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A NOTE ON BOUNDARY COMPONENTS OF ARITHMETIC QUOTIENTS OF THE GROUP SL_2 OVER AN ALGEBRAIC NUMBER FIELD

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Dedicated to M. Tadić on the occasion of his 70th birthday

Abstract. Given an algebraic number field k, we consider quotients X_G/Γ associated with arithmetic subgroups Γ of the special linear algebraic k-group $G = SL_2$. The group G is k-simple, of k-rank one, and split over k. The Lie group G_∞ of real points of the $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}(G)$, obtained by restriction of scalars, is the finite direct product $G_\infty = \prod_{v \in V_{k,\infty}} G_v = SL_2(\mathbb{R})^s \times SL_2(\mathbb{C})^t$, where the product ranges over the set $V_{k,\infty}$ of all archimedean places of k, and s (resp. t) denotes the number of real (resp. complex) places of k. The corresponding symmetric space is denoted by X_G.

Using reduction theory, one can construct an open subset $Y_\Gamma \subset X_G/\Gamma$ such that its closure $\overline{Y}_\Gamma$ is a compact manifold with boundary $\partial\overline{Y}_\Gamma$, and the inclusion $\overline{Y}_\Gamma \longrightarrow X_G/\Gamma$ is a homotopy equivalence. The connected components $Y^{[P]}$ of the boundary $\partial\overline{Y}_\Gamma$ are in one-to-one correspondence with the finite set of Γ-conjugacy classes of minimal parabolic k-subgroups of G. We are concerned with the geometric structure of the boundary components. Each component carries the natural structure of a fibre bundle. We prove that the basis of this bundle is homeomorphic to the torus T^{s+t-1} of dimension $s+t-1$, has the compact fibre T^m of dimension $m = s + 2t = [k : \mathbb{Q}]$, and its structure group is $SL_m(\mathbb{Z})$. Finally, we determine the cohomology of $Y^{[P]}$.

1. Introduction

Given an algebraic number field k, we consider quotients X_G/Γ associated with arithmetic subgroups Γ of the special linear algebraic k-group $G = SL_2$. This group is k-simple, k-split, and of k-rank one. The Lie group G_∞ of real points of the $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}(G)$, obtained by restriction of scalars, is the finite direct product $G_\infty = \prod_{v \in V_{k,\infty}} G_v = SL_2(\mathbb{R})^s \times SL_2(\mathbb{C})^t$, where

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the product ranges over the set $V_{k,\infty}$ of all archimedean places of k, and s (resp. t) denotes the number of real (resp. complex) places of k. The corresponding symmetric space is denoted by X_G. In fact, there is a G_∞-invariant Riemannian metric on X_G, and, if Γ is torsion-free, the homogenous space X_G/Γ carries the structure of a Riemannian manifold of finite volume.

Via reduction theory there exists an open subset $Y_\Gamma \subset X_G/\Gamma$ such that its closure $\overline{Y}_\Gamma$ is a compact manifold with boundary $\partial\overline{Y}_\Gamma$, and the inclusion $\overline{Y}_\Gamma \longrightarrow X_G/\Gamma$ is a homotopy equivalence. The connected components $Y^{[P]}$ of the boundary $\partial\overline{Y}_\Gamma$ are parametrised by the finite set of Γ-conjugacy classes of minimal parabolic k-subgroups of G. We are concerned with the geometric structure of the boundary components. Induced by a Levi decomposition $P = LN$ (with N the unipotent radical of P), each component carries the structure of a fibre bundle $N_\infty/(N_\infty \cap \Gamma) \longrightarrow Y^{[P]} \longrightarrow Z_L/\Gamma_L$ where the basis is a locally symmetric space originating with the Levi subgroup L. We prove that the basis of this bundle is homeomorphic to the torus T^{s+t-1} of dimension $s+t-1$, has the compact fibre T^m of dimension $m = [k : \mathbb{Q}]$, and its structure group is $SL_m(\mathbb{Z})$. Finally, we determine the cohomology of $Y^{[P]}$, thereby giving a proof of Proposition 1.1 in [4]. The action of the fundamental group Γ_L on $N_\infty/(N_\infty \cap \Gamma)$ extends to a natural action on the cohomology $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})$ of the fibre. This gives rise to a local coefficient system, to be denoted $\mathsf{H}^*(F_b, \mathbb{C})$, on the pathwise connected base space $B = Z_L/\Gamma_L$. Here $F_b \cong N_\infty/(N_\infty \cap \Gamma)$ denotes the fibre over $b \in B$. We obtain

$$H^*(Y^{[P]}, \mathbb{C}) \cong H^*(Z_L/\Gamma_L, \mathsf{H}^*(F_b, \mathbb{C})) \cong \\ H^*(Z_L/\Gamma_L, \mathbb{C}) \otimes H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})^{\Gamma_L},$$

where the term $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})^{\Gamma_L}$ denotes the space of elements in $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})$ which are invariant under the action of Γ_L. If k has a real embedding, that is, $s > 0$, the only Γ_L-invariant subspaces in $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})$ are in degree 0 and m. If $s = 0$, it may happen that one finds Γ_L-invariant classes in degree t as well.

Notation and conventions

Let k be an algebraic number field, and let $\mathcal{O}_k$ denote its ring of integers. The set of places of k will be denoted by V_k, and $V_{k,\infty}$ (resp. $V_{k,f}$) refers to the subsets of archimedean (resp. non-archimedean) places of k. Given a place $v \in V_k$, the completion of k with respect to v is denoted by k_v. For a finite place $v \in V_{k,f}$ we write $\mathcal{O}_{k,v}$ for the valuation ring in k_v. If the field k is fixed, we write $V = V_k$ etc.

Suppose the extension $k/\mathbb{Q}$ has degree $m = [k : \mathbb{Q}]$. Let Σ be the set of distinct embeddings $\sigma_i : k \to \mathbb{C}$, $1 \leq i \leq m$. Among these embeddings some factor through $k \to \mathbb{R}$. Let $\sigma_1, ..., \sigma_s$ denote these real embeddings $k \to \mathbb{R}$. Given one of the remaining embeddings $\sigma : k \to \mathbb{C}, \sigma(k) \not\subset \mathbb{R}$, to be called imaginary, there is the conjugate one $\bar{\sigma} : k \to \mathbb{C}$, defined by $x \mapsto \overline{\sigma(x)}$, where $\bar{z}$ denotes the usual complex conjugation of the complex number z. Then the number of imaginary embeddings is an even number, which we denote by $2t$. We number the $m = s + 2t$ embeddings $\sigma_i : k \to \mathbb{C}$, $i = 1, ..., m$ in such a way that, as above, σ_i is real for $1 \leq i \leq s$, and $\overline{\sigma}_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$.

The set V_∞ of archimedean places of k is naturally identified with the set of embeddings $\{\sigma_i\}_{1 \leq i \leq s+t} \subset \Sigma$. We denote by σ_v the embedding which corresponds to $v \in V_{k,\infty}$.

Let $\mathbb{A}_k$ (resp. $\mathbb{I}_k$) be the ring of adèles (resp. the group of idèles) of k. We denote by $\mathbb{A}_{k,\infty} = \prod_{v \in V_{k,\infty}} k_v$ the archimedean component of the ring $\mathbb{A}_k$, and by $\mathbb{A}_{k,f}$ the finite adèles of k. There is the usual decomposition of $\mathbb{A}_k$ into the archimedean and the non-archimedean part $\mathbb{A} = \mathbb{A}_{k,\infty} \times \mathbb{A}_{k,f}$.

2. Reduction theory for the algebraic k-group SL_2

2.1. *The group SL_2.* Given an algebraic number field k, the group of k-rational points of the connected reductive k-algebraic group GL_2 coincides with the group $GL(2, k)$ of (2×2)-matrices with entries in k. The group $Z(k)$ of k-rational points of the centre Z of GL_2 is given by the group $Z(k) = \{g = \mathrm{diag}(\lambda, \lambda) \mid \lambda \in k^\times\}$ of scalar diagonal matrices. We fix the maximal k-split torus S in GL_2 given by

$$S(k) = \left\{ g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix} \right) \mid \lambda, \mu \in k^\times \right\}.$$

Let $\Phi_k = \Phi(GL_2, S) \subset X^*(S)$ be the set of roots of GL_2 with respect to S. A basis of Φ_k is given by the non-trivial character $\alpha : S/k \to \mathbb{G}_m/k$, defined by the assignment $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix} \right) \mapsto \lambda\mu^{-1}$. We denote by Q_0 the minimal parabolic k-subgroup of GL_2 which is determined by $\{\alpha\}$. We have a Levi decomposition of Q_0 into the semi-direct product $Q_0 = SN_0$ of its unipotent radical N_0 by S.

The derived group of the general linear group GL_2 over k is the special linear k-group SL_2; it is a k-simple simply connected algebraic group of k-rank one. We fix the maximal k-split torus L_0 of SL_2, whose k-rational points are given by $L_0(k) = SL_2(k) \cap S(k)$, hence,

$$L_0(k) = \left\{ g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right) \mid \lambda \in k^\times \right\}.$$

A basis for the set of roots for SL_2 with respect to L_0 is given by the restriction of α on L_0, denoted by the same letter. The minimal parabolic k-subgroup which corresponds to α is denoted by P_0 with Levi decomposition $P_0 = L_0 N_0$ of its unipotent radical N_0 by L_0. We call P_0 the standard minimal parabolic subgroup of SL_2. Any minimal parabolic k-subgroup of SL_2 is k-conjugate under SL_2 to P_0, and we have a Levi decomposition $P = LN$.

2.2. *Reduction theory.* Given an algebraic number field k, we write G for the algebraic k-group SL_2. For every archimedean place $v \in V_{k,\infty}$, together with the corresponding embedding $\sigma_v : k \longrightarrow \overline{k}$, there are given a field $k_v = \mathbb{R}$ or $\mathbb{C}$ and a real Lie group $G_v = G^{\sigma_v}(k_v)$. The group

$$G_\infty = \prod_{v \in V_{k,\infty}} G_v, \tag{2.1}$$

viewed as the topological product of the groups G_v, $v \in V_{k,\infty}$, is isomorphic to the group of real points $(\mathrm{Res}_{k/\mathbb{Q}}G)(\mathbb{R})$ of the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}G$ obtained from G by restriction of scalars. In G_∞, we identify $G(k)$ with the set of elements $(g^{\sigma_v})_{v \in V_{k,\infty}}$ where $g \in G(k)$. In an analogous way, if H is an algebraic k-subgroup of G, we denote by H_∞ the group of real points $(\mathrm{Res}_{k/\mathbb{Q}}H)(\mathbb{R})$ of the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}H$.

We denote by s (resp. t) the number of real (resp. complex) places of k. Thus, the degree m of the extension $k/\mathbb{Q}$ equals $m = s + 2t$. Then the real Lie group G_∞ is given as the finite direct product

$$G_\infty \cong SL_2(\mathbb{R})^s \times SL_2(\mathbb{C})^t. \tag{2.2}$$

For each place $v \in V_{k,\infty}$, let X_v be the symmetric space associated with G_v, described as the space of maximal compact subgroups of G_v. In fact, all of these are conjugate to one another, thus, we may write $X_v = K_v \backslash G_v$ for any maximal compact subgroup $K_v \subset G_v$. If $v \in V_{k,\infty}$ is a real place, X_v is the hyperbolic 2-plane H^2, and, if $v \in V_{k,\infty}$ is a complex place, X_v is the hyperbolic 3-space H^3. Since X_v is diffeomorphic to $\mathbb{R}^{d(G_v)}$ where $d(G_v) = \dim G_v - \dim K_v$, the space X_v is contractible. We define

$$X_G := \prod_{v \in V_{k,\infty}} X_v \cong (\mathrm{H}^2)^s \times (\mathrm{H}^3)^t$$

as the product of the symmetric spaces X_v, and we let $d(G) = \sum_{v \in V_{k,\infty}} d(G_v)$. Since the real Lie group G_∞ acts properly from the right on X_G, a given arithmetic subgroup Γ of $G(k)$, being viewed as a discrete, thus closed subgroup of G_∞, acts properly on X_G as well. If Γ is torsion-free, the action of Γ on X_G is free, and the quotient X_G/Γ is a smooth manifold of dimension $d(G)$. There is a G_∞-invariant Riemannian metric on X_G. Given an arithmetic subgroup Γ of $G(k)$, we are interested in the homogenous space X_G/Γ. If Γ is torsion-free, the space X_G/Γ carries the structure of a Riemannian manifold of finite volume.

Since G_∞ is not compact and the k-group G is k-simple simply connected, the group G has the strong approximation property (see [7]). Therefore, $G(k)$ is dense in the locally compact group $G(\mathbb{A}_{k,f})$, or, equivalently, $G_\infty G(k)$ is dense in $G(\mathbb{A}_k)$.

Given any proper ideal $\mathfrak{a} \subset \mathcal{O}_k$ the corresponding principal congruence subgroup of level $\mathfrak{a}$ is defined by

$$\Gamma(\mathfrak{a}) := \ker(SL_2(\mathcal{O}_k) \longrightarrow SL_2(\mathcal{O}_k/\mathfrak{a})). \tag{2.3}$$

Using [11, Prop. 4.4.4], if for every prime number p, the ideal $\mathfrak{a}^{p-1}$ does not divide the principal ideal $p\mathcal{O}_k$ in $\mathcal{O}_k$, the arithmetic group $\Gamma(\mathfrak{a})$ is torsion-free. Therefore, for almost all choices of the ideal $\mathfrak{a}$ the group $\Gamma(\mathfrak{a})$ is torsion-free.

Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$, and let $v_0 \in V_{k,f}$ be the corresponding non-archimedean place of k. Given a proper ideal $\mathfrak{a} \subset \mathcal{O}_k$ let $\nu_\mathfrak{p}(\mathfrak{a})$ be the maximal exponent e such that $\mathfrak{p}^e$ divides the ideal $\mathfrak{a}$. Thus, we have $\mathfrak{a}\mathcal{O}_{v_0} = \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}\mathcal{O}_{v_0}$. For each $v \in V_{k,f}$, the kernel $K_v(\mathfrak{a})$ of the natural homomorphism $G(\mathcal{O}_v) \longrightarrow G(\mathcal{O}_v/\mathfrak{a}\mathcal{O}_v)$ is an open compact subgroup of $G(\mathcal{O}_v)$. This implies that the direct product $K(\mathfrak{a}) := \prod_{v\in V_{k,f}} K_v(\mathfrak{a})$ is an open compact subgroup of $G(\mathbb{A}_{k,f})$, and we have $\Gamma(\mathfrak{a}) = G(k) \cap K(\mathfrak{a})$. Using the strong approximation property of the algebraic k-group G, we have the continuous map $G_\infty \longrightarrow K(\mathfrak{a})\backslash G(\mathbb{A}_k)/G(k)$, defined by $g \mapsto K(\mathfrak{a})gG(k)$. It gives rise to a homeomorphism $K(\mathfrak{a})\backslash G(\mathbb{A}_k)/G(k) \tilde{\longrightarrow} G_\infty/\Gamma(\mathfrak{a})$ which is equivariant under the action of G_∞.

2.3. *Reduction theory - the boundary components.* Since the k-rank of the algebraic k-group $G = SL_2$ is one, all proper parabolic k-subgroups of G are minimal, all of these are conjugate under $G(k)$. Given any arithmetic subgroup $\Gamma \subset G(k)$, this conjugacy class falls into finitely many Γ-conjugacy classes (see [1, Prop. 15.6]). In the case of the group $\Gamma = SL_2(\mathcal{O}_k)$, the cardinality of this set is equal to the class number h_k of k (see [13, Prop. 20]).

We consider the standard minimal parabolic k-subgroup $P_0 = L_0N_0$ of G. Any k-character $\chi : L_0 \longrightarrow \mathbb{G}_m$ induces a homomorphism

$$\chi_\infty : L_{0,\infty} \longrightarrow \mathbb{G}_{m,\infty} \cong (\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t.$$

Given an archimedean place $v \in V_{k,\infty}$, we denote by $|\cdot|_v$ the absolute value on $k_v = \mathbb{R}$ if v is real resp. the square of the absolute value on $k_v = \mathbb{C}$ if v is complex. The norm homomorphism is defined by

$$|\cdot| : \mathbb{G}_{m,\infty} \cong \prod_{v\in V_{k,\infty}} k_v^\times = (\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t \longrightarrow \mathbb{R}^\times_{>0}, \quad (g_v)_{v\in V_{k,\infty}} \mapsto \prod_{v\in V_{k,\infty}} |g_v|_v.$$

The compositum $|\cdot| \circ \chi$ can be canonically extended to a homomorphism $|\chi| : P_{0,\infty} \longrightarrow \mathbb{R}^\times_{>0}$. We apply this construction to the positive simple root $\alpha : L_0 \longrightarrow \mathbb{G}_m$, and, we define $P^{(1)}_{0,\infty} := \{p \in P_{0,\infty} \mid |\alpha|(p) = 1\}$. Given any point $x \in X_G$, let $K_x \subset G_\infty$ be the corresponding maximal compact subgroup of the Lie group G_∞, then $P^{(1)}_{0,\infty} \cap K_x = P_{0,\infty} \cap K_x$. Moreover, since the image of the arithmetic group Γ under α is an arithmetic subgroup of $\mathbb{G}_m(k)$, thus, contained in $\mathcal{O}_k^\times$, we have $|\alpha|(\gamma) = 1$ for every $\gamma \in P_{0,\infty} \cap \Gamma$. It follows that $P_{0,\infty} \cap \Gamma = P^{(1)}_{0,\infty} \cap \Gamma$. Given any other minimal parabolic subgroup P of G, there is a $g \in G(k)$ such that $gP(k)g^{-1} = P_0(k)$. Therefore, we can define $P_\infty^{(1)}$ via conjugation.

In the specific case of the algebraic k-group G of k-rank one, the general results in [5, Sect. 1.2] in reduction theory take the following form; a different approach is carried through in [1, Thm. 17.10].

THEOREM 2.1. *Given a torsion-free arithmetic subgroup $\Gamma \subset G(k)$, there exists an open subset $Y_\Gamma \subset X_G/\Gamma$ such that its closure $\overline{Y}_\Gamma$ is a compact manifold with boundary $\partial\overline{Y}_\Gamma$, and the inclusion $\overline{Y}_\Gamma \longrightarrow X_G/\Gamma$ is a homotopy equivalence. The connected components of the boundary $\partial\overline{Y}_\Gamma$ are in one-to-one correspondence with the finite set, to be denoted $\mathcal{P}/\Gamma$, of Γ-conjugacy classes of minimal parabolic k-subgroups of G. If P is a representative for a Γ-conjugacy class of minimal parabolic k-subgroups of G, we denote the corresponding connected component in $\partial\overline{Y}_\Gamma$ by $Y^{[P]}$. Then we have as a disjoint union*

$$\partial\overline{Y}_\Gamma = \coprod_{[P]\in\mathcal{P}/\Gamma} Y^{[P]}.$$

and the boundary component $Y^{[P]}$ is diffeomorphic to the double coset space $(K \cap P_\infty^{(1)})\backslash P_\infty^{(1)}/(P_\infty^{(1)} \cap \Gamma)$ where $K \subset G_\infty$ is a maximal compact subgroup.

We are interested in the geometric structure of such a boundary component $Y^{[P]}$. The canonical morphism $P \longrightarrow P/N = L$ onto the maximal k-split torus L gives rise to a surjective morphism $p : P_\infty^{(1)} \longrightarrow L_\infty^{(1)}$. The image $K_L := p(K \cap P_\infty^{(1)})$ of $K \cap P_\infty^{(1)}$ under this projection is a maximal compact subgroup in $L_\infty^{(1)}$. We write $Z_L := K_L\backslash L_\infty^{(1)}$ for the associated manifold of right cosets. The preimage of a point in $L_\infty^{(1)}$ is diffeomorphic to N_∞.

The image Γ_L of $P_\infty^{(1)} \cap \Gamma$ under p is a discrete torsion-free subgroup of $L_\infty^{(1)}$. The group Γ_L acts properly and freely on Z_L, and the double coset space Z_L/Γ_L is a manifold with universal cover Z_L. The projection $p : P_\infty^{(1)} \longrightarrow L_\infty^{(1)}$ induces a surjection

$$\pi : (K \cap \Gamma)\backslash P_\infty^{(1)}/(P_\infty^{(1)} \cap \Gamma) \longrightarrow Z_L/\Gamma_L; \tag{2.4}$$

it is a locally trivial fibration whose fibre is $N_\infty/(N_\infty \cap \Gamma)$. This fibre is compact (see, e.g. [11, Sect. 9.3]).

PROPOSITION 2.2. *Given a representative P for a Γ-conjugacy class of minimal parabolic k-subgroups of G, the corresponding boundary component $Y^{[P]} \cong (K \cap P_\infty^{(1)})\backslash P_\infty^{(1)}/(P_\infty^{(1)} \cap \Gamma)$ admits the structure of a fibre bundle which is equivalent to the fibre bundle*

$$(Z_L \times_{\Gamma_L} N_\infty/(N_\infty \cap \Gamma), Z_L, N_\infty/(N_\infty \cap \Gamma)). \tag{2.5}$$

This bundle is associated by the natural action of Γ_L *on the compact fibre* $N_\infty/(N_\infty \cap \Gamma)$*, induced by inner automorphisms, to the universal covering* $Z_L \longrightarrow Z_L/\Gamma_L$*.*

PROOF. The action of the group $P_\infty^{(1)} \cap \Gamma$ on $K_L \backslash P_\infty^{(1)}$ is proper and free. Since P is the normaliser of N in G, the group $N_\infty \cap \Gamma$ is a normal subgroup in $P_\infty^{(1)} \cap \Gamma$. Therefore, the quotient group $\Gamma_{P/N} := (P_\infty^{(1)} \cap \Gamma)/(N_\infty \cap \Gamma)$ acts properly and freely on $K_L \backslash P_\infty^{(1)}/(N_\infty \cap \Gamma)$. In view of the decomposition $P_\infty^{(1)} = L_\infty^{(1)} N_\infty$ as a semi-direct product, induced by the semi-direct product $P = LN$, this space can be viewed as the product space

$$K_L \backslash P_\infty^{(1)}/(N_\infty \cap \Gamma) \xrightarrow{\sim} K_L \backslash L_\infty^{(1)}/(N_\infty \cap \Gamma) \times N_\infty/(N_\infty \cap \Gamma). \tag{2.6}$$

We have that P is the normaliser of N, thus, the group $P_\infty^{(1)} \cap \Gamma$ acts via inner automorphisms on N_∞. It follows, since N is commutative, that there is an induced action of the quotient group $\Gamma_{P/N}$ via diffeomorphisms on the space $N_\infty/(N_\infty \cap \Gamma)$. The group $\Gamma_{P/N}$ is isomorphic to Γ_L. In view of (2.6), the fibration in question is equivalent to the fibre bundle

$$(Z_L \times_{\Gamma_L} N_\infty/(N_\infty \cap \Gamma), Z_L, N_\infty/(N_\infty \cap \Gamma)) \tag{2.7}$$

which is associated by the natural action of Γ_L on $N_\infty/(N_\infty \cap \Gamma)$, induced by inner automorphisms, to the universal covering $Z_L \longrightarrow Z_L/\Gamma_L$. □

2.4. *The geometric structure of the boundary components.* Let Γ be a torsion-free arithmetic subgroup of $G(k) = SL_2(k)$. Given a representative P for a Γ-conjugacy class of minimal parabolic k-subgroups of G we seek to understand the base space and the fibre of the fibre bundle structure of the boundary component $Y^{[P]}$ of $\partial \overline{Y}_\Gamma$. For any natural number $n > 0$, we denote by $T^n = (S^1)^n$ the n-dimensional torus.

THEOREM 2.3. *The boundary component* $Y^{[P]} = Z_L \times_{\Gamma_L} N_\infty/(N_\infty \cap \Gamma)$ *is the total space of a fibre bundle with fibre* $N_\infty/(N_\infty \cap \Gamma)) \cong T^m$*, base space* $Z_L/\Gamma_L \cong T^{s+t-1}$*, and structure group* Γ_L*. Hence it is a torus bundle over a torus. The structure group* Γ_L *of the fibre bundle is a totally disconnected commutative group.*

PROOF. First, with regard to the fibre $N_\infty/(N_\infty \cap \Gamma)$, we may assume that P is the standard minimal parabolic k-subgroup $P_0 = L_0 N_0$ whose group of k-points is $P_0(k) = \left\{ g = \left(\begin{smallmatrix} x & y \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in k^\times, y \in k \right\}$. The group of k-points of its unipotent radical is commutative, and, since $m = s + 2t$ we obtain as additive groups

$$N_{0,\infty} \cong \mathrm{Res}_{k/\mathbb{Q}}(k_a)(\mathbb{R}) \cong \mathbb{R}^m$$

The group $N_{0,\infty} \cap \Gamma$ as a discrete subgroup of $N_{0,\infty}$ forms a complete lattice in $\mathbb{R}^m$, thus the claim follows.

Second, we deal with the universal cover $Z_L := K_L\backslash L_\infty^{(1)}$ of the base space Z_L/Γ_L of the fibration (2.4). We may assume that $P = P_0$ is the standard minimal parabolic k-subgroup of G. We have the identification

$$L_{0,\infty} = \left\{(g_v)_v = \left(\begin{smallmatrix} (t_v)_v & 0 \\ 0 & (t_v)_v^{-1}\end{smallmatrix}\right) \mid t_v \in \mathbb{R}^\times \text{ if } v \text{ real},\ t_v \in \mathbb{C}^\times \text{ if } v \text{ complex}\right\} \xrightarrow{\sim} (\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t,$$

where $v \in V_{k,\infty}$ ranges ovre the archimedean places of k. Passing over to the group $L_{0,\infty}^{(1)}$, we obtain a diffeomorphism

$$L_{0,\infty}^{(1)} \xrightarrow{\sim} \left\{(g_v)_v = \left(\begin{smallmatrix} (t_v)_v & 0 \\ 0 & (t_v)_v^{-1}\end{smallmatrix}\right) \mid t_v \in k_v^\times,\ \prod_{v\in V_{k,\infty}} |t_v|_v = 1\right\}.$$

Recall that, given an archimedean place $v \in V_{k,\infty}$, we denote by $|\cdot|_v$ the absolute value on $k_v = \mathbb{R}$ if v is real resp. the square of the absolute value on $k_v = \mathbb{C}$ if v is complex.

The assignement $(x_1, \ldots, x_s, z_1, \ldots, z_t) \mapsto (|x_1|, \ldots, |x_s|, |z_1|^2, \ldots, |z_t|^2)$ defines a surjective homomorphism $L_{0,\infty} = (\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t. \longrightarrow (\mathbb{R}^\times_{>0})^{s+t}$. It gives rise to a surjective homomorphism

$$\psi^{(1)} : L_{0,\infty}^{(1)} \cong ((\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t)^{(1)} \longrightarrow ((\mathbb{R}^\times_{>0})^{s+t})^{(1)}$$

where

$$((\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t)^{(1)} := \{(h_v)_{v\in V_{k,\infty}} \in ((\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t) | \prod_{v\in V_{k,\infty}} |h_v|_v = 1\},$$

and $((\mathbb{R}^\times_{>0})^s)^{(1)}$ is accordingly defined as

$$((\mathbb{R}^\times_{>0})^s)^{(1)} := \{(x_v)_{v\in V_{k,\infty}} \in (\mathbb{R}^\times_{>0})^{s+t} | \prod_{v\in V_{k,\infty}} x_v = 1\}$$

.

We summarise the aforesaid considerations in the diagram

$$\begin{array}{ccccccc}
L_\infty^{(1)} & \longrightarrow & (\prod_{v\in V_{k,\infty}} k_v^\times)^{(1)} & \xrightarrow{\psi^{(1)}} & (\prod_{v\in V_{k,\infty}} \mathbb{R}^\times_{>0})^{(1)} & \xrightarrow{\mathrm{Log}^{(1)}} & \mathcal{H} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_\infty & \longrightarrow & (\prod_{v\in V_{k,\infty}} k_v^\times) & \xrightarrow{\psi} & (\prod_{v\in V_{k,\infty}} \mathbb{R}^\times_{>0}) & \xrightarrow{\mathrm{Log}} & \mathbb{R}^{s+t},
\end{array}$$

where the map $\mathrm{Log} : (\prod_{v\in V_{k,\infty}} \mathbb{R}^\times_{>0}) \longrightarrow \mathbb{R}^{s+t}$ is defined by the assignment $(x_1, \ldots, x_{s+t}) \mapsto (\log x_1, \ldots, \log x_{s+t})$, where $\mathcal{H}$ denotes the hypersurface $\mathcal{H} := \{r = (r_i) \in \mathbb{R}^{s+t} \mid \sum_i r_i = 0\}$ in $\mathbb{R}^{s+t}$, and where the vertical arrows are the natural inclusions. The map $\psi^{(1)}$ resp. $\mathrm{Log}^{(1)}$ is obtained via restriction from the map ψ resp. Log. The horizontal arrows in both exterior squares of the diagram are isomorphisms. The kernel of $\psi^{(1)}$ is the unique maximal compact subgroup in $((\mathbb{R}^\times)^s \times (\mathbb{C}^\times)^t)^{(1)}$, given as the product of s copies of $\{\pm 1\}$ and t copies of S^1. We obtain $\ker(\psi^{(1)}) = K_L$. It follows that $Z_L = K_L \backslash L^{(1)}_{0,\infty} \cong \mathbb{R}^{s+t-1}$.

By assumption the arithmetic group $\Gamma \subset SL_2(\mathcal{O}_k)$ is torsion-free. It follows, since $\ker \psi^{(1)}$ is compact, $\Gamma_L \cap \ker \psi^{(1)} = \{0\}$. Therefore, $\psi^{(1)}$ maps Γ_L isomorphically onto a discrete torsion-free subgroup of $(\prod_{v \in V_{k,\infty}} \mathbb{R}^\times_{>0})^{(1)}$. The arithmetic group Γ_L may be viewed as a sugbroup of $\mathcal{O}_k^\times$. As worked out in the usual proof of the Dirichlet theorem on the unit groups of number fields (see, e.g., [3]), in our context of the diagram, the map induced by the inclusion $\mathcal{O}_k^\times \longrightarrow L_0(k)$, maps $\mathcal{O}_k^\times$ isomophically onto a complete lattice in the hyperplane $\mathcal{H}$. Therefore, $(\mathrm{Log}^{(1)} \circ \psi^{(1)})(\Gamma_L)$ is a complete lattice in $\mathcal{H}$. It follows that the base space Z_L/Γ_L of the fibre bundle is the torus T^{s+t-1}. $\square$

Remark 2.4. Given a totally real quadratic number field, we are concerned with Hilbert modular surfaces as dealt with in [6]. In this case the boundary components $Y^{[P]}$ occur in the disguise of boundaries of neighbourhoods of cusp singularities.

3. Torus bundles over tori

In order to determine the cohomology of a boundary component $Y^{[P]}$ it is useful to describe an inductive construction of fibre bundles whose fibre is a torus T^m and whose basis is a torus T^r, and whose structure group is the group $SL_m(\mathbb{Z})$ of automorphisms of the free $\mathbb{Z}$-module $\mathbb{Z}^m$ of determinant one.

First, using [15, Chap. 18], in particular, the notions and notations introduced there, we recall the classification of fibre bundles over the 1-sphere. Let (E, S^1, F, π) be a fibre bundle over S^1 with totally disconnected structure group G. Up to equivalence, this bundle is in normal form. Thus we can describe it in the following way: we cut the 1-sphere S^1 into two (closed) hemispheres E_1 and E_2 whose intersection consists of exactly two antipodal points x_0 and x_1 in S^1. Then we can choose two open neighbourhoods $V_1, V_2 \subset S^1$ such that $E_i \subset V_i, i = 1, 2$, and such that the change of coordinates g_{12} satisfies $g_{12}(x_0) = e \in G$. Then the group element $g_{12}(x_1) \in G$ describes the gluing process of the fibre over the point x_1; this element is called the characteristic homeomorphism of the given bundle. By [15, 18.3], two fibre bundles (E, S^1, F, π) and (E', S^1, F, π') over S^1 with characteristic homeomorphisms A and A' whose fibre F and structure group G coincide are equivalent if and only if there are an element $g \in G$ and a path $\omega : I \longrightarrow G$

in G such that $\omega(0) = A$ and $\omega(1) = gA'g^{-1}$. Since G is totally disconnected the characteristic homeomorphism is determined by a suitable chosen generator of the fundamental group $\pi_1(S^1)$. The bundle is equivalent to the bundle $((I \times F)/\sim, S^1, F, p)$ where the equivalence relation $\sim$ is given by $(1, x) \sim (0, Ax), x \in F$, and the projection map $p : (I \times F)/\sim \longrightarrow I/(1 \sim 0)$ is defined by the assignment $(t, x) \mapsto t, t \in I, x \in F$.

Second, let $\mathsf{A} := \{A_1, A_2, \ldots, A_r\}$ be a subset of elements in $SL_m(\mathbb{Z})$ which commute with one another. We inductively construct, following [9], fibre bundles $T(A_1, \ldots, A_i)$, $1 \le i \le r$, over $T^i = \mathbb{R}^i/\mathbb{Z}^i$ with fibre T^m and structure group $SL_m(\mathbb{Z})$. The matrices A_i, $1 \le i \le r$, induce homeomorphisms of T^m which will also be denoted by A_i.

We set $T(A_1) = (I \times T^m)/\sim$, where $(1, x) \sim (0, A_1 x)$, $x \in T^m$, and the projection $\pi : T(A_1) \longrightarrow S^1$ is given by $(t, x) \mapsto t$. The matrices $A_i, 1 < i \le r$, act on $T(A_1)$ via $(t, x) \mapsto (t, A_i x)$ in a natural way.

Suppose that the torus bundle $(T(A_1, \ldots, A_k), T^k, T^m, \pi_k)$ is constructed, and the matrices A_i with $k + 1 \le i \le r$ act on $T(A_1, \ldots, A_k)$. We define

$$T(A_1, \ldots, A_{k+1}) = (I \times T(A_1, \ldots, A_k))/\sim$$

where $(1, y) \sim (0, A_{k+1} y)$, $y \in T(A_1, \ldots, A_k)$, and the projection

$$\pi_{k+1} : T(A_1, \ldots, A_{k+1}) \longrightarrow S^1 \times T^k = T^{k+1}$$

is given by the assignment $(t, y) \mapsto (t, \pi_k(y))$. The matrices $A_{k+2}, \ldots, A_r$ act on $T(A_1, \ldots, A_{k+1})$ via $(t, y) \mapsto (t, A_i y)$, $k + 2 \le i \le r$. Since the matrices $A_1, A_2, \ldots, A_r$ commute with one another, this is well defined. The total space is endowed with the orientations induced by the canonical orientations on $\mathbb{R}^m$ and $\mathbb{R}^j$. One checks by induction that the induced action of the matrix A_j, $j > k$, on the bundle is fibrewise.

For the sake of completeness we note the fact that this construction exhausts up to equivalence all torus bundles over tori T^r with structure group $SL_m(\mathbb{Z})$. Via induction one proves (see [9, Thm. 4.3]) the following result.

Proposition 3.1. *Let (E, T^r, T^m, ρ) be a torus bundle over the torus T^r with structure group $SL_m(\mathbb{Z})$. Then this bundle is equivalent to the bundle*

$$(\mathbb{R}^r \times_{\pi_1(T^r)} T^m, T^r, T^m, \pi)$$

associated to the universal covering $\mathbb{R}^r \longrightarrow \mathbb{R}^r/\mathbb{Z}^r$ by the natural action of the fundamental group of the basis on the fibre T^m where $\pi_1(T^r) = \mathbb{Z}^r$ acts on $\mathbb{R}^r$ via right translations. If we denote by $A_1, A_2, \ldots, A_r$ elements in $SL_m(\mathbb{Z})$ which correspond to the action of suitably chosen generators of the fundamental group $\pi_1(T^r)$ then the bundle $T(A_1, A_2, \ldots, A_r)$ is equivalent to the bundle $(\mathbb{R}^r \times_{\pi_1(T^r)} T^m, T^r, T^m, \pi)$. The matrices $A_1, A_2, \ldots, A_r$ are uniquely determined up to conjugation in $SL_m(\mathbb{Z})$.

4. Digression: semi-simple endomorphisms

We review some basic facts regarding semi-simple endomorphisms of finite-dimensional vector spaces over a field K. This notion plays a conclusive role in the actual computation of the cohomology of torus bundles over tori. This allows us to determine the cohomology of a boundary component.

Definition 4.1. *Let V be a finite-dimensional vector space over a field K. We call an endomorphism $\alpha \in \mathrm{End}_K(V)$ semi-simple if every α-stable subspace $U \subset V$, that is, $\alpha U \subset U$, has a complementary α-stable subspace. In other words, equivalently, V viewed as a $K[X]$-module, with X acting as α, is semi-simple.*

By the classification of finitely generated modules over the polynomial ring $K[X]$, such a V is isomorphic to a direct sum of modules of the form $k[X]/(f^m)$ where $f \in K[X]$ is some irreducible polynomial. Therefore V is semi-simple if and only if each of these direct summands is semi-simple, that is, $m = 1$. It follows that an endomorphism $\alpha \in \mathrm{End}_K(V)$ is semi-simple if and only if its minimum polynomial is the product of relatively prime irreducible polynomials.

The semi-simplicity of an endomorphism $\alpha \in \mathrm{End}_K(V)$ is preserved by passage to an α-invariant K-subspace $W \subset V$, as well as to the quotient space V/W.

The following technical observation is useful. Let V, W be two finite-dimensional vector spaces over a field K. In view of the isomorphism

$$\mathrm{End}_K(V) \otimes_K \mathrm{End}_K(W) \tilde{\longrightarrow} \mathrm{End}_K(V \otimes_K W),$$

given an endomorphism $\omega \in \mathrm{End}_K(V \otimes_K W)$, there exist $\phi_i \in \mathrm{End}_K(V)$, $\psi_j \in \mathrm{End}_K(W)$ such that $\omega = \sum \phi_i \otimes \psi_j$. A straightforward argument shows: if $\phi \in \mathrm{End}_K(V)$, $\psi \in \mathrm{End}_K(W)$ are semi-simple endomorphisms, then the endomorphism $\omega = \phi \otimes \psi \in \mathrm{End}_K(V \otimes_K W)$ is semi-simple.

Let L/K be a field extension. Given an endomorphism $\alpha \in \mathrm{End}_K(V)$, let $\alpha_L := \mathrm{id} \otimes \alpha \in \mathrm{End}_L(V_L)$ be the endomorphism of $V_L := L \otimes_K V$ induced by extension of scalars. If α_L is semi-simple, then α is also semi-simple, and if L/K is separable, then the converse is correct.

We observe the following result (see [2, Chap. VII, §5, No. 8, Prop. 15]):

Proposition 4.2. *Given an endomoprhism $\alpha \in \mathrm{End}_K(V)$ with minimum polynomial $m_\alpha \in K[X]$, the following assertions are equivalent:*

- *For every field extension L/K, the endomorphism α_L is semi-simple.*
- *There exists a field extension L/K such that the endomorphism α_L is diagonalisable.*
- *The minimum polynomial m_α is separable over K.*

Definition 4.3. *An endomorphism $\alpha \in \mathrm{End}_K(V)$ is called absolutely semi-simple if one of the equivalent conditions in Proposition 4.2 is valid.*

Clearly, a necessary and sufficient condition for α to be absolutely semi-simple is that the irreducible factors of the minimum polynomial m_α have no multiple roots in the algebraic closure $\bar{K}$ of K.

More generally, we consider a family $\mathcal{A}$ of K-endomorphisms of a given finite-dimensional K-vector space V. We say that the family $\mathcal{A}$ is diagonalisable if there exists a basis $\mathsf{v} = \{v_i\}_{i \in I}$ of V such that the matrix $M_{\alpha,\mathsf{v}}$ for each $\alpha \in \mathcal{A}$ with respect to v has diagonal form. If $\mathcal{A} = \{\alpha\}$ consists of a single element, we say that α is diagonalisable.

The following observation is decisive for the subsequent result: Let $\alpha, \beta \in \mathrm{End}_K(V)$ be two endomorphisms of V which commute with one another, and let V_λ be any eigenspace for α. Then, for all $v \in V_\lambda$, we have $\alpha(\beta(v)) = \beta(\alpha(v)) = \beta(\lambda v) = \lambda\beta(v)$. Thus, V_λ is stable under β.

Proposition 4.4. *Let $\mathcal{A}$ be a family of K-endomorphisms of a given finite-dimensional K-vector space V. Then $\mathcal{A}$ is diagonalisable if and only if all elements in $\mathcal{A}$ are diagonalisable and commute with one another.*

Combining this result with the characterisations of an absolutely semi-simple endomorphism in Proposition 4.2 we obtain

Proposition 4.5. *Let $\mathcal{A}$ be a family of K-endomorphisms of a given finite-dimensional K-vector space V. There exists a field extension L/K such that the set $\mathcal{A}_L := \{\alpha_L \mid \alpha \in \mathcal{A}\} \subset \mathrm{End}_L(V_L)$ is diagonalisable if and only if the endomorphisms in $\mathcal{A}$ are absolutely semi-simple and commute with one another.*

Proposition 4.6. *Let V be a finite-dimensional vector space over a field K. Let $\mathcal{A} = \{\phi_a\}$ be a finite family of semi-simple endomorphisms $\phi_a \in \mathrm{End}_K(V)$ which commute pairwise with one another. We denote by A the subalgebra of the endomorphism algebra $\mathrm{End}_K(V)$ generated by $\mathcal{A}$ and the identity Id_V. Then V decomposes as a direct sum $V = V^A \oplus U$ into the subspace $V^A = \{v \in V \mid \phi(v) = v$ for all $\phi \in A\}$ and a complementary subspace U.*

Proof. The proof proceeds by induction over the number of generators of A. The case of a single generator is taken care by the very definition of a semi-simple endomorphism. Let A be generated by the set $\{\phi_1, \dots, \phi_n, \mathrm{Id}_V\} \subset \mathcal{A}$, and let A' be the subalgebra of $\mathrm{End}_K(V)$ generated by $\phi_1, \dots, \phi_{n-1}$ and Id_V. By induction hypothesis, the subspace $V^{A'}$ admits a direct complement U' such that $V = V^{A'} \oplus U'$. Since for all $1 \leq i \leq n-1$, $\phi_n \circ \phi_i = \phi_i \circ \phi_n$, the restriction of ϕ_n to $V^{A'}$ is well defined and $(\phi_n)_{|V^{A'}}$ is semi-simple. Thus, there exists a direct complement U'' of V^A in $V^{A'}$. We put $U := U'' \oplus U'$. Then we have $V = V^A \oplus U$. ∎

Corollary 4.7. *Let V be a finite-dimensional vector space over a field K. Given an absolutely semi-simple endomorphism $\phi \in \mathrm{End}_K(V)$, there is a canonical identification* $\ker(\phi - \mathrm{Id}) \xrightarrow{\sim} \mathrm{coker}(\phi - \mathrm{Id})$.

Proof. Setting $V^\phi = \{v \in V \mid \phi(v) = v\}$, since ϕ is absolutely semi-simple, we have the direct sum decomposition $V = V^\phi \oplus U$ where $U = \mathrm{im}(\phi - \mathrm{Id})$. This implies the assertion. □

To be in the position to determine the cohomology of torus bundles over tori as constructed above we determine the cohomology ring of an n-torus.

Let R be a commutative ring with identity element, and let $\{a_1, \ldots, a_n\}$ be a finite set of n symbols. We write $\mathsf{I} = \{1, \ldots, n\}$. Then the exterior algebra $\bigwedge_R[a_1, \ldots, a_n]$ is defined as the free R-module with generators $a_{i_1} \cdots a_{i_k}$, for all k-tuples $(i_1, \ldots . i_k)$ of indices in I with $i_1 < \ldots < i_k$, $1 \le k \le n$, endowed with the associative and distributive multiplication determined by the rules $a_i^2 = 0, i = 1, \ldots, n$, and $a_i a_j - a_j a_i = 0$ if $i \neq j$, $i, j = 1, \ldots, n$. If we put $\bigwedge_R^0[a_1, \ldots, a_n] := R$, then $\bigwedge_R^*[a_1, \ldots, a_n]$ becomes a graded commutative ring with the scalar 1 as unit element. For a fixed index p, $1 \le p \le n$, $\bigwedge_R^p[a_1, \ldots, a_n]$ denotes the free R-submodule with basis $a_{i_1} \cdots a_{i_p}$ for all $i_1 < \ldots < i_p$. The generators $a_1, \ldots, a_n$ have degree one. The R-rank of $\bigwedge_R^p[a_1, \ldots, a_n]$ is $\binom{n}{p}$.

If $R = \mathbb{Z}$, we identify the elements $a_1, \ldots, a_n$ with the standard basis $e_1, \ldots, e_n$ of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, and we write $\Lambda^*(\mathbb{Z}^n)$ for the corresponding exterior algebra.

Proposition 4.8. *Given the n-dimensional torus T^n its cohomology ring with coefficients in any commutative field R is given as the exterior algebra $H^*(T^n, R) = \bigwedge_R[a_1, \ldots, a_n]$.*

Proof. The cohomology of the sphere S^1 is $R[a]/(a^2)$ as a ring, and the underlying cohomology group is free. We view the n-torus T^n as the n-fold product of the sphere S^1. Then the Künneth formula [14, VI, 12.16] yields that the cohomology of T^n is the graded tensor product of n copies of $R[a]/(a^2)$. Therefore we obtain $H^*(T^n, R) = \bigwedge_R[a_1, \ldots, a_n]$. □

Corollary 4.9. *An endomorphism $A \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ of the free $\mathbb{Z}$-module $\mathbb{Z}^n$ induces a unique map $A : T^n = \mathbb{R}^n/\mathbb{Z}^n \longrightarrow T^n = \mathbb{R}^n/\mathbb{Z}^n$. Then the ring homomorphism $A^* : H^*(T^n, R) \longrightarrow H^*(T^n, R)$ induced on the cohomology ring $H^*(T^n, R)$ coincides with the unique extension of A to a homomorphism $\Lambda^*(A)$ on the exterior algebra $\Lambda^*(R^n)$ with $\Lambda^*(A)(1) = 1$.*

5. The cohomology of a boundary component

The boundary component $Y^{[P]}$ in X_G/Γ attached to a Γ-conjugacy class of minimal parabolic k-subgroups of G has, up to equivalence, the structure of the fibre bundle

$$(Z_L \times_{\Gamma_L} N_\infty/(N_\infty \cap \Gamma), Z_L/\Gamma_L, N_\infty/(N_\infty \cap \Gamma)) \tag{5.1}$$

associated by the natural action of Γ_L on the compact fibre $N_\infty/(N_\infty \cap \Gamma)$, induced by inner automorphisms, to the universal covering $Z_L \longrightarrow Z_L/\Gamma_L$. This fibre bundle with fibre $N_\infty/(N_\infty \cap \Gamma) \cong T^m$, where $m = s + 2t$ is the degree of the extension $k/\mathbb{Q}$, and base space $Z_L/\Gamma_L \cong T^r$, where $r = s+t-1$ is the $\mathbb{Z}$-rank of the unit group $\mathcal{O}_k^\times$ of the underlying algebraic number field k. We will see that this fibre bundle falls into the realm of torus bundles over tori with structure group $SL_m(\mathbb{Z})$ discussed in Section 3.

The action of the fundamental group Γ_L on $N_\infty/(N_\infty \cap \Gamma)$ extends to an action on the cohomology $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})$ of the fibre. This gives rise to a local coefficient system, to be denoted $\mathsf{H}^*(F_b, \mathbb{C})$ on the pathwise connected base space $B = Z_L/\Gamma_L$. Here $F_b \cong N_\infty/(N_\infty \cap \Gamma)$ denotes the fibre over $b \in B$.

THEOREM 5.1. *Let P be a representative for a Γ-conjugacy class of minimal parabolic k-subgroups of G. The cohomology of the corresponding boundary component $Y^{[P]}$ is given as*

$$H^*(Y^{[P]}, \mathbb{C}) \cong H^*(Z_L/\Gamma_L, \mathsf{H}^*(F_b, \mathbb{C})) \cong H^*(Z_L/\Gamma_L, \mathbb{C}) \otimes H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})^{\Gamma_L}, \tag{5.2}$$

where $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})^{\Gamma_L}$ denotes the space of elements in the cohomology $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})$ which are invariant under Γ_L.

PROOF. We may assume that $P = P_0$ is the standard minimal parabolic k-subgroup. The k-rational points of its unipotent radical are given by $N_0(k) = \left\{ g = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \mid x \in k \right\}$. Moreover, upon identifying $N_0(k)$ with k, we see that $N_0(k) \cap \Gamma = \Delta$ is a complete $\mathbb{Z}$-lattice in k. Passing over to the real points of the group $\mathrm{Res}_{k/\mathbb{Q}}(N_0)$, we obtain $N_{0,\infty} \cong \prod_{v \in V_{k,\infty}} k_v$. Therefore, the underlying structure as a vector space over $\mathbb{R}$, endowed with the Euclidean topology, is $N_{0,\infty}^+ \cong \prod_{v \in V_{k,\infty}} k_v \cong \mathbb{R}^m$. The group $N_{0,\infty}^+ \cap \Gamma$ is a discrete subgroup of maximal rank in $N_{0,\infty}^+ \cong \mathbb{R}^m$. It follows that $N_{0,\infty}^+ \cap \Gamma$ is freely generated over $\mathbb{Z}$ by m vectors $u_1, \ldots, u_m$ which are linearly independent over $\mathbb{R}$. We fix such a basis $\mathsf{u} = \{u_1, \ldots, u_m\}$ of $\mathbb{R}^m$. With regard to the basis u, the action of $N_{0,\infty}^+ \cap \Gamma$ on $N_{0,\infty}^+ \cong \mathbb{R}^m$ is the standard action of $\mathbb{Z}^m$ on $\mathbb{R}^m$.

It follows that we can describe the action of the fundamental group Γ_L on the fibre $N_{0,\infty}/(N_{0,\infty} \cap \Gamma)$ in terms of matrices with integral entries. It is induced by the operation of Γ_L on $N_{0,\infty}$ via inner automorphisms. The group Γ_L is a subgroup of the unit group $\mathcal{O}_k^\times$, hence, viewed as a finitely generated $\mathbb{Z}$-module, it is of rank $s + t - 1$. Given a set $\{\alpha_1, \ldots, \alpha_{s+t-1}\}$ of generators, each of them acts on $N_{0,\infty} \cong \mathbb{R}^m$ with respect to the basis u by an integral matrix $A_i \in GL_m(\mathbb{Z})$, $i = 1, \ldots, s+t-1$, since α_i leaves $N_{0,\infty} \cap \Gamma$ invariant. Since Γ_L is commutative, the matrices A_i, $i = 1, \ldots, s+t-1$,, commute with one another.

We view Γ_L via the diagonal embedding $\mathcal{O}_k^\times \longrightarrow \prod_{v \in V_{k,\infty}} k_v^\times$ as a subgroup of $k_\infty^\times$. Then the action of an element $\epsilon \in \Gamma_L$ is given as a matrix over $\mathbb{C}$ with respect to a suitable basis by a diagonal form $\mathrm{diag}(\epsilon_{(1)}^2, \ldots, \epsilon_{(m)}^2)$ where $\epsilon_{(j)}$ denotes the j-th component of $\epsilon \in k_\infty^\times$. Since $\epsilon \in \Gamma_L \subset \mathcal{O}_k^\times$ is a unit, the determinant of this matrix is one. Therefore, ϵ acts orientation-preserving on the fibre $N_{0,\infty}/(N_{0,\infty} \cap \Gamma)$. In addition, the endomorphisms induced by generators of Γ_L are semi-simple endomorphisms.

Following the construction (and notation) introduced in Section 3, we have to determine the cohomology of a torus bundle $T(A_1, \ldots, A_r)$, where $r := s+t-1$, with fibre T^m and basis T^r, determined by integral matrices $A_i \in SL_m(\mathbb{Z})$, $i = 1, \ldots, r$. We proceed by induction over the dimension r of the basis.

A decisive tool in the argument is the Wang sequence in cohomology for fibre bundles over the 1-sphere (see [8, Lemma 8.4.] or [14, Chap. 8, Sect. 5, Cor. 6]). It relates the cohomology of the total space to the cohomology of the fibre, accentuating the role of the characteristic homeomorphism.

Proposition 5.2. *Let (E, S^1, F, π) be a fibre bundle over S^1 with totally disconnected structure group G and characteristic homeomorphism $A \in G$. Then there is an exact sequence*

(5.3)

$$\longrightarrow H^q(E,R) \xrightarrow{j^*} H^q(F,R) \xrightarrow{A^*-\mathrm{Id}} H^q(F,R) \xrightarrow{\delta^*} H^{q+1}(E,R) \longrightarrow$$

of cohomology groups where the coefficients are in any field R. The map $j : F \longrightarrow E$ is the natural inclusion, and δ^* is induced by the boundary operator in a Mayer-Vietoris Sequence attached to a suitable excisive couple of subsets of E. The endomorphism $H^q(F,R) \longrightarrow H^q(F,R)$ is given by $A^* - \mathrm{Id}$.

As an application to our case of interest this result has the following consequence:

Corollary 5.3. *Let (E, S^1, F, π) be a fibre bundle over S^1 with totally disconnected structure group G and characteristic homeomorphism $A \in G$. Suppose that the endomorphism $H^*(A) = A^* : H^*(F,\mathbb{Q}) \longrightarrow H^*(F,\mathbb{Q})$ induced by A is semi-simple, then we have*

$$H^n(E,\mathbb{Q}) = \bigoplus_{p+q=n} H^p(S^1,\mathbb{Q}) \otimes H^q(F,\mathbb{Q})^{A^*} \tag{5.4}$$

where $H^q(F,\mathbb{Q})^{A^*}$ denotes the subspace of elements in $H^q(F,\mathbb{Q})$ invariant under the endomorphism A^*.

Choose a prime ℓ so that the endomorphism $A^* : H^*(F, \mathbb{Z}_\ell) \longrightarrow H^*(F, \mathbb{Z}_\ell)$ induced by A is semi-simple. Then the analogous result is correct for the cohomology $H^n(E, \mathbb{Z}_\ell)$ with coefficients in the finite field $\mathbb{Z}_\ell$.

PROOF. We simultaneously prove both results, and we accordingly write R for the field of coefficients. We may assume that the given bundle is of the form $((I \times F)/\sim, S^1, F, p)$ where the equivalence relation $\sim$ is given by $(1, x) \sim (0, Ax), x \in F$, and the projection $p' : (I \times F)/\sim \to I/(1 \sim 0)$ is defined by the assignment $(t, x) \mapsto t, t \in I, x \in F$. The Wang sequence in Proposition 5.2 gives a short exact sequence

$$0 \longrightarrow \mathrm{coker}(H^{q-1}(A) - Id) \longrightarrow H^q(E, R) \longrightarrow \ker(H^q(A) - Id) \longrightarrow 0.$$

This sequence splits, and one gets a direct sum decomposition

$$H^q(E, R) = \ker(H^q(A) - Id) \oplus \mathrm{coker}(H^{q-1}(A) - Id).$$

This isomorphism is not canonical but depends on the choice of a basis. However, the endomorphism A^* is semi-simple, thus there is a canonical identification $\mathrm{coker}(H^{q-1}(A) - Id) = \ker(H^{q-1}(A) - Id)$. Taking into account that

$$\ker(H^{q-1}(A) - Id) \cong \ker(H^{q-1}(A) - Id) \otimes H^1(S^1, R)$$

resp.

$$\ker(H^q(A) - Id) \cong \ker(H^q(A) - Id) \otimes H^0(S^1, R),$$

together with the identity $\ker(H^q(A) - Id) = H^*(F)^A$, brings the final result. □

The bundle $T(A_1)$ is obtained by the action of A_1 on the fibre T^m. The induced endomorphism $\Lambda^*(A_1) =: A_1^*$ on the cohomology of the fibre $H^*(T^m, \mathbb{Q}) = \Lambda^*(\mathbb{Q}^m)$ is semi-simple. Therefore, by Corollary 5.3, we have

$$H^n(T(A_1), \mathbb{Q}) = \bigoplus_{p+q=n} H^p(S^1, \mathbb{Q}) \otimes H^q(T^m, \mathbb{Q})^{A_1^*}.$$

The matrices $A_j, j > 1$, act on $T(A_1)$ via $(t, x) \mapsto (t, A_j x)$, $x \in T^m$. Since the action is fibrewise the induced homomorphism in cohomology is of the form

$$\alpha_j : s \otimes y \mapsto s \otimes ((A_j)^* y),$$

where $s \in H^p(S^1, \mathbb{Q})$ and $y \in H^q(T^m, \mathbb{Q})^{A_1^*}$. We observe that the restriction of the semi-simple endomorphism A_j^* on $H^q(T^m, \mathbb{Q})^{A_1^*}$ is semi-simple. Note that the endomorphisms $\alpha_j, j > 1$ are semi-simple.

We have the following induction hypothesis:

$$H^n(T(A_1, A_2, \ldots, A_{i-1}), \mathbb{Q}) = \bigoplus_{q+r=n} H^q(T^{i-1}, \mathbb{Q}) \otimes H^r(T^m, \mathbb{Q})^{A_1^*, A_2^*, \ldots, A_{i-1}^*},$$

and the endomorphism induced by A_j, $j > i-1$, on $H^n(T(A_1, A_2, \ldots, A_{i-1}), \mathbb{Q})$ is given by the assignment $\alpha_j : s \otimes y \mapsto s \otimes ((A_j)^* y)$, where $s \in H^q(T^{i-1}, \mathbb{Q})$

and $y \in H^r(T^m, \mathbb{Q})^{A_1^* A_2^*, \dots, A_{i-1}^*}$. The endomorphisms α_j, $j > i-1$, are semi-simple.

By construction $T(A_1, \dots, A_i) = (I \times T(A_1, \dots, A_{i-1}))/\sim$ where $(1, y) \sim (0, A_i y)$. We obtain by assigning $(t, x) \mapsto t$ a locally trivial fibration

$$\pi : T(A_1, \dots, A_i) \longrightarrow S^1$$

over S^1 with fibre $T(A_1, \dots, A_{i-1})$. The characteristic homeomorphism of this bundle over S^1 is the morphism induced by the action of A_i on $T(A_1, \dots, A_{i-1})$. By induction hypothesis the corresponding homomorphism in cohomology is semi-simple, thus the cohomology result for bundles over S^1 yields eventually the assertion. Indeed, using the induction hypothesis and the compatibility of the tensor product with direct sums, we have

$$\begin{aligned}
&H^n(T(A_1, A_2, \dots, A_i), \mathbb{Q}) \cong \\
&\cong \bigoplus_{q+p=n} (H^q(S^1, \mathbb{Q}) \otimes H^p(T(A_1, A_2, \dots, A_{i-1}), \mathbb{Q})^{A_i^*}) \\
&\cong \bigoplus_{q+p=n} (H^q(S^1, \mathbb{Q}) \otimes \bigoplus_{a+b=p} (H^a(T^{i-1}, \mathbb{Q}) \otimes H^b(T^m, \mathbb{Q})^{A_1^*, A_2^*, \dots, A_{i-1}^*})^{A_i^*}) \\
&= \bigoplus_{q+p=n} (H^q(S^1, \mathbb{Q}) \otimes \bigoplus_{a+b=p} (H^a(T^{i-1}, \mathbb{Q}) \otimes H^b(T^m, \mathbb{Q})^{A_1^*, A_2^*, \dots, A_{i-1}^*, A_i^*})) \\
&= \bigoplus_{q+p=n} (\bigoplus_{a+b=p} (H^q(S^1, \mathbb{Q}) \otimes H^a(T^{i-1}, \mathbb{Q}) \otimes H^b(T^m, \mathbb{Q})^{A_1^*, A_2^*, \dots, A_{i-1}^*, A_i^*})) \\
&\cong \bigoplus_{u+b=n} (H^u(T^i, \mathbb{Q}) \otimes H^b(T^m, \mathbb{Q})^{A_1^*, A_2^*, \dots, A_{i-1}^*, A_i^*}).
\end{aligned}$$

One verifies that the endomorphism induced by A_j, $j > i$, on a single summand of the cohomology $H^n(T(A_1, A_2, \dots, A_i), \mathbb{Q})$ is given by the assignment $\alpha_j : z \otimes y \mapsto z \otimes ((A_j)^* y)$, where $z \in H^u(T^i, \mathbb{Q})$ and $y \in H^b(T^m, \mathbb{Q})^{A_1^*, A_2^*, \dots, A_i^*}$. The endomorphisms α_j, $j > i$, are semi-simple. □

REMARK 5.4. The same result is correct if we replace the coefficient system $\mathbb{Q}$ by a finite field $\mathbb{Z}_\ell = \mathbb{Z}/\ell\mathbb{Z}$ where we have to suppose that the prime number ℓ is admissible with regard to the integral matrices $A_1, A_2, \dots, A_r$, that is, the endomorphisms $A_{j,\ell} \in \mathrm{End}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell^m)$ induced by A_j are absolutely semi-simple. This is the case for almost all prime numbers.

It is not difficult to describe the space $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C})^{\Gamma_L}$ of elements in the cohomology of the fibre which are invariant under the action of Γ_L. Recall that $H^*(N_\infty/(N_\infty \cap \Gamma), \mathbb{C}) \cong \Lambda^*(\mathbb{C}^m)$. Let $\Sigma = \{\sigma_1, \dots, \sigma_m\}$ be the set of embeddings $k \longrightarrow \mathbb{C}$. For each subset $J \subset \Sigma$, we find a one-dimensional subspace $U_J \subset \Lambda^{m-|J|}(\mathbb{C}^m)$ such that $u \in \Gamma_L$ acts on U_J via multiplication by $\prod_{\sigma \in J} \sigma(u^2)$. The direct sum of these subspaces exhaust $\Lambda^*(\mathbb{C}^m)$. Since the elements u in $\Gamma_L \subset \mathcal{O}_k^*$ are units, and u acts via u^2, we have that the product

$\prod_{\sigma\in\Sigma}\sigma(u^2)$ over all embeddings in Σ is equal to 1. Therefore, in order to identify the subsets $J\subset\Sigma$ with $\prod_{\sigma\in J}\sigma(u^2)=1$, we have to ensure that also $\prod_{\sigma\in J^c}\sigma(u^2)=1$ where σ ranges over all elements in the complement J^c of J in Σ. Clearly the empty set $J=\emptyset$ and the set $J=\Sigma$ fulfil these requirements, and the corresponding Γ_L-invariant spaces are equal to $U_\emptyset=\Lambda^m(\mathbb{C}^m)$ respectively $U_\Sigma=\Lambda^0(\mathbb{C}^m)$, thus, one-dimensional.

If k has a real embedding, that is, $s>0$, these subspaces are the only Γ_L-invariant subspaces in $H^*(N_\infty/(N_\infty\cap\Gamma),\mathbb{C})$. If $s=0$, it may happen that one finds Γ_L-invariant classes in $H^t(N_\infty/(N_\infty\cap\Gamma),\mathbb{C})$.

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Opaska o komponentama ruba aritmetičkih kvocijenata grupe SL_2 nad poljem algebarskih brojeva

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Sažetak. Za polje algebarskih brojeva k, promatramo kvocijente X_G/Γ pridružene aritmetičkim podgrupama Γ specijalne linearne algebarske grupe $G = SL_2$ definirane nad k. Grupa G je prosta, ranga jedan i rascjepiva nad k. Liejeva grupa G_∞ realnih točaka $\mathbb{Q}$-grupe $\mathrm{Res}_{k/\mathbb{Q}}(G)$, dobivene restrikcijom skalara, je konačni direktni produkt $G_\infty = \prod_{v \in V_{k,\infty}} G_v = SL_2(\mathbb{R})^s \times SL_2(\mathbb{C})^t$, gdje produkt prolazi po skupu $V_{k,\infty}$ svih arhimedskih mjesta od k, a s (odnosno t) označava broj realnih (odnosno kompleksnih) mjesta od k. Odgovarajući simetrični prostor je označen s X_G. Koristeći teoriju redukcije, može se konstruirati otvoreni podskup $Y_\Gamma \subset X_G/\Gamma$ čiji zatvarač $\overline{Y}_\Gamma$ je kompaktna mnogostrukost s rubom $\partial\overline{Y}_\Gamma$, pri čemu je ulaganje $\overline{Y}_\Gamma \longrightarrow X_G/\Gamma$ homotopska ekvivalencija. Komponente povezanosti $Y^{[P]}$ ruba $\partial\overline{Y}_\Gamma$ su u bijekciji sa skupom klasa Γ-konjugiranosti minimalnih paraboličkih k-podgrupa od G koji je konačan. Zanima nas geometrijska struktura komponenata ruba. Svaka komponenta ima prirodnu strukturu svežnja vlakana. U radu je dokazano da je taj svežanj homeomorfan torusu T^{s+t-1} dimenzije $s+t-1$, ima kompaktna vlakna T^m dimenzije $m = s + 2t = [k : \mathbb{Q}]$ te strukturnu grupu $SL_m(\mathbb{Z})$. Na kraju, određena je kohomologija komponenti $Y^{[P]}$.

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