_1^{x-1}, (p_1-1)^{y-x-1}])^{-}{}_C \sqcup (\widetilde{\underline{p}}_{\le p_1-2})^{+-}{}_C\\ &= [p_1+1, p_1^{x-2}, (p_1-1)^{y-x}] \sqcup ([p_1-1] \sqcup [p_{y+1}, \dots, p_r]^{-})_C\\ &= [p_1+1, p_1^{x-2}, p_1^{y-x+1}] \sqcup ([p_{y+1}, \dots, p_r])^{-}{}_C\\ &= ([p_1+1, p_1^{x-2}, p_1^{y-x+1}] \sqcup [p_{y+1}, \dots, p_r])^{-}{}_C\\ &= (\widetilde{\underline{p}}^{\flat})^{+-}{}_C.\end{aligned}$$

This completes the proof of the lemma for type $B$.

**Type C**: For a partition $\underline{\mathfrak{p}}$ of type $C$, its Barbasch-Vogan dual is given by

$$d_{BV}(\underline{\mathfrak{p}}) = \underline{\mathfrak{p}}^{+}{}_B{}^{*}.$$

Thus Assumption (ii) is equivalent to

$$\underline{p}^{+}{}_B = \underline{q}^{+}{}_B. \tag{3.3}$$

For simplicity, for a partition $\underline{\mathfrak{p}}$, we denote

$$\underline{\mathfrak{p}}_O := \begin{cases} \underline{\mathfrak{p}}_B & \text{if } |\underline{\mathfrak{p}}| \text{ is odd},\\ \underline{\mathfrak{p}}_D & \text{if } |\underline{\mathfrak{p}}| \text{ is even}.\end{cases}$$

Then Lemma 2.1 in this case can be rephrased as

$$\underline{\mathfrak{p}}_O = \begin{cases} (\underline{\mathfrak{p}}_{>x})^{-}{}_O \sqcup (\underline{\mathfrak{p}}_{\le x})^{+}{}_O & \text{if } l(\underline{\mathfrak{p}}_{>x}) + |\underline{\mathfrak{p}}_{>x}| \text{ is odd},\\ (\underline{\mathfrak{p}}_{>x})_O \sqcup (\underline{\mathfrak{p}}_{\le x})_O & \text{if } l(\underline{\mathfrak{p}}_{>x}) + |\underline{\mathfrak{p}}_{>x}| \text{ is even}.\end{cases}$$

First, we deal with the case that $l(\underline{p}_{>p_x}) + |\underline{p}_{>p_x}|$ is odd. In this case, $\underline{p}_{>p_x}$ is non-empty, and $l((\underline{p}^{+})_{>p_x}) + |(\underline{p}^{+})_{>p_x}|$ is even. Then we may replace

$\underline{p}$ and $\underline{q}$ with $\underline{p}_{\leq p_x}$ and $\underline{q}_{\leq p_x}$ respectively, and assume $p_1 = p_x$. After the replacement, (3.3) becomes

$$\underline{p}_O = \underline{q}_O, \tag{3.4}$$

and $\underline{p}$ and $\underline{q}$ are still of type $C$.

Since $\underline{q}_O$ contains at most $x-1$ copies of $p_1$, in order that (3.4) holds, we must have $p_1$ is even, $x$ is odd and $p_{x+1} < p_1$. Therefore, we may write

$$\underline{p} = [p_1^x, (p_1-1)^{y-x-1}] \sqcup [p_y, \ldots, p_r].$$

Note that $y-x-1$ is even since $\underline{p}$ is of type $C$. These parity conditions give

$$\underline{p}_O = [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup [p'_y, p'_{y+1}, \cdots],$$

where $0 \leq p'_y \leq p_y + 1 \leq p_1 - 1$, and $p'_{y+1} \leq p_y < p_1 - 1$. On the other hand, write

$$\underline{q} = [p_1^{x-1}, (p_1-1)^{2l}] \sqcup \underline{q}_{<p_1-1}.$$

Then we have

$$\underline{q}_O = [p_1^{x-1}, (p_1-1)^{2l}] \sqcup (\underline{q}_{<p_1-1})_O.$$

Comparing the multiplicity of $p_1 - 1$ in both sides of (3.4), we obtain $2l = y - x + 1$ and $p'_y = p_y + 1 = p_1 - 1$. This verifies both Parts (a) and (b) in this case.

For Part (c), $\underline{q}^\sharp$ and $\underline{p}^\flat$ are of the form

$$\underline{q}^\sharp = [p_1^x, (p_1-1)^{y-x-1}, p_1 - 2] \sqcup \underline{q}_{<p_1-1},$$
$$\underline{p}^\flat = [p_1^{x-1}, (p_1-1)^{y-x+1}] \sqcup [p_{y+1}, \ldots, p_r],$$

after the replacement. It suffices to show that $(\underline{q}^\sharp)_O = \underline{q}_O$ and $(\underline{p}^\flat)_O = \underline{p}_O$. Indeed,

$$\begin{aligned}(\underline{q}^\sharp)_O &= ([p_1^x, (p_1-1)^{y-x-1}])^-{}_O \sqcup ([p_1-2] \sqcup \underline{q}_{<p_1-1})^+{}_O \\ &= [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup [p_1-1] \sqcup (\underline{q}_{<p_1-1})_O \\ &= \underline{q}_O.\end{aligned}$$

Similarly,

$$\begin{aligned}\underline{p}_O &= ([p_1^x, (p_1-1)^{y-x-1}])^-{}_O \sqcup ([p_1-2] \sqcup [p_{y+1}, \ldots, p_r])^+{}_O \\ &= [p_1^{x-1}, (p_1-1)^{y-x}] \sqcup [p_1-1] \sqcup ([p_{y+1}, \ldots, p_r])_O \\ &= ([p_1^{x-1}, (p_1-1)^{y-x+1}])_O \sqcup ([p_{y+1}, \ldots, p_r])_O \\ &= (\underline{p}^\flat)_O.\end{aligned}$$

This completes the proof of the case that $l(\underline{p}_{>p_x}) + |\underline{p}_{>p_x}|$ is odd.

Next, we deal with the case that $l(\underline{p}_{>p_x}) + |\underline{p}_{>p_x}|$ is even. If $\underline{p}_{>p_x}$ is non-empty, then $l((\underline{p}^+)_{>p_x}) + |(\underline{p}^+)_{>p_x}|$ is odd, and hence we may replace $\underline{p}$ and $\underline{q}$

with $\underline{p}_{\leq p_x}$ and $\underline{q}_{\leq p_x}$ respectively and assume $p_1 = p_x$. After the replacement, (3.3) still holds, and $\underline{p}$, $\underline{q}$ are still of type $C$.

Write

$$\underline{p} = [p_1^x, p_{x+1}, \dots],\ \underline{q} = [p_1^{x-1}, q_x, \dots],$$

where $p_{x+1} \leq p_1$ and $q_x < p_1$. Let

$$\ell_p := l((\underline{p}^+{}_O)_{\geq p_1}),\ \ell_q := l((\underline{q}^+{}_O)_{\geq p_1}).$$

Note that (3.3) implies $\ell_p = \ell_q$.

If $p_1$ is odd, then $\ell_q \leq x$ and $\ell_p \geq x$, where $\ell_p = x$ only if $p_{x+1} < p_1$. Thus $\ell_p = \ell_q$ implies that $p_{x+1} < p_1$, but then one of $\underline{p}, \underline{q}$ is not of type $C$, a contradiction. Therefore, $p_1$ must be even. Then $\ell_q \leq x - 1$ and $\ell_p \geq x - 1$, where $\ell_p = x - 1$ only if $p_{x+1} < p_1$ and $x - 1$ is odd. Thus we may rewrite

$$\begin{aligned}
\underline{p} &= [p_1^x, (p_1 - 1)^{y-x-1}] \sqcup [p_y, \dots, p_r],\\
\underline{q} &= [p_1^{x-1}, (p_1 - 1)^{2l}] \sqcup \underline{q}_{<p_1-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\underline{p}^+{}_O &= [p_1 + 1, p_1^{x-2}, (p_1 - 1)^{y-x}] \sqcup [p'_y, p'_{y+1}, \dots],\\
\underline{q}^+{}_O &= [p_1 + 1, p_1^{x-2}, (p_1 - 1)^{2l}] \sqcup (\underline{q}_{<p_1-1})_O,
\end{aligned}$$

where $0 \leq p'_y \leq p_y + 1 \leq p_1 - 1$ and $p'_{y+1} \leq p_y < p_1 - 1$. As $\underline{p}$ is of type $C$, $y_1 - x_1 - 1$ is even, and hence comparing the multiplicity of $p_1 - 1$ in both sides of (3.3) gives $2l = y - x + 1$ and $p'_y = p_y + 1 = p_1 - 1$. This verifies both Parts (a) and (b).

For Part (c), it suffices to show $(\underline{q}^\sharp)^+{}_O = \underline{q}^+{}_O$ and $(\underline{p}^\flat)^+{}_O = \underline{p}^+{}_O$. Indeed,

$$\begin{aligned}
(\underline{q}^\sharp)^+{}_O &= ([p_1 + 1, p_1^{x-1}, (p_1 - 1)^{y-x-1}])^-{}_O \sqcup ([p_1 - 2] \sqcup \underline{q}_{<p_1-1})^+{}_O\\
&= [p_1 + 1, p_1^{x-2}, (p_1 - 1)^{y-x}] \sqcup [p_1 - 1] \sqcup (\underline{q}_{p_1-1})_O\\
&= \underline{q}^+{}_O.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{p}^+{}_O &= ([p_1 + 1, p_1^{x-1}, (p_1 - 1)^{y-x-1}])^-{}_O \sqcup ([p_1 - 2] \sqcup [p_{y+1}, \dots, p_r])^+{}_O\\
&= [p_1 + 1, p_1^{x-2}, (p_1 - 1)^{y-x}] \sqcup [p_1 - 1] \sqcup ([p_{y+1}, \dots, p_r])_O\\
&= ([p_1 + 1, p_1^{x-2}, (p_1 - 1)^{y-x+1}])_O \sqcup ([p_{y+1}, \dots, p_r])_O\\
&= (\underline{p}^\flat)^+{}_O.
\end{aligned}$$

This completes the proof of the lemma for type $C$.

**Type D**: For a partition $\underline{\mathfrak{p}}$ of type $D$, its Barbasch-Vogan dual is given by

$$d_{BV}(\underline{\mathfrak{p}}) = \underline{\mathfrak{p}}^*{}_D = \underline{\mathfrak{p}}^{+-}{}_C{}^*,$$

where the last equality holds if $\underline{p}$ is of type $D$ (see [1, Lemma 3.3]). Thus Assumption (ii) is equivalent to

$$\underline{p}^{+-}{}_{C} = \underline{q}^{+-}{}_{C}. \tag{3.5}$$

First, we deal with the case that $|\underline{p}_{>p_x}|$ is odd. In this case, both $\underline{p}_{>p_x}$ and $\underline{p}_{\le p_x}$ are non-empty, and hence $|(\underline{p}^{+-})_{>p_x}|$ is even. Then if we denote $\widetilde{\underline{p}} := \underline{p}_{\le p_x}$ and $\widetilde{\underline{q}} := \underline{q}_{\le p_x}$, which are of type $B$, then (3.5) becomes

$$\widetilde{\underline{p}}^{-}{}_{C} = \widetilde{\underline{q}}^{-}{}_{C}. \tag{3.6}$$

Therefore, Parts (a) and (b) in this case follow from Parts (a) and (b) for type $B$, which are already established above. Moreover, if we define $\widetilde{\underline{q}^{\sharp}} := (\underline{q}^{\sharp})_{\le p_x}$ and $\widetilde{\underline{p}^{\flat}} := (\underline{p}^{\flat})_{\le p_x}$, then Part (c) holds if $(\widetilde{\underline{q}^{\sharp}})^{-}{}_{C} = \widetilde{\underline{q}}^{-}{}_{C}$ and $(\widetilde{\underline{p}^{\flat}})^{-}{}_{C} = \widetilde{\underline{p}}^{-}{}_{C}$, which are also verified in Part (c) for type $B$. This completes the proof of this case.

Next, we deal with the case that $|\underline{p}_{>p_x}|$ is even. If $\underline{p}_{>p_x}$ is non-empty, then $|(\underline{p}^{+-})_{>p_x}|$ is odd, and hence we may replace $\underline{p}$ and $\underline{q}$ with $\underline{p}_{\le p_x}$ and $\underline{q}_{\le p_x}$ respectively and assume $p_1 = p_x$. After the replacement, (3.5) still holds, and $\underline{p}$, $\underline{q}$ are still of type $D$. Then Parts (a), (b) and (c) are already verified in the second case of the proof of type $B$, where (3.2) holds. This completes the proof of the lemma. $\square$

In the following example, we denote $\underline{q}^{\sharp} = \underline{q}^{\sharp}(\underline{p}, \underline{q})$ and $\underline{p}^{\flat} = \underline{p}^{\flat}(\underline{p}, \underline{q})$ to keep track of the pair $(\underline{p}, \underline{q})$ where $\underline{p} \succcurlyeq \underline{q}$.

Example 3.2. *Let $\underline{\mathfrak{p}} = [7^2, 5^2, 3^2, 1^2]$, a special partition of type $D$. Consider the following partitions in $d_{BV}^{-1}(\underline{\mathfrak{p}})$:*

$$\underline{p_1} = [7^2, 5, 4^2, 2^2, 1],\ \underline{p_2} = [7, 6^2, 5, 3, 2^2, 1],\ \underline{p_3} = [7, 6^2, 4^2, 3, 1^2],$$

*where $\underline{p_1} \succcurlyeq \underline{p_2} \succcurlyeq \underline{p_3}$. Then*

$$\begin{aligned}
\underline{p_4} &:= \underline{q}^{\sharp}(\underline{p_1}, \underline{p_2}) = [7^2, 5^2, 3, 2^2, 1],\\
\underline{p_5} &:= \underline{q}^{\sharp}(\underline{p_1}, \underline{p_3}) = [7^2, 5, 4^2, 3, 1^2],\\
\underline{p_6} &:= \underline{q}^{\sharp}(\underline{p_2}, \underline{p_3}) = [7, 6^2, 5, 3^2, 1^2],\\
\underline{p_7} &:= \underline{q}^{\sharp}(\underline{p_4}, \underline{p_5}) = [7^2, 5^2, 3^2, 1^2] = \underline{q}^{\sharp}(\underline{p_5}, \underline{p_6}) = \underline{q}^{\sharp}(\underline{p_4}, \underline{p_6}),\\
\underline{p_8} &:= \underline{p}^{\flat}(\underline{p_1}, \underline{p_2}) = [7, 6^2, 4^2, 2^2, 1] = \underline{p}^{\flat}(\underline{p_1}, \underline{p_3}) = \underline{p}^{\flat}(\underline{p_2}, \underline{p_3}).
\end{aligned}$$

*Indeed,* $d_{BV}^{-1}(\underline{\mathfrak{p}}) = \{\underline{p_1}, \ldots, \underline{p_8}\}$*. We visualize the dominance ordering in the following picture.*

$$\begin{tikzcd}
& \underline{p_7} & \\
\underline{p_4} \arrow[ur] & \underline{p_5} \arrow[u] & \underline{p_6} \arrow[ul] \\
\underline{p_1} \arrow[u] \arrow[ur] & \underline{p_2} \arrow[ul] \arrow[ur] & \underline{p_3} \arrow[ul] \arrow[u] \\
& \underline{p_8} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

In the above example, there exists a unique maximal/minimal element in $d_{BV}^{-1}(\underline{\mathfrak{p}})$. This is not a coincidence as shown in the following corollary.

Corollary 3.3. *Let* $(X, X') \in \{(B, C), (C, B), (D, D)\}$ *and take a* $\underline{\mathfrak{p}} \in \mathcal{P}_{X'}$ *such that* $d_{BV}^{-1}(\underline{\mathfrak{p}})$ *is non-empty. Here* $d_{BV} = d_{BV,(X,X')}$*. Then the followings hold.*

*(a) For arbitrary* $\underline{p_1}, \underline{p_2} \in d_{BV}^{-1}(\underline{\mathfrak{p}})$*, there exist* $\underline{q}^{\uparrow}, \underline{q}^{\downarrow} \in d_{BV}^{-1}(\underline{\mathfrak{p}})$ *such that*

$$\underline{q}^{\uparrow} \geq \underline{p_i}, \;\; \underline{q}^{\downarrow} \leq \underline{p_i}$$

*for* $i = 1, 2$*.*

*(b) The set* $d_{BV}^{-1}(\underline{\mathfrak{p}})$ *has a unique maximal element and a unique minimal element under the dominance order. Moreover, the unique maximal element is exactly* $d_{BV,(X',X)}(\underline{\mathfrak{p}})$*.*

Proof. For Part (a), we apply induction on

$$t(\underline{p_1}, \underline{p_2}) := \max(l(\underline{p_1}), l(\underline{p_2})) + 1 - x(\underline{p_1}, \underline{p_2}) \geq 0.$$

Note that $t = 0$ if and only if $\underline{p_1} = \underline{p_2}$, where the conclusion trivially holds.

Suppose that $t(\underline{p_1}, \underline{p_2}) = k > 0$ and that the conclusion is verified for every pair $(\underline{p_1}', \underline{p_2}') \in d_{BV}^{-1}(\underline{\mathfrak{p}}) \times d_{BV}^{-1}(\underline{\mathfrak{p}})$ with $t(\underline{p_1}', \underline{p_2}') < k$. We may assume $\underline{p_1} \not\geq \underline{p_2}$ so that $\underline{p_1}^{\flat}$ and $\underline{p_2}^{\sharp}$ is defined in Lemma 3.1(c). Then by the definition of $\underline{p_1}^{\flat}$ and $\underline{p_2}^{\sharp}$ there, we have strict inequality

$$x(\underline{p_1}, \underline{p_2}^{\sharp}) > x(\underline{p_1}, \underline{p_2}), \;\; x(\underline{p_1}^{\flat}, \underline{p_2}) > x(\underline{p_1}, \underline{p_2}),$$

and

$$\max(l(\underline{p_1}), l(\underline{p_2})) = \max(l(\underline{p_1}^{\flat}), l(\underline{p_2})) = \max(l(\underline{p_1}), l(\underline{p_2}^{\sharp})).$$

Therefore, we have

$$t(\underline{p_1}, \underline{p_2}^{\sharp}) < t(\underline{p_1}, \underline{p_2}), \;\; t(\underline{p_1}^{\flat}, \underline{p_2}) < t(\underline{p_1}, \underline{p_2}).$$

Thus by the induction hypothesis, there exist $\underline{q}^{\uparrow}$ and $\underline{q}^{\downarrow}$ such that

$$\begin{cases} \underline{q}^{\uparrow} \geq \underline{p_1}, \\ \underline{q}^{\uparrow} \geq \underline{p_2}^{\sharp} \geq \underline{p_2}, \end{cases} \qquad \begin{cases} \underline{q}^{\downarrow} \leq \underline{p_1}^{\flat} \leq \underline{p_1}, \\ \underline{q}^{\downarrow} \leq \underline{p_2}. \end{cases}$$

This completes the proof of Part (a).

For Part (b), the uniqueness of the maximal and minimal element follows from Part (a). For the second part, we recall that for any partition $\underline{p}$ of type $X$,

(i) $d_{BV,(X,X')}(\underline{p}) = d_{BV,(X,X')} \circ d_{BV,(X',X)} \circ d_{BV,(X,X')}(\underline{p})$, and
(ii) $d_{BV,(X',X)} \circ d_{BV,(X,X')}(\underline{p}) \geq \underline{p}$.

See [3, Proposition A2, Corollary A3]. Let $\underline{p} \in d_{BV}^{-1}(\underline{\mathfrak{p}})$. By (i),

$$\begin{aligned} \underline{\mathfrak{p}} &= d_{BV,(X,X')}(\underline{p}) = d_{BV,(X,X')} \circ d_{BV,(X',X)} \circ d_{BV,(X,X')}(\underline{p}) \\ &= d_{BV,(X,X')}(d_{BV,(X',X))}(\underline{\mathfrak{p}})), \end{aligned}$$

and hence $d_{BV,(X',X)}(\underline{\mathfrak{p}})$ is in $d_{BV}^{-1}(\underline{\mathfrak{p}})$. By (ii),

$$d_{BV}(\underline{\mathfrak{p}}) = d_{BV,(X',X)} \circ d_{BV,(X,X')}(\underline{p}) \geq \underline{p}.$$

This completes the proof of the corollary. $\square$

Next, given a partition $\underline{p} \in \mathcal{P}_X$, where $X \in \{B, C, D\}$, we describe the necessary and sufficient conditions on $\underline{q}$ such that $\underline{p} \geq \underline{q}$ and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$ in the following two lemmas.

LEMMA 3.2. *Let $X \in \{B, C, D\}$. Suppose $\underline{p} = [p_1, \ldots, p_r], \underline{q} = [q_1, \ldots, q_s] \in \mathcal{P}_X(n)$ satisfy that $\underline{p} \geq \underline{q}$ and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$. Then there exists a sequence of pair of positive integers $\{(x_i, y_i)\}_{i=1}^{\alpha}$ where*

*(a) $1 \leq x_i < y_i \leq r+1$,*
*(b) $p_{x_i} = p_{x_i+1} + 1 = \cdots = p_{y_i - 1} + 1 = p_{y_i} + 2$, where we set $p_{r+1} = 0$,*
*(c) the sequence $(p_{x_1}, \ldots, p_{x_\alpha})$ is strictly decreasing,*

*such that $\underline{q}$ can be obtained from $\underline{p}$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=1}^{\alpha}$ in $\underline{p}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=1}^{\alpha}$.*

PROOF. Note that $\underline{p} \geq \underline{q}$ implies that $\underline{p} \succcurlyeq \underline{q}$. We fix $\underline{\mathfrak{p}} := d_{BV}(\underline{p})$, and apply induction on

$$t(\underline{p}, \underline{q}) := l(\underline{p}) + 1 - x(\underline{p}, \underline{q}).$$

Note that under the assumption that $\underline{p} \geq \underline{q}$, $t(\underline{p}, \underline{q}) = 0$ if and only if $\underline{p} = \underline{q}$, where the conclusion holds trivially.

Suppose that $t(\underline{p}, \underline{q}) = k > 0$ and that the conclusion is verified for every pair $(\underline{p}', \underline{q}') \in d_{BV}^{-1}(\underline{\mathfrak{p}}) \times d_{BV}^{-1}(\underline{\mathfrak{p}})$ with $\underline{p}' \geq \underline{q}'$ and $t(\underline{p}', \underline{q}') < k$. Recall that in Lemma 3.1, we define $x = x(\underline{p}, \underline{q}) < y \leq r+1$ and define

$$\underline{p}^{\flat} := [q_1, \ldots, q_y, p_{y+1}, \ldots, p_r],$$

which is also in $d_{BV}^{-1}(\underline{\mathfrak{p}})$ by Part (c) of that Lemma. By definition, we have

$$\underline{p} \geq \underline{p}^{\flat} \geq \underline{q},$$

and

$$x(\underline{p}^{\flat}, \underline{q}) > x(\underline{p}, \underline{q}).$$

Note that $\underline{p}^{\flat}$ is obtained from $\underline{p}$ by replacing $p_x, p_y$ in $\underline{p}$ with $p_x - 1, p_y + 1$.

If $l(\underline{p}^{\flat}) > l(\underline{p})$, then it is not hard to see from the construction that $y = r + 1$ and $\underline{p}^{\flat} = \underline{q}$, and hence the conclusion holds with $\alpha = 1$. If $l(\underline{p}^{\flat}) = l(\underline{p})$, then $t(\underline{p}^{\flat}, \underline{q}) < t(\underline{p}, \underline{q})$, and hence the induction hypothesis implies that there exist $\{(x_i, y_i)\}_{i=2}^{\alpha}$ such that $\underline{q}$ can be obtained from $\underline{p}^{\flat}$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=2}^{\alpha}$ in $\underline{p}^{\flat}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=2}^{\alpha}$. Note that if $\alpha \geq 2$, then

$$p_{x_2} \leq p_{y+1} = p_x - 2 < p_x.$$

Therefore, the conclusion holds with $(x_1, y_1) = (x, y)$. This completes the proof of the lemma. ∎

LEMMA 3.3. *Let $X \in \{B, C, D\}$. Suppose $\underline{p} = [p_1, \ldots, p_r] \in \mathcal{P}_X(n)$, and $\{(x_i, y_i)\}_{i=1}^{m}$ is a sequence of pairs of integers satisfies Conditions (a),(b) and (c) in Lemma 3.2. If we define $\underline{q}$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=1}^{\alpha}$ in $\underline{p}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=1}^{\alpha}$, then $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$ if and only if the following holds for all $1 \leq i \leq \alpha$, where we denote $m_i$ the multiplicity of $p_{x_i}$ in $\underline{p}$.*

*(i) If $X = B$, then $p_{x_i}$ is odd and $m_i + |\underline{p}_{>p_{x_i}}|$ is odd.*
*(ii) If $X = C$, then $p_{x_i}$ is even and $m_i + l(\underline{p}_{>p_{x_i}}) + |\underline{p}_{>p_{x_i}}|$ is even.*
*(iii) If $X = D$, then $p_{x_i}$ is odd and $m_i + |\underline{p}_{>p_{x_i}}|$ is even.*

PROOF. For $1 \leq j \leq \alpha$, we denote $\underline{q}_j$ by replacing $\{p_{x_i}, p_{y_i}\}_{i=1}^{j}$ in $\underline{p}$ with $\{p_{x_i} - 1, p_{y_i} + 1\}_{i=1}^{j}$. Also, we set $\underline{q}_0 = \underline{p}$. Then it suffices to show the conditions are equivalent to $d_{BV}(\underline{q}_j) = d_{BV}(\underline{q}_{j+1})$ for $j = 0, \ldots, \alpha - 1$. Note that Conditions (a), (b) and (c) in Lemma 3.2 imply that for $1 \leq i \leq \alpha$,

$$|\underline{p}_{>p_{x_i}}| = |(\underline{q}_{i-1})_{>p_{x_i}}|, \quad l(\underline{p}_{>p_{x_i}}) = l((\underline{q}_{i-1})_{>p_{x_i}}).$$

Therefore, we may assume $\alpha = 1$.

The necessary direction is already shown in the proof of Lemma 3.1. Also, the computation of the sufficient direction is identical to the verification of the equality $d_{BV}(\underline{p}^{\flat}) = d_{BV}(\underline{p})$ in the proof of Lemma 3.1(c), which we omit. This completes the proof of the lemma. ∎

Finally, we give a complete description of the set $d_{BV}^{-1}(\underline{\mathfrak{p}})$ using its unique maximal element $d_{BV}(\underline{\mathfrak{p}})$.

Theorem 3.4. *Let $(X, X') \in \{(B, C), (C, B), (D, D)\}$ and $\underline{\mathfrak{p}}$ be a special partition of type $X'$. Write $d_{BV}(\underline{\mathfrak{p}}) =: \underline{p} = [p_1^{m_1}, \ldots, p_r^{m_r}]$, and define a subset $I \subseteq \{1, \ldots, r\}$ type by type as follows. (We set $p_{r+1} = 0$ and $m_{r+1} = 1$.)*

*(i) When $X = B$,*

$$I := \left\{1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is odd, and } m_i+\sum_{j=1}^{i-1} m_j p_j \text{ is odd.}}\right\}.$$

*(ii) When $X = C$,*

$$I := \left\{1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is even, and } m_i+\sum_{j=1}^{i-1} m_j (p_j+1) \text{ is even.}}\right\}.$$

*(iii) When $X = D$,*

$$I := \left\{1 \leq i \leq r \;\middle|\; \substack{p_{i+1}=p_i-2 \text{ or } p_{i+2}=p_i-2, \\ p_i \text{ is odd, and } m_i+\sum_{j=1}^{i-1} m_j p_j \text{ is even.}}\right\}.$$

*For any subset $J \subseteq I$, we define $\underline{p}_J$ from $\underline{p}$ by reducing the multiplicity of $p_j$ and $p_j - 2$ by 1 and increasing the multiplicity of $p_j - 1$ by 2 for each $j \in J$.*

*Then the following map is a bijection of partially ordered sets*

$$\begin{aligned} (2^I, \geq) &\longrightarrow (d_{BV}^{-1}(\underline{\mathfrak{p}}), \geq) \\ J &\longmapsto \underline{p}_J, \end{aligned}$$

*where $(2^I, \geq)$ is the power set of $I$ with the partial ordering defined by $J_1 \geq J_2$ if $J_1 \subseteq J_2$.*

Proof. This follows directly by applying Lemmas 3.2 and 3.3 to $\underline{p} = d_{BV}(\underline{\mathfrak{p}})$, the unique maximal element in $d_{BV}^{-1}(\underline{\mathfrak{p}})$. □

We remark that Case (iii) in above Theorem can be viewed as the inverse of [5, Lemma 6.3.9].

Example 3.5. *We explain Theorem 3.4 on Example 3.2. We have $d_{BV}(\underline{\mathfrak{p}}) = [7^2, 5^2, 3^2, 1^2] =: \underline{p}$. Then $I = \{1, 2, 3\}$, and $\underline{p_k} = \underline{p}_{J_k}$, where*

$$J_1 = \{2, 3\}, J_2 = \{1, 3\}, J_3 = \{1, 2\}, J_4 = \{3\},$$
$$J_5 = \{2\}, J_6 = \{1\}, J_7 = \emptyset, J_8 = \{1, 2, 3\}.$$

The following corollary is a useful criterion to argue $d_{BV}(\underline{p}) \neq d_{BV}(\underline{q})$.

Corollary 3.6. *Suppose $\underline{p} = [p_1, \ldots, p_r] \geq \underline{q} = [q_1, \ldots, q_s]$ are of the same type and $d_{BV}(\underline{p}) = d_{BV}(\underline{q})$. Then for any $1 \leq t \leq r$, we have*

$$0 \leq \sum_{i=1}^{t} p_i - \sum_{i=1}^{t} q_i \leq 1.$$

Proof. This follows from the explicit description in Theorem 3.4. □

Finally, for $(X, X') \in \{(B, C), (C, B), (D, D)\}$ and a special $\mathcal{O}' \in \mathcal{N}_{X'}$, we relate $d_{BV}^{-1}(\mathcal{O}')$ with $d_{BV}^{-1}(\underline{p}_{\mathcal{O}'})$ in the following proposition.

PROPOSITION 3.7. *Let $(X, X') \in \{(B, C), (C, B), (D, D)\}$. For each special $\mathcal{O}' \in \mathcal{N}_{X'}$, we have the following.*

*(a) If $\underline{\mathfrak{p}} := \underline{p}_{\mathcal{O}'}$ is not very even of type $D$, then any $\underline{p} \in d_{BV}^{-1}(\underline{\mathfrak{p}})$ is not very even, and*

$$d_{BV}^{-1}(\mathcal{O}') = \{\mathcal{O}_{\underline{p}} \mid \underline{p} \in d_{BV}^{-1}(\underline{\mathfrak{p}})\}.$$

*(b) If $\underline{\mathfrak{p}} := \underline{p}_{\mathcal{O}'}$ is very even of type $D$, then*

$$d_{BV}^{-1}(\mathcal{O}') = \{d_{BV}(\mathcal{O}')\},$$

*which is a singleton.*

PROOF. It suffices to show that if $\underline{\mathfrak{p}}$ is very even of type $D$, then $d_{BV}^{-1}(\underline{\mathfrak{p}}) = \{d_{BV}(\underline{\mathfrak{p}})\}$. Indeed, it is not hard to see that $\underline{\mathfrak{p}}^*$ is also very even of type $D$. Therefore, we may write

$$d_{BV}(\underline{\mathfrak{p}}) = \underline{\mathfrak{p}}^* = \underline{p} = [p_1^{m_1}, \dots, p_r^{m_r}],$$

where $p_i$ and $m_i$ are all even. Then the index set $I$ defined in Theorem 3.4 is the empty set, and hence $d_{BV}^{-1}(\underline{\mathfrak{p}}^*)$ is a singleton. This completes the proof of the proposition. $\square$

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## Jiangova slutnja i vlakna Barbasch-Vogan dualnosti

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Sažetak. Poznata Shahidijeva slutnja kaže da temperirani $L$–paketi imaju generičke članove. Prirodna generalizacija Shahidijeve slutnje na ne-temperiranine lokalne Arthurove pakete je Jiangova slutnja koja karakterizira odnos između strukture lokalnih Arthurovih parametara i gornjih granica valnih fronti reprezentacija u lokalnim Arthurovim paketima. Jiangova slutnja temelji se na dualnosti Barbascha i Vogana. U ovom radu, najprije dajemo pregled recentnih rezultata vezanih za Jiangovu slutnju, a zatim eksplicitno opisujemo vlakna Barbasch-Vogan dualnosti za klasične grupe.

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