

ON SWAN EXPONENTS OF SYMMETRIC AND EXTERIOR SQUARE GALOIS REPRESENTATIONS

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*Dedicated to Marko Tadić, on his 70th birthday.**

ABSTRACT. Let F be a local non-Archimedean field and E a finite Galois extension of F , with Galois group G . If ρ is a representation of G on a complex vector space V , we may compose it with any tensor operation R on V , and get another representation $R \circ \rho$. We study the relation between the Swan exponents $\text{Sw}(\rho)$ and $\text{Sw}(R \circ \rho)$, with a particular attention to the cases where R is symmetric square or exterior square. Indeed those cases intervene in the local Langlands correspondence for split classical groups over F , via the formal degree conjecture, and we present some applications of our work to the explicit description of the Langlands parameter of simple cuspidal representations. For irreducible ρ our main results determine $\text{Sw}(\text{Sym}^2 \rho)$ and $\text{Sw}(\wedge^2 \rho)$ from $\text{Sw}(\rho)$ when the residue characteristic p of F is odd, and bound them in terms of $\text{Sw}(\rho)$ when p is 2. In that case where p is 2 we conjecture stronger bounds, for which we provide evidence.

1. INTRODUCTION

Let p be a prime number, F a non-Archimedean local field of residue characteristic p , and E a finite Galois extension of F with Galois group $G = \text{Gal}(E/F)$. Suppose that ρ is a finite-dimensional representation of G on a complex vector space V (in this paper, we refer to such ρ simply as a *Galois representation*). For any algebraic representation R of $\text{GL}_{\mathbb{C}}(V)$ on a finite-dimensional complex vector space V' , by composing R with ρ , we obtain

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*The first author met Marko Tadić already in 1971, at the International Mathematical Olympiads in Slovakia. But we realized that only later, when discussing mathematics and more, which happened on many nice occasions.

another Galois representation $R \circ \rho$ of G .

$$G \begin{array}{c} \xrightarrow{\rho} \\ \xrightarrow{R \circ \rho} \\ \xrightarrow{R} \end{array} \text{GL}_{\mathbb{C}}(V) \longrightarrow \text{GL}_{\mathbb{C}}(V')$$

Given that the *Swan exponent* is one of the most fundamental invariants of Galois representations (see Section 2.1 for the definition), it is natural to ask if there is an interesting relationship between the Swan exponents $\text{Sw}(R \circ \rho)$ and $\text{Sw}(\rho)$. This is the problem of our interest.

We first introduce several preceding results on this problem. Bushnell–Henniart–Kutzko investigated the case where ρ is irreducible and $R = \text{Std} \otimes \text{Std}^{\vee}$, i.e., $R \circ \rho = \rho \otimes \rho^{\vee}$, where ρ^{\vee} denotes the contragredient representation of ρ , in [13]. They established an explicit formula of $\text{Sw}(\rho \otimes \rho^{\vee})$. (In fact, their result is more general; for any two irreducible Galois representations ρ_1 and ρ_2 , they obtained an explicit formula of $\text{Sw}(\rho_1 \otimes \rho_2)$). Based on the result of Bushnell–Henniart–Kutzko, Anandavardhanan–Mondal ([2]) established a similar explicit formula in the case where R is the symmetric square Sym^2 , the exterior square \wedge^2 , or the Asai representation “Asai”, by assuming that $p \neq 2$.

However, in these cases, the Swan exponent $\text{Sw}(R \circ \rho)$ cannot be expressed only in terms of $\text{Sw}(\rho)$. The description requires the information of a simple stratum (in the sense of Bushnell–Kutzko, [14]) associated to the supercuspidal representation corresponding to ρ under the local Langlands correspondence (cf. [13, 6.5, Theorem] and [2, Theorem 6.1]).

Based on this observation, we settle for looking for bounds estimating $\text{Sw}(R \circ \rho)$ from $\text{Sw}(\rho)$, sometimes even getting exact formulas. Concerning this direction, let us mention the work of Bushnell–Henniart [11] and also the work of Kiliç [27], in which upper bounds and lower bounds of $\text{Sw}(\rho_1 \otimes \rho_2)$ are investigated. For example, some of the inequalities proved in *loc. cit.* are as follows:

THEOREM 1.1 ([11, Theorems AS and CS]). *Let ρ_1 and ρ_2 be Galois representations. We put $n_1 := \dim(\rho_1)$ and $n_2 := \dim(\rho_2)$.*

1. *If ρ_1 is minimal in the sense that $\text{Sw}(\rho_1) \leq \text{Sw}(\rho_1 \otimes \chi)$ for any character χ , then we have*

$$\text{Sw}(\rho_1 \otimes \rho_2) \geq \frac{1}{2} \max\{n_2 \text{Sw}(\rho_1), n_1 \text{Sw}(\rho_2)\}.$$

2. *We have*

$$\text{Sw}(\rho_1 \otimes \rho_2) \leq n_2 \text{Sw}(\rho_1) + n_1 \text{Sw}(\rho_2) - \min\{\text{Sw}(\rho_1), \text{Sw}(\rho_2)\}.$$

THEOREM 1.2 ([27, Theorem 9.3]). *If ρ is a minimal (in the sense as in Theorem 1.1) Galois representation on V , then, for any algebraic representation R of $\mathrm{GL}_{\mathbb{C}}(V)$, we have*

$$\mathrm{Sw}(\rho \otimes (R \circ \rho)) \geq \frac{1}{2} \dim(R \circ \rho) \cdot \mathrm{Sw}(\rho).$$

Now let us explain our main results. Our principal interest is in the case where $R = \mathrm{Sym}^2$ or $R = \wedge^2$. Our first main result is the following:

PROPOSITION 1.3 (Proposition 2.3). *Suppose that $p \neq 2$. For any Galois representation ρ , we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) - \mathrm{Sw}(\wedge^2 \rho) = \mathrm{Sw}(\rho).$$

Note that if ρ is self-dual, then we have $\rho \otimes \rho^\vee \cong \rho \otimes \rho \cong \mathrm{Sym}^2 \rho \oplus \wedge^2 \rho$. Thus, by combining Proposition 1.3 with the explicit formula of $\mathrm{Sw}(\rho \otimes \rho)$ by Bushnell–Henniart–Kutzko mentioned above, it is possible to get a precise formula of $\mathrm{Sw}(\mathrm{Sym}^2 \rho)$ and $\mathrm{Sw}(\wedge^2 \rho)$. Especially, when ρ furthermore satisfies some additional conditions, the resulting formula is particularly simple as follows:

COROLLARY 1.4 (Corollary 2.4). *Suppose that $p \neq 2$. Let ρ be a $2n$ -dimensional irreducible self-dual Galois representation such that $(\mathrm{Sw}(\rho), 2n) = 1$. Then we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) = n \cdot \mathrm{Sw}(\rho) \quad \text{and} \quad \mathrm{Sw}(\wedge^2 \rho) = (n - 1) \cdot \mathrm{Sw}(\rho).$$

In fact, the equality of Proposition 1.3 has been already obtained in [2], thus not new. Our contribution is that we gave an alternative and simpler proof. See Remark 2.5 for more comments on Proposition 1.3 and Corollary 1.4.

We note that the assumption that $p \neq 2$ is crucially used in the proof of Proposition 1.3. In fact, the equality of Proposition 1.3 does not hold in general when $p = 2$. For example, we have the following:

PROPOSITION 1.5 (Proposition 2.8). *Suppose that $p = 2$. Let ρ be a $(2n + 1)$ -dimensional irreducible self-dual Galois representation such that $(\mathrm{Sw}(\rho), 2n + 1) = 1$. Then we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) - \mathrm{Sw}(\wedge^2 \rho) = 0.$$

Hence, by the same argument as in the case where $p \neq 2$ (i.e., using the Bushnell–Henniart–Kutzko formula), we obtain the following:

COROLLARY 1.6 (Corollary 2.10). *With the same assumptions as in Proposition 1.5, we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) = \mathrm{Sw}(\wedge^2 \rho) = n \cdot \mathrm{Sw}(\rho).$$

On the other hand, there is also an example such that $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho)$ is not zero (for example, take ρ to be a 1-dimensional character such that ρ^2 is wildly ramified). Then what can we expect about the quantity $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho)$ when $p = 2$? Our second main result is the following:

THEOREM 1.7 (Theorem 2.6). *Suppose that $p = 2$. For any Galois representation ρ , we have*

$$0 \leq \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) \leq 2 \text{Sw}(\rho).$$

Note that the left inequality of Theorem 1.7 is “sharp” in the sense that there exists ρ which attains the equality (Proposition 1.5; see also Proposition 2.7). On the other hand, the right inequality is not sharp whenever ρ is not tame (Proposition 2.7). In fact, we expect that the following inequality holds and is sharp.

CONJECTURE 1.8. *Assume $p = 2$. For any Galois representation ρ , we have*

$$(\star) \quad \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) \leq \text{Sw}(\rho).$$

In this paper, we verify Conjecture 1.8 in several special cases and also discuss attempts to the conjecture in the general case.

1. In Section 3.1, we prove the inequality (\star) for any 2-dimensional ρ .
2. In Section 3.2, we prove the inequality (\star) for any ρ of the form $\text{Ind}_{F'/F} \chi$, where F'/F is cyclic and χ is a character of F'^{\times} .
3. In Section 3.3, we discuss an approach based on an analysis of the structure of the ramification subgroups. We also obtain definitive results in some specific cases (Propositions 3.9 and 3.10).
4. In Section 3.4, we discuss an approach via induction on the order of the Galois group G . Especially, we show that if the conjecture is true, then the inequality (\star) must be strict when ρ is orthogonal (3.13).
5. In Section 3.5, we discuss an approach via induction on $\dim(\rho)$.

We finally mention that, in fact, our motivation originates from investigating the *explicit local Langlands correspondence for classical groups*. When \mathbf{G} is a connected reductive group over F , the local Langlands correspondence attaches to a smooth irreducible representation π of $\mathbf{G}(F)$ an L -parameter ϕ (L -parameters are a variant of Galois representations, see Section 4.1). When π is discrete series, it is expected that a certain explicit identity is satisfied between the formal degree of π and the special value of the adjoint γ -factor of ϕ (the *formal degree conjecture*, see Section 4.2). The point is that, when \mathbf{G} is a classical group, the L -parameter ϕ can be naturally regarded as a Galois representation (say ρ) and then the adjoint γ -factor of ϕ is related to the Swan exponent $\text{Sw}(R \circ \rho)$ for some tensor operation R . For example, if $\mathbf{G} = \text{SO}_{2n+1}$

(resp. Sp_{2n} or SO_{2n}), then R is given by Sym^2 (resp. \wedge^2).

$$\begin{array}{ccccc}
 \pi: \text{disc. ser. of } \mathbf{G}(F) & \xleftarrow{\text{LLC}} & \phi: L\text{-par.} & \longleftrightarrow & \rho: \text{Gal. rep.} \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 \text{formal degree of } \pi & \xleftarrow{\text{FDC}} & \text{adjoint } \gamma\text{-factor of } \phi & \longleftrightarrow & \mathrm{Sw}(R \circ \rho)
 \end{array}$$

Thus, for a given L -parameter ϕ , computing $\mathrm{Sw}(R \circ \rho)$ is related to knowing the formal degree of π , which can be a clue to determine the representation π exactly (and vice versa). In Section 4, we present a few results which are obtained based on this philosophy.

1.1. *Notation.* For a non-Archimedean local field F , we write \mathcal{O}_F , \mathfrak{p}_F , and k_F for the ring of integers, the maximal ideal, and the residue field of F , respectively. For any $r \in \mathbb{Z}_{>0}$, we let $U_F^r := 1 + \mathfrak{p}_F^r$.

2. MAIN RESULTS

2.1. *Swan exponent of Galois representations.* We first recall the definition of the Swan exponent of a local Galois representation. The contents of this subsection are based on [32, Chapter VI] and [31, Chapter 19].

Let F be a non-Archimedean local field with residue characteristic $p > 0$. Let E be a finite Galois extension of F . Let $G := \mathrm{Gal}(E/F)$ and $\{G_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be the filtration by the lower ramification subgroups of G ; recall that $\{G_i\}_{i \geq 0}$ is decreasing and satisfies $G_i = \{1\}$ for sufficiently large i (see [32, Chapter IV]). We write g_i for the cardinality of G_i .

Suppose that ρ is a representation of G on a finite-dimensional \mathbb{C} -vector space V . We simply call such ρ a *Galois representation (of G)*. Then the *Swan exponent* of ρ is defined by

$$\mathrm{Sw}(\rho) := \sum_{i \geq 1} \frac{g_i}{g_0} \dim(V/V^{G_i}).$$

Similarly, the *Artin exponent* of ρ is defined by

$$\mathrm{Art}(\rho) := \sum_{i \geq 0} \frac{g_i}{g_0} \dim(V/V^{G_i}).$$

Let us review several basic properties of $\mathrm{Sw}(\rho)$ which will be useful to us.

2.1.1. *Swan exponent as a scalar product.* The Swan exponent of ρ can be also thought of as the scalar product of the character $\mathrm{Tr}(\rho)$ with the character \mathcal{S} of the Swan representation “ sw_G ” (see [31, Chapter 19]):

$$\mathrm{Sw}(\rho) = \langle \mathrm{Tr}(\rho), \mathcal{S} \rangle.$$

2.1.2. *Swan exponent via upper ramification filtration.* For a Galois representation ρ , we define the number $\text{slope}(\rho)$ by

$$\text{slope}(\rho) := \inf\{s \in \mathbb{R}_{\geq 0} \mid \rho \text{ is trivial on } G^s\},$$

where $\{G^r\}_{r \geq 0}$ is the filtration of G by the upper ramification subgroups (see [32, Section IV.3] for the definition of the upper ramification subgroups). When ρ is irreducible, we have

$$\text{Sw}(\rho) = \dim(\rho) \cdot \text{slope}(\rho).$$

Indeed, it is obvious when ρ is trivial on $G_0 = G^0$, so let us consider the case where ρ is non-trivial on G_0 . As each G_i is a normal subgroup of G , V^{G_i} is a subspace of V stable under the action of G . Thus, by letting $m \in \mathbb{Z}_{\geq 0}$ be the largest integer such that G_m is non-trivial, the irreducibility of ρ implies that

$$V^{G_i} = \begin{cases} 0 & \text{if } 0 \leq i \leq m, \\ V & \text{if } m < i. \end{cases}$$

Hence we have

$$\text{Sw}(\rho) = \dim(V) \sum_{i=1}^m \frac{g_i}{g_0}.$$

The sum on the right-hand side is nothing but the value $\varphi_{E/F}(m)$ of the Herbrand function $\varphi_{E/F}$ with respect to $G = \text{Gal}(E/F)$ at m (see [32, Section IV.3]). By the definition of the upper ramification filtration, we have

$$G^{\varphi_{E/F}(m)} = G_m \quad \text{and} \quad G^{\varphi_{E/F}(m)+\varepsilon} = G_{m+1}$$

for any sufficiently small positive number ε . In other words, we have $\text{slope}(\rho) = \varphi_{E/F}(m) = \text{Sw}(\rho)/\dim(\rho)$.

Note that, by the definition of the slope, we have the following:

LEMMA 2.1. *Let ρ_1 and ρ_2 be irreducible Galois representations of G . Then we have*

$$\text{slope}(\rho_1 \otimes \rho_2) \leq \max\{\text{slope}(\rho_1), \text{slope}(\rho_2)\},$$

where the equality holds if $\text{slope}(\rho_1) \neq \text{slope}(\rho_2)$.

2.1.3. *Behavior under a tamely ramified extension.* Let F' be a tamely ramified extension of F contained in E . Note that this condition is equivalent to that the corresponding subgroup $G' := \text{Gal}(E/F')$ of $G = \text{Gal}(E/F)$ contains G_1 . Let $\{G'_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be the filtration of G' by the lower ramification subgroups of G' . By noting that $G' \supset G_1$ and $G'_i = G' \cap G_i$ for any $i \in \mathbb{Z}_{\geq 0}$ ([32, Chapter IV, Proposition 2]), we have $G'_i = G_i$ for any $i \in \mathbb{Z}_{\geq 1}$. This implies that

$$\text{Sw}(\rho|_{G'}) = \frac{g_0}{g'_0} \cdot \text{Sw}(\rho),$$

where g'_0 denotes the cardinality of G'_0 . In particular, by letting F' be the maximal tamely ramified extension of F in E so that $G' = G_1$, we have

$$\text{Sw}(\rho|_{G_1}) = \frac{g_0}{g_1} \cdot \text{Sw}(\rho) = e_{F'/F} \cdot \text{Sw}(\rho),$$

where $e_{F'/F}$ denotes the ramification index of F'/F .

2.1.4. *Behavior under the induction.* Let F' be an extension of F contained in E . For any finite-dimensional representation σ of the Galois group $\text{Gal}(E/F')$, we have

$$\text{Sw}(\text{Ind}_{F'/F} \sigma) = f_{F'/F} \cdot \text{Sw}(\sigma) + (v_F(\mathfrak{d}_{F'/F}) - [F' : F] + f_{F'/F}) \cdot \dim(\sigma),$$

where $f_{F'/F}$ denotes the residue degree of F'/F and $\mathfrak{d}_{F'/F}$ is the discriminant of F'/F (see [32, Corollary of Proposition 4, Chapter VI], in which the formula is stated in terms of Artin exponents).

In particular, we have the following:

LEMMA 2.2. *Suppose that F'/F is a ramified quadratic extension of dyadic fields. Then we have*

$$\text{Sw}(\text{Ind}_{F'/F} \sigma) = \text{Sw}(\sigma) + \text{Sw}(\omega_{F'/F}) \cdot \dim(\sigma),$$

where $\omega_{F'/F}$ denotes the quadratic character of F^\times corresponding to F'/F .

PROOF. By applying the above formula, we have

$$\text{Sw}(\text{Ind}_{F'/F} \sigma) = \text{Sw}(\sigma) + (v_F(\mathfrak{d}_{F'/F}) - 1) \cdot \dim(\sigma).$$

We have $v_F(\mathfrak{d}_{F'/F}) = \text{Sw}(\omega_{F'/F}) + 1$ by [32, Corollary 2 of Proposition 6, Chapter VI] (note that the symbol $f(-)$ in *loc. cit.* denotes the Artin exponent), hence we get the assertion. \square

2.2. *Difference of Sym^2 and \wedge^2 Swan exponents: the case where $p \neq 2$.* In the rest of this section, we investigate the difference of the Swan exponents of the symmetric and exterior squares of a Galois representation. We first consider the case where the residue characteristic p of F is odd.

Let us start with the following observation. For any class function $f: G \rightarrow \mathbb{C}$, we define $\Psi^2 f: G \rightarrow \mathbb{C}$ by $\Psi^2 f(g) := f(g^2)$. Note that then $\Psi^2 f$ is again a class function on G . When ρ is a finite-dimensional representation of G whose dimension is d , for $g \in G$, we have

$$\text{Tr}(\text{Sym}^2 \rho)(g) - \text{Tr}(\wedge^2 \rho)(g) = \sum_{i=1}^d \alpha_i^2 = \text{Tr}(\rho)(g^2) = \Psi^2 \text{Tr}(\rho)(g),$$

where $\alpha_1, \dots, \alpha_d \in \mathbb{C}^\times$ are the eigenvalues of $\rho(g)$. In other words, $\Psi^2 \text{Tr}(\rho)$ is the character of the virtual representation $\text{Sym}^2 \rho - \wedge^2 \rho$ of G . Let us write $\Psi^2 \rho := \text{Sym}^2 \rho - \wedge^2 \rho$.

PROPOSITION 2.3. *Suppose that $p \neq 2$. For any Galois representation ρ , we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) - \mathrm{Sw}(\wedge^2 \rho) = \mathrm{Sw}(\rho).$$

PROOF. If we replace F with the maximal tamely ramified extension F' of F in E , then each term in the equality in the assertion is multiplied by $e_{F'/F}$ (see Section 2.1.3). Thus, we may assume that $G = G_1$ from the beginning. Note that then G is a finite p -group. If we let p^r be the order of G , then ρ is defined over $\mathbb{Q}(\mu_{p^r})$, where μ_{p^r} denotes the subset of \mathbb{C} consisting of p^r -th roots of unity (see [31, Proposition 33]).

Let $g \in G$ and $\alpha_1, \dots, \alpha_d \in \mathbb{C}^\times$ be the eigenvalues of $\rho(g)$, where $d := \dim(\rho)$. Let $\tau \in \mathrm{Gal}(\mathbb{Q}(\mu_{p^r})/\mathbb{Q}) \cong (\mathbb{Z}/p^r\mathbb{Z})^\times$ be the element corresponding to $2 \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, i.e., $\tau(\alpha) = \alpha^2$ for any $\alpha \in \mu_{p^r}$. As every eigenvalue α_i belongs to μ_{p^r} , the class function $\Psi^2 \mathrm{Tr}(\rho)$ is equal to the character $\mathrm{Tr}(\tau\rho)$ of the τ -twist of ρ . Hence we have $\Psi^2 \rho = \tau\rho$. Since the Galois twist does not change the Swan exponent, we get $\mathrm{Sw}(\mathrm{Sym}^2 \rho) - \mathrm{Sw}(\wedge^2 \rho) = \mathrm{Sw}(\Psi^2 \rho) = \mathrm{Sw}(\tau\rho) = \mathrm{Sw}(\rho)$. \square

By considering the last non-trivial lower ramification subgroup, we can also prove the above theorem in the following way:

ANOTHER PROOF OF PROPOSITION 2.3. As in the above proof, let us assume that $G = G_1$ and ρ is a non-trivial irreducible representation of G . Let G_m be the last non-trivial lower ramification group of G , i.e., $G_{m+1} = \{1\}$. Then G_m is central in G by [32, Chapter IV, Proposition 10]. Hence G_m acts on ρ via a character χ . Then G_m acts on both $\mathrm{Sym}^2 \rho$ and $\wedge^2 \rho$ via χ^2 . As G_m is a p -group, where p is odd, χ^2 is non-trivial. This means that any irreducible constituent of $\mathrm{Sym}^2 \rho$ and $\wedge^2 \rho$ has the same slope as ρ . Thus, by noting that $\dim(\mathrm{Sym}^2 \rho) - \dim(\wedge^2 \rho) = \dim(\rho)$, we get $\mathrm{Sw}(\mathrm{Sym}^2 \rho) - \mathrm{Sw}(\wedge^2 \rho) = \mathrm{Sw}(\rho)$ (see Section 2.1.2). \square

COROLLARY 2.4. *Suppose that $p \neq 2$. Let ρ be a $2n$ -dimensional irreducible self-dual Galois representation such that $(\mathrm{Sw}(\rho), 2n) = 1$. Then we have*

$$\mathrm{Sw}(\mathrm{Sym}^2 \rho) = n \cdot \mathrm{Sw}(\rho) \quad \text{and} \quad \mathrm{Sw}(\wedge^2 \rho) = (n - 1) \cdot \mathrm{Sw}(\rho).$$

PROOF. As ρ is self-dual, we have $\rho \otimes \rho^\vee \cong \rho \otimes \rho \cong \mathrm{Sym}^2 \rho \oplus \wedge^2 \rho$. By the explicit formula of Bushnell–Henniart–Kutzko ([13, 6.5, Theorem], see also [12, 3.5]), the assumption that $(\mathrm{Sw}(\rho), 2n) = 1$ implies that

$$\mathrm{Sw}(\rho \otimes \rho^\vee) = (2n - 1) \mathrm{Sw}(\rho).$$

Thus, by combining this equality with Proposition 2.3, we get the assertion. \square

- REMARK 2.5. 1. It is known that there is no irreducible self-dual Galois representation whose dimension is odd and greater than 1 if the residue characteristic of F is odd (see, e.g., [30, page 84, Proposition 4]). Hence the assumption that the dimension of ρ is even in Corollary 2.4 automatically follows from the irreducibility and self-duality assumptions.
2. Proposition 2.3 is not new; it has been already obtained in the work of Anandavardhanan–Mondal ([2, Theorem 1.2]). Thus the point here is that our proof is different from theirs.
3. Since the explicit formula of Bushnell–Henniart–Kutzko works for general irreducible ρ , it is possible to get an explicit formula of $\text{Sw}(\text{Sym}^2 \rho)$ and $\text{Sw}(\wedge^2 \rho)$ for any irreducible ρ as in the manner of the proof of Corollary 2.4. (We put various assumptions on ρ just because then the formula is simplified.) In fact, this is exactly how Anandavardhanan–Mondal established an explicit formula of $\text{Sw}(\text{Sym}^2 \rho)$ and $\text{Sw}(\wedge^2 \rho)$.

2.3. *Difference of Sym^2 and \wedge^2 Swan exponents: the case where $p = 2$.* We next consider the case where the residue characteristic of F is $p = 2$.

THEOREM 2.6. *Suppose that $p = 2$. For any Galois representation ρ of G , we have*

$$(2.1) \quad 0 \leq \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) \leq 2 \text{Sw}(\rho).$$

PROOF. By replacing G with G_1 , we may assume that $G = G_1$ (the same argument as in the proof of the case where $p \neq 2$). We first note that the quantity $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho)$ is given by applying the operation Ψ^2 to $\text{Tr}(\rho)$ and then taking the scalar product with \mathcal{S} (see Section 2.1.1). In particular, $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho)$ is additive in ρ . Thus it is enough to prove the inequalities when ρ is irreducible. Since the assertion is obvious when ρ is trivial, we may also assume that ρ is non-trivial.

We put $d := \dim(\rho)$. We note that $\dim(\text{Sym}^2 \rho) - \dim(\wedge^2 \rho) = \dim(\rho)$. By the definition of the Swan exponent, it suffices to show that, for any $i > 0$,

$$0 \leq d - \dim((\text{Sym}^2 \rho)^{G_i}) + \dim((\wedge^2 \rho)^{G_i}) \leq 2d - 2 \dim(\rho^{G_i}),$$

or equivalently

$$(2.2) \quad -d \leq \dim((\wedge^2 \rho)^{G_i}) - \dim((\text{Sym}^2 \rho)^{G_i}) \leq d - 2 \dim(\rho^{G_i}).$$

By noting that this is additive in ρ and depends only on G_i , we may assume that $G = G_i$ and ρ is irreducible non-trivial by the same argument as above. Then we have $\rho^G = 0$ and

$$\dim((\text{Sym}^2 \rho)^G) + \dim((\wedge^2 \rho)^G) = \dim((\rho \otimes \rho)^G) = \begin{cases} 1 & \text{if } \rho \text{ is self-dual,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence only $0, \pm 1$ are the possibilities of the value of

$$\dim((\wedge^2 \rho)^G) - \dim((\text{Sym}^2 \rho)^G),$$

which gives the inequalities (2.2). \square

Let us now discuss the cases where the equalities of (2.1) are attained. Note that all terms in (2.1) are 0 when ρ is tame, i.e., trivial on G_1 .

PROPOSITION 2.7. *Suppose that $p = 2$. Let ρ be a Galois representation of G which is not tame. Then we have*

$$0 \leq \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) < 2\text{Sw}(\rho).$$

The equality $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = 0$ holds if and only if the restriction of ρ on G_1 decomposes into the sum of quadratic characters.

PROOF. By the same argument as in the proof of Theorem 2.6, it suffices to show the inequalities

$$(2.3) \quad -d \leq \dim((\wedge^2 \rho)^G) - \dim((\text{Sym}^2 \rho)^G) < d$$

and discuss the condition so that the left equality holds by assuming $G = G_1$ and ρ is a non-trivial irreducible representation of G whose dimension is d . The proof of Theorem 2.6 shows that the inequalities of (2.3) always hold and the equality on the left never holds when $d > 1$. When $d = 1$, i.e., ρ is a (non-trivial) character, we have $\dim((\wedge^2 \rho)^G) = 0$ and

$$\dim((\text{Sym}^2 \rho)^G) = \begin{cases} 0 & \text{if } \rho^2 \text{ is non-trivial,} \\ 1 & \text{if } \rho^2 \text{ is trivial.} \end{cases}$$

This completes the proof. \square

As an application of the above proposition, we have the following.

PROPOSITION 2.8. *Suppose that $p = 2$. Let ρ be a $(2n + 1)$ -dimensional irreducible self-dual Galois representation such that $(\text{Sw}(\rho), 2n + 1) = 1$. Then we have*

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = 0.$$

PROOF. Since p is prime to $\dim(\rho) = 2n + 1$, the classification result of self-dual irreducible Galois representations due to Bushnell–Henniart implies that ρ is of the form $\text{Ind}_{L/F}(\chi)$, where L/F is a tamely ramified extension of degree $2n + 1$ and χ is a quadratic character of L^\times ([10, Section 3]). In other words, there exist a subgroup H of G containing G_1 and a quadratic character χ of H such that $\rho \cong \text{Ind}_H^G(\chi)$. Thus we have $\rho|_{G_1} \cong \bigoplus_{g \in G/H} {}^g \chi|_{G_1}$. Hence the assumption in the last sentence of Proposition 2.7 is satisfied and we get the assertion. \square

REMARK 2.9. Note that the assumption that $\dim(\rho)$ is odd is not automatic when $p = 2$ and indispensable in the above proof so that the result of Bushnell–Henniart [10] is available.

By the same argument as in the case where $p \neq 2$ (i.e., using the Bushnell–Henniart–Kutzko formula), we obtain the following:

COROLLARY 2.10. *With the same assumptions as in Proposition 2.8, we have*

$$\text{Sw}(\text{Sym}^2 \rho) = \text{Sw}(\wedge^2 \rho) = n \cdot \text{Sw}(\rho).$$

3. A CONJECTURE AND SOME EVIDENCE

Let us keep the notation as in the previous section.

CONJECTURE 3.1. *Assume $p = 2$. For any Galois representation ρ , we have*

$$(3.1) \quad \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) \leq \text{Sw}(\rho).$$

We remark that, by the same argument as in the proof of Theorem 2.6, the above conjecture is equivalent to the following:

CONJECTURE 3.2. *Assume $p = 2$ and $G = G_1$. For any irreducible faithful Galois representation ρ , we have*

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) \leq \text{Sw}(\rho).$$

In the following, let us suppose that $p = 2$ and $G = G_1$. In particular, $\dim(\rho)$ is a power of 2. Conjecture 3.2 is obvious when $\dim(\rho) = 1$, so let us investigate the case where $\dim(\rho) > 1$ in the following. Note that then G is necessarily non-abelian since any irreducible representation of an abelian group is 1-dimensional.

REMARK 3.3. Note that the proof of Theorem 2.6 used only a descending filtration by subgroups, but no particular property of that filtration. The stronger conjecture (Conjecture 3.1), if true, necessarily uses strong properties of the filtration by ramification subgroups, as can be seen in the examples in this section.

3.1. *The 2-dimensional case.* We first investigate the case where $\dim(\rho) = 2$. Since any irreducible representation of a p -group is monomial, i.e., induced from a 1-dimensional representation of a subgroup ([31, Theorems 14 and 16]), we may write $\rho \cong \text{Ind}_{G'}^G(\chi)$, where $G' = \text{Gal}(E/F')$ for a quadratic separable (ramified) extension F' of F and χ is a character of G' . We also see χ as a character of F'^{\times} by class field theory. Let $\omega := \omega_{F'/F}$ be the quadratic character of F^{\times} defining F'/F . Let $\chi^{\circ} := \chi|_{F^{\times}}$. We put $a := \text{Sw}(\chi)$, $b := \text{Sw}(\chi^2)$, $c := \text{Sw}(\chi^{\circ})$, $d := \text{Sw}(\omega\chi^{\circ})$, and $s := \text{Sw}(\omega)$.

LEMMA 3.4. *We have the following relations:*

1. $d \leq \max\{c, s\}$, where equality holds when $c \neq s$,
2. $a \geq 2c$,
3. $0 < s \leq 2e$, where $e := \text{val}_F(2)$,
4. $a \geq s$.

PROOF. The inequality (1) is obvious.

Since F'/F is a ramified quadratic extension, we get the inequality (2).

As F'/F is wildly ramified, we have $s > 0$. On the other hand, as used in the proof of Lemma 2.2, we have $v_F(\mathfrak{d}_{F'/F}) = s + 1$. By the upper bound formula for $v_F(\mathfrak{d}_{F'/F})$ (see [32, Chapter III, Section 6, Remarks (1)]), we get $v_F(\mathfrak{d}_{F'/F}) \leq 1 + 2e$. Thus we obtain $s \leq 2e$, hence the inequalities (3).

Since $\rho \cong \text{Ind}_{G'}^G(\chi)$ is irreducible, we necessarily have $\chi^\sigma \neq \chi$, where σ is the non-trivial element of $\text{Gal}(F'/F)$. If $a < s$, then σ acts trivially on $F'^{\times}/1 + \mathfrak{p}_{F'}^{a+1}$, which implies that $\chi^\sigma \cdot \chi^{-1}$ is trivial, hence a contradiction. Thus we get the inequality (4). \square

LEMMA 3.5. *We have $b \leq \max\{a/2, a - 2e\} (< a)$. More precisely, we have*

- $b = a - 2e$ when $a > 4e$,
- $b = a/2$ when $a < 4e$ and a is even,
- $b \leq a/2$ otherwise.

PROOF. The first assertion simply follows from that $(1 + \mathfrak{p}_{F'}^b)^2 \subset 1 + \mathfrak{p}_{F'}^{\min\{b+2e, 2b\}}$.

To show the latter assertion, let us examine the behavior of the squaring map on $\mathcal{O}_{F'}^\times$. We have $(1+x)^2 = 1+2x+x^2$ for any $x \in F'$. If $v_{F'}(x) = i \geq 1$, then $v_{F'}(x^2) = 2i$ and $v_{F'}(2x) = i + 2e$ (recall that $e = v_F(2)$). It follows that $v_{F'}(2x+x^2) \geq \min\{2i, i+2e\}$ with equality when $2i \neq i+2e$. Recall that, by definition, $a \in \mathbb{Z}_{>0}$ is the integer such that χ is trivial on $1 + \mathfrak{p}_{F'}^{a+1}$, but not on $1 + \mathfrak{p}_{F'}^a$. (Note that we have $a > 0$ by Lemma 3.4 (3) and (4).)

Let us first assume $a > 4e$ and take $i = a - 2e$. Then we have $2i > i + 2e$, hence $\min\{2i, i + 2e\} = a$. Moreover, we have $(1+x)^2 = 1+2x \pmod{\mathfrak{p}_{F'}^{a+1}}$. Consequently, there exists an element $x \in F'$ with valuation i such that $\chi(1+x)^2 \neq 1$, thus $b(:= \text{Sw}(\chi^2))$ is at least $a - 2e$. On the other hand, if we take $i = a - 2e + 1$, then we have $\min\{2i, i + 2e\} \geq a + 1$. Hence we have $\chi(1+x)^2 = 1$ for any element $x \in F'$ with valuation i , which implies that $b = a - 2e$.

Let us then assume $a < 4e$ and a is even. We take $i = a/2$. Then we have $2i < i + 2e$ and $\min\{2i, i + 2e\} = 2i = a$. Hence we see that $(1+x)^2 = 1+x^2 \pmod{\mathfrak{p}_{F'}^{a+1}}$. By noting that the squaring map is bijective on $k_{F'}^\times$, we can choose $x \in F'$ with valuation i satisfying $\chi(1+x)^2 \neq 1$. Similarly with $i = a/2 + 1$, we see that $\chi(1+x)^2 = 1$ for any element $x \in F'$ with valuation i . Thus we get $b = a/2$.

When $a < 4e$ and a is odd or $a = 4e$, we can only assert that $b \leq a/2$. \square

By Lemma 2.2, we have $\text{Sw}(\rho) = a + s$. Note that we have

$$\wedge^2(\text{Ind}_{F'/F}(\chi)) \cong \omega\chi^\circ \quad \text{and} \quad \text{Sym}^2(\text{Ind}_{F'/F}(\chi)) \cong \text{Ind}_{F'/F}(\chi^2) \oplus \chi^\circ$$

(see Proposition A.1). Hence we get

$$\begin{aligned} \text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) &= \text{Sw}(\text{Ind}_{F'/F}(\chi^2)) + \text{Sw}(\chi^\circ) - \text{Sw}(\omega\chi^\circ) \\ &= b + s + c - d, \end{aligned}$$

where we again used Lemma 2.2 in the second equality. We investigate the conjectural inequality (3.1) by a case-by-case argument:

1. If $c > s$, then $d = c$ by Lemma 3.4 (1). Thus, by using Lemma 3.5, we have

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = b + s + c - c = b + s < a + s = \text{Sw}(\rho).$$

Note that this is the strict inequality.

2. If $c < s$, then $d = s$ by Lemma 3.4 (1). Thus, by using Lemma 3.5, we have

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = b + s + c - s = b + c < a + s = \text{Sw}(\rho).$$

Note that this is the strict inequality.

3. If $c = s$, we have $d \leq c$ by Lemma 3.4 (1). Let us show that the weak inequality

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = b + s + c - d \leq a + s = \text{Sw}(\rho),$$

or equivalently, $a - b - c + d \geq 0$ holds in this case.

- (a) Suppose that $a > 4e$, hence $b = a - 2e$ by Lemma 3.5. Since we have $s \leq 2e$ by Lemma 3.4 (3), we get $b + c = a - 2e + s \leq a$, which implies that $a - b - c + d \geq 0$. Note that we have the strict inequality unless $d = 0$ and $s = 2e$.
- (b) Suppose that $a < 4e$ and a is even, hence $b = a/2$ by Lemma 3.5. Since we have $a \geq 2c$ by Lemma 3.4 (2), we get $b + c \leq a$, which implies that $a - b - c + d \geq 0$. Note that we have the strict inequality unless $d = 0$ and $a = 2c$.
- (c) Suppose that $a \leq 4e$, hence $b \leq a/2$ by Lemma 3.5. Thus the same discussion as in the previous case works. Note that we have the strict inequality unless $d = 0$ and $a = 2b = 2c$.

Note that $d = 0$ means that ρ has trivial determinant (recall that we are assuming $G = G_1$), i.e. is symplectic.

Thus we obtain the following:

THEOREM 3.6. *Conjecture 3.2 is true if $\dim(\rho) = 2$. Moreover, the inequality is strict unless ρ is symplectic.*

REMARK 3.7. Let us present an example where the equality $\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = \text{Sw}(\rho)$ holds. Take F'/F to have $s = 1$, and take χ such that $a = 2$ and $\chi^\circ = \omega$; then we have $c = 1, b = 1, d = 0$.

3.2. *The case where ρ is induced along a cyclic extension.*

PROPOSITION 3.8. *Suppose that ρ is an irreducible Galois representation of the form $\rho = \text{Ind}_{F'/F}(\chi)$, where F'/F is cyclic and χ is a character of F'^{\times} . Then the inequality (3.1) holds.*

PROOF. Let $\rho = \text{Ind}_{F'/F}(\chi)$, where F'/F is cyclic and χ is a character of F'^{\times} . In other words, F' is a subfield of E corresponding to a normal subgroup G' of G such that $G/G' \cong \text{Gal}(F'/F)$ is cyclic, say order $2n$. Let c be a generator of G/G' . Let H be a subgroup of G which is the preimage of $\langle c^n \rangle \subset G/G'$. Let L be the subfield of F' corresponding to H .

By the Frobenius formula for induced representations, we have

$$\text{Tr}(\Psi^2 \rho)(g) = \text{Tr}(\rho)(g^2) = \begin{cases} \sum_{\tau \in G/G'} \chi^{\tau}(g^2) & \text{if } g^2 \in G', \\ 0 & \text{otherwise.} \end{cases}$$

Note that, as G/G' is cyclic, we have $g^2 \in G'$ if and only if $g \in H$. Similarly, we have

$$\text{Tr}(\text{Ind}_{F'/F}(\chi^2))(g) = \begin{cases} \sum_{\tau \in G/G'} (\chi^2)^{\tau}(g) & \text{if } g \in G', \\ 0 & \text{otherwise.} \end{cases}$$

We let χ° denote the character of H corresponding to $\chi|_{L^{\times}}$ under local class field theory. By letting ω be the non-trivial character of H/G' , we have

$$\text{Tr}(\text{Ind}_{L/F}(\chi^{\circ}) - \text{Ind}_{L/F}(\chi^{\circ}\omega))(g) = \begin{cases} 0 & \text{if } g \in G', \\ 2 \sum_{\tau \in G/H} (\chi^{\circ})^{\tau}(g) & \text{if } g \in H \setminus G', \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by noting that $\chi(g^2) = \chi^{\circ}(g)$ for any $g \in H \setminus G'$ (see [9, 29.1, Transfer theorem]), we get

$$\text{Tr}(\Psi^2 \rho) = \text{Tr}(\text{Ind}_{F'/F}(\chi^2) + \text{Ind}_{L/F}(\chi^{\circ}) - \text{Ind}_{L/F}(\chi^{\circ}\omega)).$$

Hence

$$\begin{aligned} \text{Sym}^2 \rho - \wedge^2 \rho &\cong \text{Ind}_{F'/F}(\chi^2) + \text{Ind}_{L/F}(\chi^{\circ}) - \text{Ind}_{L/F}(\chi^{\circ}\omega) \\ &\cong \text{Ind}_{L/F}(\text{Ind}_{F'/L}(\chi^2) + \chi^{\circ} - \chi^{\circ}\omega). \end{aligned}$$

On the other hand, we also have $\rho \cong \text{Ind}_{L/F}(\text{Ind}_{F'/L}(\chi))$. Hence, by Lemma 2.2, it suffices to show the inequality

$$(3.2) \quad \text{Sw}(\text{Ind}_{F'/L}(\chi^2)) + \text{Sw}(\chi^{\circ}) - \text{Sw}(\chi^{\circ}\omega) \leq \text{Sw}(\text{Ind}_{F'/L}(\chi))$$

(note that $\dim(\text{Sym}^2 \rho) - \dim(\wedge^2 \rho) = \dim(\rho)$).

As observed in the proof of Theorem 3.6, this inequality (3.2) is nothing but the inequality (3.1) for $\text{Ind}_{F'/L}(\chi)$, which we have already proved. \square

3.3. *Approach based on the structure of ramification subgroups.* We next investigate the conjectural inequality by looking at the last two steps of the ramification filtration. Let us assume that $G = G_1 \neq \{1\}$ and ρ is an irreducible faithful representation of G . We let $n = \dim(\rho)$, a power of 2. Since the cases where $n = 1, 2$ are already treated, let us assume that $n > 2$ (most considerations also work when $n = 2$). Note that the center $Z(G)$ of G is cyclic by Lemma A.4.

Let $Z \subset G$ be the last non-trivial lower ramification subgroup. Since $G = G_1$, Z is contained in the center $Z(G)$ of G (this follows from [32, Chapter IV, Proposition 10]). By noting that Z has an \mathbb{F}_2 -vector space structure ([32, Chapter IV, Section 2, Corollary 3]) and that $Z(G)$ is cyclic, the order of Z is 2 and $Z \neq G$ since $n > 1$.

Let us look at the smallest ramification subgroup H containing Z properly. The quotient $V = H/Z$ is a non-trivial \mathbb{F}_2 -vector space, equipped with a quadratic form $q: V \rightarrow Z \cong \mathbb{F}_2; v \mapsto v^2$ and an alternating bilinear form $b: V \times V \rightarrow Z \cong \mathbb{F}_2; (v_1, v_2) \mapsto v_1 v_2 v_1^{-1} v_2^{-1}$. (See [32, Chapter IV, Proposition 10].) The center Y of H is the inverse image of

$$V^0 := \{v_0 \in V \mid b(v_0, v) = 0 \text{ for any } v \in V\}$$

in H . Note that $q|_{V^0}: V^0 \rightarrow Z$ is a group homomorphism. We have either that $q|_{V^0}$ is trivial (then Y has exponent 2) or that $q|_{V^0}$ is non-trivial, hence $\text{Ker}(q|_{V^0})$ is a hyperplane in V^0 (then Y has elements of order 4).

The group G acts by conjugation on H and Y . As we have $G = G_1$, G acts trivially on $V = H/Z$ (again by [32, Chapter IV, Proposition 10]); this action preserves b and q . Similarly, the action of G on $V^0 = Y/Z$ is trivial, hence any $g \in G$ defines a homomorphism $\phi_g: V^0 \rightarrow Z$ given by $v \mapsto g v g^{-1} v^{-1}$. Since ϕ_g is multiplicative on g , we get a homomorphism

$$\phi: G \rightarrow \text{Hom}_{\mathbb{F}_2}(V^0, Z); \quad g \mapsto \phi_g,$$

which is trivial on H . We define a subspace W of V^0 by $W := \bigcap_{g \in G} \text{Ker } \phi_g$. If we let $I \subset Y$ be the inverse image of W in Y , then we have $I = H \cap Z(G)$. As $Z(G)$ is cyclic, we see that I is either Z or cyclic of order 4.

$$\begin{array}{ccccc} H & \longrightarrow & V := H/Z & \xrightarrow{q} & Z \\ \cup & & \cup & \nearrow & \\ Y & \longrightarrow & V^0 & \xrightarrow{q|_{V^0}} & \\ \cup & & \cup & & \\ I & \longrightarrow & W & & \\ \cup & & \cup & & \\ Z & \longrightarrow & 0 & & \end{array}$$

3.3.1. Let us first consider the case where I is cyclic of order 4.

PROPOSITION 3.9. *When I is of order 4, the strict inequality holds in Conjecture 3.2.*

PROOF. We write $Z = G^\alpha$ and $H = G^{\alpha'}$ via the upper numbering. As ρ is irreducible and faithful, we have $\text{slope}(\rho) = \alpha$, hence $\text{Sw}(\rho) = n\alpha$ (see Section 2.1.2). Since I acts on ρ as a faithful 1-dimensional character, any generator of I acts by -1 on $\rho \otimes \rho$. This implies that any irreducible constituent of $\rho \otimes \rho \cong \text{Sym}^2 \rho \oplus \wedge^2 \rho$ has slope α' . Hence we have $\text{Sw}(\text{Sym}^2 \rho) = n(n+1)\alpha'/2$ and $\text{Sw}(\wedge^2 \rho) = n(n-1)\alpha'/2$. Thus we get

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = n\alpha' < n\alpha = \text{Sw}(\rho),$$

which is the strict inequality. □

3.3.2. We now assume that $I = Z$, which means that $W = 0$. Note that $\text{Im}(\phi)$ is a subspace of the dual $(V^0)^\vee$ of V^0 , and its orthogonal in V^0 is W by the definition of W . Hence we have $\text{Im}(\phi) = \text{Hom}_{\mathbb{F}_2}(V^0, Z)$. Now an irreducible representation of H non-trivial on Z is determined by its central character χ , which can be any character on Y which is non-trivial on Z (see Section A.3). The action of g in G on χ is obtained by multiplying χ by ϕ_g . An irreducible component of the restriction of ρ to H is thus labelled by such a character χ , and G acts transitively on (the classes of) such irreducible components.

PROPOSITION 3.10. *When $I = Z$ and ρ restricts irreducibly to H , the inequality holds. Moreover, the inequality is strict unless ρ is symplectic.*

In the rest of this subsection, we prove this proposition. Since $\rho|_H$ is irreducible, the center Y of H is equal to $I = Z$ by the same reasoning as in the beginning of this subsection. Hence we have $V^0 = 0$. In particular V is a symplectic vector space (meaning that the alternating form on it is non-degenerate). That representation is self-dual because the central character is the order 2 character of Z .

We write $Z = G^\alpha$ and $H = G^{\alpha'}$ as above (thus $\alpha' < \alpha$). Then we have $\text{Sw}(\rho) = n\alpha$. Since $\rho|_H$ is irreducible and self-dual, $\rho \otimes \rho|_H$ contains the trivial representation exactly once. This implies that exactly one irreducible constituent of $\rho \otimes \rho$ has slope smaller than α' . Moreover, such a constituent must be a 1-dimensional character.

1. If ρ is self-dual, then $\rho \otimes \rho$ contains the trivial representation with multiplicity one.

- (a) If ρ is orthogonal (i.e., the trivial representation appears in $\text{Sym}^2 \rho$), we have $\text{Sw}(\text{Sym}^2 \rho) = \alpha'(n(n+1)/2 - 1)$ and $\text{Sw}(\wedge^2 \rho) = \alpha'n(n-1)/2$, hence

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = (n-1)\alpha' < n\alpha = \text{Sw}(\rho).$$

- (b) If ρ is symplectic (i.e., the trivial representation appears in $\wedge^2 \rho$), we have $\text{Sw}(\text{Sym}^2 \rho) = \alpha'n(n+1)/2$ and $\text{Sw}(\wedge^2 \rho) = \alpha'(n(n-1)/2 - 1)$, hence

$1)/2 - 1)$, hence

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = (n + 1)\alpha'.$$

2. If ρ is not self-dual, then the above observation on the slopes of the irreducible constituents of $\rho \otimes \rho$ implies that $\text{Sw}(\text{Sym}^2 \rho) = \alpha'(n(n + 1)/2 - 1) + \gamma$ and $\text{Sw}(\wedge^2 \rho) = \alpha'(n(n - 1)/2 - 1) + \delta$ with $0 \leq \gamma \leq \alpha'$ and $0 \leq \delta \leq \alpha'$. Moreover, as $G = G_1$, δ cannot be 0 (if $\delta = 0$, then it means that $\wedge^2 \rho$ contains the trivial character of G , hence ρ is self-dual). Thus we get

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = n\alpha' + \gamma - \delta < (n + 1)\alpha'.$$

LEMMA 3.11. *We have $Z(G)H = G$.*

PROOF. For $g \in G$ and $v \in V = H/Z$ with a lift $h \in H$, the commutator $ghg^{-1}h^{-1}$ does not depend on the choice of the lift h and belongs to Z since $G = G_1$ acts trivially on V . Then $\Phi_g: V \rightarrow Z; v \mapsto ghg^{-1}h^{-1}$ is a linear homomorphism because we have, for $h, k \in H$, $ghkg^{-1}k^{-1}h^{-1} = (ghg^{-1}h^{-1})h(gkg^{-1}k^{-1})h^{-1}$ and h commutes with $gkg^{-1}k^{-1}$ which is in Z . Moreover, for $g, j \in G$, we have $gjhj^{-1}g^{-1}h^{-1} = g(jhj^{-1}h^{-1})g^{-1}(ghg^{-1}h^{-1})$, and g commutes with $jhj^{-1}h^{-1}$, hence $gjhj^{-1}g^{-1}h^{-1} = (jhj^{-1}h^{-1})(ghg^{-1}h^{-1})$. In other words, we have a group homomorphism

$$\Phi: G \rightarrow \text{Hom}_{\mathbb{F}_2}(V, Z); \quad g \mapsto \Phi_g.$$

Note that since V is symplectic, the restriction of Φ to H is surjective. Let J be the kernel of Φ . The intersection of H and J is just Z , and $G = JH$. Moreover, J is the commutant of H in G . Thus, for any $j \in J$, $\rho(j)$ gives an H -automorphism of $\rho|_H$. Since $\rho|_H$ is irreducible, this implies that $\rho(j)$ is a scalar multiplication by Schur's lemma. Hence we get a homomorphism $\rho|_J: J \rightarrow \mathbb{C}^\times$. Noting that ρ is faithful, we see that J is abelian. Therefore, J is central in G , which implies that $J = Z(G)$. \square

LEMMA 3.12. *We have $n\alpha \geq (n + 1)\alpha'$*

PROOF. The same proof as that of [19, Théorème 1.8] works in the present setting. For the sake of completeness, let us reproduce it here.

By Proposition A.5, we have $\rho|_H \cong \text{Ind}_X^H(\chi)$, where X is a subgroup of H such that X contains Z and X/Z is a maximal totally isotropic subspace of H/Z and χ is a character of X . Let us put $\tilde{X} := Z(G)X$ and $\tilde{Y} := Z(G)Y$. Then, since we have $G = Z(G)H$ by Lemma 3.11, there exists an extension of $\tilde{\chi}$ of χ to \tilde{X} satisfying $\rho \cong \text{Ind}_{\tilde{X}}^G(\tilde{\chi})$. By the induction formula for Swan exponents (Section 2.1.4; note that $[G: \tilde{X}] = n$), we have

$$(3.3) \quad \text{Sw}(\rho) = \text{Sw}(\tilde{\chi}) + v_F(\mathfrak{d}_{E\tilde{X}/F}) - n + 1,$$

where $E^{G'}$ denotes the fixed field of G' in E for any subgroup G' of G .

$$\begin{array}{ccc} G & \supset & H = G^{\alpha'} \\ \cup & & \cup \\ \tilde{X} & \supset & X \\ \cup & & \cup \\ \tilde{Y} & \supset & Y \\ \cup & & \cup \\ Z(G) & \supset & Z = G^\alpha \end{array}$$

To compute the right-hand side of (3.3), we introduce a few notations. For a finite Galois extension L/K of non-Archimedean local fields, we put

$$\alpha(L/K) := \inf\{u \mid \text{Gal}(L/K)^u = \{1\}\},$$

$$\beta(L/K) := \inf\{v \mid \text{Gal}(L/K)_v = \{1\}\}.$$

Note that we have

$$\varphi_{L/K}(\beta(L/K)) = \alpha(L/K) \quad \text{and} \quad \psi_{L/K}(\alpha(L/K)) = \beta(L/K),$$

where $\varphi_{L/K}$ and $\psi_{L/K}$ are the Herbrand functions. We remark that, in the case where L/K is ramified, we have $\alpha(L/K) = \sup\{u \mid \text{Gal}(L/K)^u \neq \{1\}\}$ and $\beta(L/K) = \sup\{v \mid \text{Gal}(L/K)_v \neq \{1\}\}$ and these numbers are non-negative.

We fix a non-trivial element $s \in G/\tilde{X} \cong \text{Gal}(E^{\tilde{X}}/F)$ and let $M \subset G$ be the preimage of $\langle s \rangle \subset G/\tilde{X}$ in G and $N \subset \tilde{X}$ be the kernel of the character $\tilde{\chi}^{s-1}$. Thus we have the chain

$$G \supset M \supset \tilde{X} \supset N \supset \tilde{Y}$$

whose indices are $[G : M] = n/2$, $[M : \tilde{X}] = 2$, $[\tilde{X} : N] = 2$, and $[N : \tilde{Y}] = n/2$. Then, by a result of Buhler ([6, Proposition 3, page 31]), we have

$$\text{Sw}(\tilde{\chi}) \geq \alpha(E^N/E^{\tilde{X}}) + \beta(E^{\tilde{X}}/E^M).$$

On the other hand, by [19, Lemme 4.4], we have

$$v_F(\mathfrak{d}_{E^{\tilde{X}}/F}) = n(\alpha(E^{\tilde{X}}/F) + 1) - \beta(E^{\tilde{X}}/F) - 1.$$

Hence, by (3.3), we have

$$(3.4) \quad \text{Sw}(\rho) \geq \alpha(E^N/E^{\tilde{X}}) + \beta(E^{\tilde{X}}/E^M) + n\alpha(E^{\tilde{X}}/F) - \beta(E^{\tilde{X}}/F).$$

As the numbering of the lower ramification filtration is consistent with that for any subgroups ([32, Chapter IV, Proposition 2]), we have

$$\beta(E^{\tilde{X}}/E^M) = \beta(E^{\tilde{X}}/F)$$

(note that there is no jump between H and Z by definition). Also, we have

$$\begin{aligned} \alpha(E^N/E^{\tilde{X}}) &= \varphi_{E^N/E^{\tilde{X}}}(\beta(E^N/E^{\tilde{X}})) \\ &= \varphi_{E^N/E^{\tilde{X}}}(\beta(E^N/F)) \\ &= \varphi_{E^N/E^{\tilde{X}}} \circ \psi_{E^N/F}(\alpha(E^N/F)) \\ &= \varphi_{E^N/E^{\tilde{X}}} \circ \psi_{E^N/E^{\tilde{X}}} \circ \psi_{E^{\tilde{X}}/F}(\alpha(E^N/F)) = \psi_{E^{\tilde{X}}/F}(\alpha(E^N/F)). \end{aligned}$$

By noting that the slope of $\psi_{E^{\tilde{X}}/F}$ is greater than or equal to 1 and that $\alpha(E^N/F) = \alpha'$ (the upper numbering is consistent with taking a quotient of the Galois group; [32, Chapter IV, Proposition 14]), we get $\alpha(E^N/E^{\tilde{X}}) \geq \alpha'$. Thus, finally noting that $\alpha(E^{\tilde{X}}/F) = \alpha'$, the above inequality (3.4) implies that

$$\text{Sw}(\rho) \geq (n + 1)\alpha'.$$

Since we have $\text{Sw}(\rho) = n\alpha$ (see Section 2.1.2), this is nothing but the claimed inequality. \square

3.4. *Approach via induction on the order of G .* We next investigate the conjectural inequality from the viewpoint of the induction on the order of G .

PROPOSITION 3.13. *Suppose that Conjecture 3.1 is true. Then the inequality (3.1) is strict when ρ is irreducible orthogonal and non-trivial on G_1 .*

PROOF. We prove the statement by induction on the order of G . Let ρ be an irreducible orthogonal representation of G on V and assume that $G = G_1$ as usual. We write $g := |G|$ and $g_i := |G_i|$. Let $m \in \mathbb{Z}_{>0}$ be the integer such that $G_m \supsetneq G_{m+1} = \{1\}$. By the definition of the Swan exponents, what we want to prove (3.1) is that

$$\sum_{i \geq 1} \frac{g_i}{g} (\dim(\text{Sym}^2 V / (\text{Sym}^2 V)^{G_i}) - \dim(\wedge^2 V / (\wedge^2 V)^{G_i})) < \sum_{i \geq 1} \frac{g_i}{g} \dim(V / V^{G_i}).$$

Since ρ is irreducible, hence $V^{G_i} = 0$ for any $i \leq m$, the above inequality is equivalent to

$$(3.5) \quad \sum_{i=1}^m \frac{g_i}{g} (\dim((\text{Sym}^2 V)^{G_i}) - \dim((\wedge^2 V)^{G_i})) > 0.$$

Now let $a \in \mathbb{Z}_{>0}$ be the integer such that $G_1 = \dots = G_a \supsetneq G_{a+1}$. Suppose that H is a subgroup of G containing G_{a+1} . Let $h := |H|$ and $h_i := |H_i|$. As the numbering of the lower ramification subgroups of G is consistent with any subgroup of G ([32, Chapter IV, Proposition 2]), we have

$$H = H_1 = \dots = H_a \supsetneq H_{a+1} (= G_{a+1}) \supsetneq \dots \supsetneq H_m (= G_m) \supsetneq H_{m+1} = \{1\}.$$

Let us take H so that $[G : H] = 2$. In this case, there are three possibilities:

1. $\rho|_H$ is irreducible, hence orthogonal;
2. $\rho|_H$ has two irreducible constituents, both of which are orthogonal;

3. $\rho|_H$ has two irreducible constituents, both of which are not self-dual.

We first consider the cases (1) and (2). We apply the induction hypothesis to $\rho|_H$ in the case (1) and to each irreducible constituent of $\rho|_H$ and then summing up them in the case (2). Then, by the same reasoning as in the first paragraph, we obtain

$$(3.6) \quad \sum_{i=1}^m \frac{h_i}{h} (\dim((\text{Sym}^2 V)^{H_i}) - \dim((\wedge^2 V)^{H_i})) > 0$$

After multiplied by h/g , the summands of the left-hand side of (3.6) for $i > a$ are identical to those of (3.5). Thus, by noting that $G = G_1 = \dots = G_a$ and $H = H_1 = \dots = H_a$, it is enough to show that

$$(3.7) \quad \dim((\text{Sym}^2 V)^G) - \dim((\wedge^2 V)^G) \geq \frac{h}{g} (\dim((\text{Sym}^2 V)^H) - \dim((\wedge^2 V)^H)).$$

As ρ is irreducible orthogonal, we have $\dim((\text{Sym}^2 V)^G) = 1$ and $\dim((\wedge^2 V)^G) = 0$. In the case (1), as $\rho|_H$ is irreducible orthogonal, we have $\dim((\text{Sym}^2 V)^H) = 1$ and $\dim((\wedge^2 V)^H) = 0$. In the case (2), as $\rho|_H$ has two irreducible orthogonal constituents, we have $\dim((\text{Sym}^2 V)^H) = 2$ and $\dim((\wedge^2 V)^H) = 0$. Since $h/g = 1/2$, we get the assertion in both cases.

We next consider the case (3). In this case, by applying Conjecture 3.1 to each irreducible constituent of $\rho|_H$ and then summing up them, we obtain

$$\sum_{i=1}^m \frac{h_i}{h} (\dim((\text{Sym}^2 V)^{H_i}) - \dim((\wedge^2 V)^{H_i})) \geq 0$$

by the same reasoning as in the first paragraph. Thus, by the same discussion as in the cases (1) and (2), it suffices to show that the inequality (3.7) holds without equality. This follows from that $\dim((\text{Sym}^2 V)^H) = 0$ and $\dim((\wedge^2 V)^H) = 0$ since two irreducible constituents of $\rho|_H$ are not self-dual. \square

Thus it is also reasonable to expect the following.

CONJECTURE 3.14. *Assume $p = 2$. For any irreducible orthogonal Galois representation ρ of G which is not tame, we have*

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) < \text{Sw}(\rho).$$

The point here is that it could be possible to approach Conjecture 3.1 by induction on the order of G as performed in the proof of Proposition 3.13. At least, by a similar consideration to the above proof, we can conclude the following:

1. If ρ is orthogonal, then the inequality for H implies the inequality for G , and even a strict one unless ρ is induced from an orthogonal representation of H .

2. If ρ is not self-dual, then the inequality for H implies the inequality for G if $\rho|_H$ is not orthogonal.
3. If ρ is symplectic, then the inequality for H implies the inequality for G only if ρ is induced from a symplectic representation of H .

From this observation, we see that the obstacle for making the induction argument (on the order of G) work is the case where ρ is symplectic.

3.5. *Approach via induction on the dimension of ρ .* Let us finish this section by giving some comments on the approach via the induction on the dimension of ρ .

Suppose that ρ is irreducible, $G = G_1$, and $d := \dim(\rho) > 2$. Again by the fact that any irreducible representation of a p -group is monomial, we may write $\rho = \text{Ind}_H^G(\sigma)$, where H is a subgroup of G of index 2 and σ is an irreducible representation of H . Let F' be the quadratic extension of F corresponding to H and ω be the quadratic character of F^\times defining F'/F . By Lemma 2.2 and Section 2.1.2, we have

$$\text{Sw}(\rho) = \text{Sw}(\sigma) + sd/2,$$

where $s = \text{Sw}(\omega)$. On the other hand, by Proposition A.1, we have

$$\wedge^2 \rho \cong \text{Ind}_H^G(\wedge^2 \sigma) \oplus \tau \otimes \omega \quad \text{and} \quad \text{Sym}^2 \rho \cong \text{Ind}_H^G(\text{Sym}^2 \sigma) \oplus \tau,$$

where τ is an extension of the representation $\sigma \otimes \sigma^\gamma$ of H ($\gamma \in G \setminus H$) to G (see Proposition A.1). Hence, again by Lemma 2.2, we get

$$\text{Sw}(\text{Sym}^2 \rho) - \text{Sw}(\wedge^2 \rho) = \text{Sw}(\text{Sym}^2 \sigma) - \text{Sw}(\wedge^2 \sigma) + sd/2 + \text{Sw}(\tau \otimes \omega) - \text{Sw}(\tau).$$

Therefore, to get the conjectural inequality, it suffices to show that

$$\text{Sw}(\text{Sym}^2 \sigma) - \text{Sw}(\wedge^2 \sigma) + \text{Sw}(\tau \otimes \omega) - \text{Sw}(\tau) \leq \text{Sw}(\sigma).$$

A favourable situation is when $\text{Sw}(\tau \otimes \omega) = \text{Sw}(\tau)$, because then the inequality for σ implies the inequality for ρ , and similarly for the strict inequality. This is the case if, for example, the slope of any irreducible constituent of τ is greater than s by Lemma 2.1. But we cannot insure this property in general. Moreover, when σ is not self-dual, hence σ^γ is the contragredient of σ , $\sigma \otimes \sigma^\gamma$ contains the trivial character of H . Thus τ contains at least one irreducible constituent whose slope is 0 (i.e., the trivial character of G), hence smaller than s . Therefore the equality $\text{Sw}(\tau \otimes \omega) = \text{Sw}(\tau)$ never holds in this case. This means that we should have some stronger inequality for σ to conclude the conjectural inequality for ρ if σ is not self-dual.

4. APPLICATION TO THE LOCAL LANGLANDS CORRESPONDENCE

4.1. *Local Langlands correspondence.* From now on, we assume that the characteristic of a non-Archimedean local field F is zero. We write W_F for the Weil group of F . Let \mathbf{G} be a split connected reductive group over F . Let $\hat{\mathbf{G}}$ denote the Langlands dual group of \mathbf{G} . We say that a homomorphism

$\phi: W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \hat{\mathbf{G}}$ is an L -parameter of \mathbf{G} if ϕ is smooth on W_F and the restriction $\phi|_{\mathrm{SL}_2(\mathbb{C})}: \mathrm{SL}_2(\mathbb{C}) \rightarrow \hat{\mathbf{G}}$ is algebraic. We let $\Pi(\mathbf{G})$ be the set of equivalence classes of irreducible smooth representations of $\mathbf{G}(F)$ and $\Phi(\mathbf{G})$ the set of $\hat{\mathbf{G}}$ -conjugacy classes of L -parameters of \mathbf{G} .

The conjectural *local Langlands correspondence* asserts that there exists a natural map

$$\mathrm{LLC}_{\mathbf{G}}: \Pi(\mathbf{G}) \rightarrow \Phi(\mathbf{G}),$$

with finite fibers. In other words, by letting $\Pi_{\phi}^{\mathbf{G}}$ be the fiber of the map $\mathrm{LLC}_{\mathbf{G}}$ at an L -parameter ϕ (called an L -packet), we have a natural partition

$$\Pi(\mathbf{G}) = \bigsqcup_{\phi \in \Phi(\mathbf{G})} \Pi_{\phi}^{\mathbf{G}},$$

where each $\Pi_{\phi}^{\mathbf{G}}$ is finite. Moreover, it is expected that each $\Pi_{\phi}^{\mathbf{G}}$ is equipped with a natural map $\Pi_{\phi}^{\mathbf{G}} \rightarrow \mathrm{Irr}(\mathcal{S}_{\phi})$, where $\mathrm{Irr}(\mathcal{S}_{\phi})$ is the set of irreducible characters of a certain finite group \mathcal{S}_{ϕ} associated to ϕ (see [24, Section 1] and also [3, Section 6] for the details).

The local Langlands correspondence has been established for several specific groups. Especially, when \mathbf{G} is GL_N , the correspondence was constructed by Harris–Taylor [18] and the first author [20]. Also, for a certain class of classical groups, the correspondence has been established; for example, quasi-split symplectic and orthogonal groups by Arthur [4]. Our motivation is to seek an explicit description of the local Langlands correspondence for these groups.

4.2. Hiraga–Ichino–Ikeda’s formal degree conjecture. We next recall the *formal degree conjecture* proposed by Hiraga–Ichino–Ikeda ([24]).

CONJECTURE 4.1 (Formal degree conjecture, [24, Conjecture 1.4]). *We assume the local Langlands correspondence for \mathbf{G} . Suppose that $\pi \in \Pi(\mathbf{G})$ is a discrete series representation whose L -parameter is ϕ . Then we have the following identity:*

$$\mathrm{deg}(\pi) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_{\phi}^{\natural}|} \cdot |\gamma(0, \mathrm{Ad} \circ \phi, \psi_F)|.$$

Here,

- $\mathrm{deg}(\pi)$ is the formal degree of π (see [24] and also [23] for the normalization of the Haar measure used here),
- $\langle -, \pi \rangle$ denotes the irreducible character of \mathcal{S}_{ϕ} associated to π via the above-mentioned map $\Pi_{\phi}^{\mathbf{G}} \rightarrow \mathrm{Irr}(\mathcal{S}_{\phi})$,
- $\mathcal{S}_{\phi}^{\natural}$ is a variant of the group \mathcal{S}_{ϕ} (see [24, Section 1] for the definition),
- $|\gamma(-, \mathrm{Ad} \circ \phi, \psi_F)|$ denotes the γ -factor of the representation $\mathrm{Ad} \circ \phi$ of $W_F \times \mathrm{SL}_2(\mathbb{C})$ with respect to a non-trivial additive character ψ_F , where Ad is the adjoint representation of $\hat{\mathbf{G}}$ on $\mathrm{Lie}(\hat{\mathbf{G}})/\mathrm{Lie}(Z(\hat{\mathbf{G}}))$.

4.3. *Relation to Swan exponents.* Now we explain why the above conjectures are related to the problem of our interest. For this, let us first review the γ -factors for L -parameters. For simplicity, we consider only the case where the $\mathrm{SL}_2(\mathbb{C})$ -part is trivial. (See [16, Section 2] for the general case.)

We first consider the case of GL_n . In this case, an L -parameter with trivial $\mathrm{SL}_2(\mathbb{C})$ -part is nothing but an n -dimensional smooth semisimple representation of W_F . Let ϕ be such a representation of W_F on a \mathbb{C} -vector space V . Then its L -factor is defined by

$$L(\phi, s) := \det(1 - q^{-s} \cdot \phi(\mathrm{Frob}) \mid V^{\phi(I_F)})^{-1},$$

where I_F denotes the inertia subgroup of W_F and Frob is any lift of the geometric Frobenius. Also, by fixing a non-trivial additive character ψ_F of F , the ε -factor $\varepsilon(\phi, s, \psi_F)$ is associated (Deligne [15] and Tate [33]). We do not recall the definition of the ε -factor, but note that it has the following relationship with the Artin exponent:

$$\varepsilon(\phi, s, \psi_F) = w \cdot q^{\mathrm{Art}(\phi)(\frac{1}{2}-s)},$$

where w is a complex number independent of s , which is a root of unity when ρ is self-dual (see [16, Section 2.3]), and ψ_F is taken to be of level zero, i.e., trivial on \mathcal{O}_F but not on \mathfrak{p}_F^{-1} . Let us recall that the γ -factor is defined by using these local factors as follows:

$$\gamma(\phi, s, \psi_F) := \varepsilon(\phi, s, \psi_F) \cdot \frac{L(\phi^\vee, 1-s)}{L(\phi, s)},$$

where ϕ^\vee is the contragredient representation of ϕ .

We next consider the case where \mathbf{G} is general. In this case, by taking a finite-dimensional algebraic representation R of $\hat{\mathbf{G}}$, we obtain an L -parameter $R \circ \phi$ of a general linear group. Thus we can consider its L -factor, ε -factor, and γ -factor. In the formal degree conjecture, R is taken to be the adjoint representation of $\hat{\mathbf{G}}$. Note that Ad is self-dual (consider the Killing form), hence so is $\mathrm{Ad} \circ \phi$. For example, when \mathbf{G} is one of GL_N , SO_N , and Sp_{2n} , $\mathrm{Ad} \circ \phi$ is described as follows. By composing ϕ with the standard representation of $\hat{\mathbf{G}}$, we may regard ϕ as a representation ρ of W_F (let us write ρ). Then, $\mathrm{Ad} \circ \phi$ is isomorphic to $R \circ \rho$ as representations of W_F , where

$$R = \begin{cases} \mathrm{Std} \otimes \mathrm{Std}^\vee - \mathbb{1} & \text{if } \mathbf{G} = \mathrm{GL}_N, \\ \mathrm{Sym}^2 & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1}, \\ \wedge^2 & \text{if } \mathbf{G} = \mathrm{Sp}_{2n} \text{ or } \mathbf{G} = \mathrm{SO}_{2n}. \end{cases}$$

Now, by the above consideration, we can rewrite the identity predicted by the formal degree conjecture as follows:

$$(4.1) \quad \deg(\pi) = \frac{\langle \mathbb{1}, \pi \rangle}{|\mathcal{S}_\phi^\natural|} \cdot q^{\frac{1}{2} \mathrm{Art}(\mathrm{Ad} \circ \phi)} \cdot \frac{|L(\mathrm{Ad} \circ \phi, 1)|}{|L(\mathrm{Ad} \circ \phi, 0)|}.$$

Our fundamental expectation is that establishing this identity is related to investigating the explicit local Langlands correspondence. This is exactly the point where our motivation for computing the Swan exponent of $\mathrm{Sym}^2 \rho$ or $\wedge^2 \rho$ comes.

For example, let us consider the case where \mathbf{G} is either SO_N or Sp_{2n} . In fact, Conjecture 4.1 in this case has been already solved ([25] for SO_{2n+1} and [5] for SO_{2n} and Sp_{2n}). Suppose that an L -parameter ϕ is given “explicitly” in the sense that we have a description of the associated representation $\rho = \mathrm{std} \circ \phi$ of W_F . Then, by computing the quantity on the right-hand side of (4.1), we can access the formal degree of $\pi \in \Pi_\phi^{\mathbf{G}}$. This enables us to narrow down the possibility of π . Of course, here the roles of ϕ and π can be swapped; we can also start from an explicitly given discrete series representation of $\mathbf{G}(F)$.

Conversely, we can also try to get the identity (4.1) by assuming the explicit local Langlands correspondence, i.e., assuming that we have an explicit description of an irreducible discrete series representation π of $\mathbf{G}(F)$ and its L -parameter ϕ .

4.4. *Some consequences for simple supercuspidal representations.* Now we explain that the above strategy indeed works for *simple supercuspidal representations*, in the sense of Gross–Reeder [16], of symplectic groups.

4.4.1. *From explicit LLC to FDC.*

COROLLARY 4.2. *Simple supercuspidal representations of symplectic groups satisfy the formal degree conjecture when $p \neq 2$.*

PROOF. We utilize the result of [29] (see [29, Section 9] for the details of the arguments in this paragraph). Let π be a simple supercuspidal representation of $\mathrm{Sp}_{2n}(F)$ with L -parameter ϕ . By [29, Theorem 7.18], as a $(2n+1)$ -dimensional representation of W_F , ϕ is the direct sum of a $2n$ -dimensional representation ϕ_0 satisfying $\mathrm{Sw}(\phi_0) = 1$ and the 1-dimensional representation $\det \circ \phi_0$. Moreover, [29, Theorem 7.18] also says that the L -packet of ϕ consists of exactly two elements, which implies that $\langle \pi, 1 \rangle = 1$ and $|\mathcal{S}_\phi^{\mathfrak{g}}| = 2$. Since the adjoint L -factor $L(s, \mathrm{Ad} \circ \phi)$ is trivial ([29, Proposition 9.22]), the right-hand side of the equality (4.1) is given by $\frac{1}{2} \cdot q^{\frac{1}{2} \mathrm{Art}(\mathrm{Ad} \circ \phi)}$. By noting that $\mathrm{Ad} \circ \phi \cong \wedge^2 \phi_0 \oplus \phi_0 \otimes (\det \circ \phi_0)$ and again using that $L(s, \mathrm{Ad} \circ \phi)$ is trivial, we see that

$$\mathrm{Art}(\mathrm{Ad} \circ \phi) = \mathrm{Sw}(\mathrm{Ad} \circ \phi) + n(2n+1) = \mathrm{Sw}(\wedge^2 \phi_0) + 1 + n(2n+1).$$

On the other hand, by definition, the simple supercuspidal representation π is constructed by the compact induction of a character of an explicit open compact subgroup of $\mathrm{Sp}_{2n}(F)$ (see [29, Section 2]). Thus, by using a formula for the formal degrees of such compactly-induced supercuspidal representations (see, e.g., [8, Theorem A.14]), we can express $\mathrm{deg}(\pi)$ in terms of the volume of the open compact subgroup. As performed in [16] (see also the end

of [29, Section 9]), we can conclude that $\deg(\pi) = \frac{1}{2} \cdot q^{n^2+n}$. Therefore, in order to get the equality (4.1), it suffices to show that $\text{Sw}(\wedge^2 \phi_0) = n - 1$. This is a special case of Corollary 2.4. \square

REMARK 4.3. In [29], the second author proved the equality $\text{Sw}(\wedge^2 \rho) = n - 1$, hence obtained Corollary 4.2, in the case where $p \nmid 2n$ or $p = p^e \cdot n'$ for $e \in \mathbb{Z}_{>0}$ and $n' \in \mathbb{Z}_{>0}$ satisfying $n' \mid (p - 1)$. In [28], Mieda obtained Corollary 4.2 for any $p \neq 2$ via a completely different method. On the other hand, as mentioned above, it was announced by Beuzart-Plessis that the formal degree conjecture for Sp_{2n} has been solved (see [5]). Hence Corollary 4.2 is not new in any case. But we would like to emphasize that our proof presented above is new.

4.4.2. *From FDC to explicit LLC.* We next consider the case where $p = 2$. In this case, we determined explicitly the L -parameter of a simple supercuspidal representation of $\text{Sp}_{2n}(F)$ in [22]. However, we cannot deduce the formal degree conjecture for simple supercuspidal representations of $\text{Sp}_{2n}(F)$ from the description in *loc. cit.* as we did in Section 4.4.1. Rather, in contrast to the case where $p \neq 2$, we utilized the formal degree conjecture in order to determine the L -parameter. Let us just present the outline of the arguments of *loc. cit.* here.

The point is starting from an irreducible $(2n + 1)$ -dimensional orthogonal representation ρ of W_F whose Swan exponent is 1. Then it associates an irreducible supercuspidal representation π of $\text{Sp}_{2n}(F)$. On the other hand, it also associates an irreducible self-dual supercuspidal representation Π of $\text{GL}_{2n+1}(F)$, which is known to be simple supercuspidal. By utilizing the twisted endoscopic character relation between Π and π , we can prove that π is either depth-zero supercuspidal or simple supercuspidal ([22, Corollary 4.5]). Since we know that $\text{Sw}(\wedge^2 \rho) = n$ ([22, Proposition 4.12], or more generally, Corollary 2.10 of this paper), the formal degree conjecture tells us the value of the formal degree of π . In fact, this information is enough for concluding that π is not depth-zero, hence simple supercuspidal (see [21, A.4]). Once we see that π is simple supercuspidal, it is not difficult to determine it exactly again by using the twisted endoscopic character relation.

REMARK 4.4. Although we only mentioned the case of Sp_{2n} , it is also possible to establish the explicit local Langlands correspondence for simple supercuspidal representations of split special orthogonal groups by utilizing the formal degree conjecture effectively. That is the content of [1].

5. SIMPLE SUPERCUSPIDAL L -PARAMETERS FOR $\text{Sp}_6(\mathbb{Q}_2)$ HAVE VALUE IN $G_2(\mathbb{C})$

This section, a remark really, is prompted by a question of Gordan Savin to the authors. After we had submitted our previous paper [22] mentioned

in Section 4.4.2, Savin asked us if the L -parameter of a simple supercuspidal representation of $\mathrm{Sp}_6(\mathbb{Q}_2)$, which takes values in $\mathrm{SO}_7(\mathbb{C})$, actually can be conjugated to take values in the subgroup $G_2(\mathbb{C})$ of $\mathrm{SO}_7(\mathbb{C})$.

That question is related to the analysis of the parameter of simple supercuspidal representations of $G_2(\mathbb{Q}_2)$ in [26, Section 4]. In *loc. cit.*, a subgroup I of $G_2(\mathbb{C})$ is introduced. The group I is a semi-direct product of a normal subgroup J , which is isomorphic to the additive 2-group \mathbb{F}_8 (where \mathbb{F}_8 is a field with 8 elements), with a subgroup which is itself a semi-direct product of \mathbb{F}_8^\times and $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$ (acting naturally on \mathbb{F}_8^\times), with their natural actions on \mathbb{F}_8 .

$$\begin{array}{ccccccc}
 1 & \longrightarrow & J (\cong \mathbb{F}_8) & \longrightarrow & I & \longrightarrow & I/J (\cong \mathbb{F}_8^\times \times \mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)) \longrightarrow 1 \\
 & & & & & \swarrow & \\
 & & & & & \text{split} &
 \end{array}$$

It is shown that the inclusion of I into $G_2(\mathbb{C})$, itself inside $\mathrm{GL}_7(\mathbb{C})$, gives the unique irreducible faithful self-dual (hence orthogonal) representation of I of dimension 7. It is also explicated in *loc. cit.* how the group I is a quotient of $W_{\mathbb{Q}_2}$.

Now take a simple supercuspidal representation π of $\mathrm{Sp}_6(\mathbb{Q}_2)$, and let σ be the corresponding representation of $W_{\mathbb{Q}_2}$, which is an orthogonal irreducible representation of dimension 7.

PROPOSITION 5.1. *The representation σ is the same as the representation described after Proposition 4.2 in loc. cit.*

PROOF. By [22] (cf. [10]), the representation σ is given by $\mathrm{Ind}_{K/\mathbb{Q}_2}(\chi)$, where K/\mathbb{Q}_2 is the tame totally ramified extension of degree 7 generated by a 7th root ϖ of 2 (remark that K is unique up to isomorphism) and χ is a quadratic character of K^\times of Swan exponent 1. Note that such χ must be as follows:

- $\chi(\varpi)$ is either 1 or -1 ,
- $\chi|_{U_K^2}$ is trivial, and
- $\chi|_{U_K^1}$ is the unique non-trivial character (say χ_1) of U_K^1 .

Under the identification

$$U_K^1/U_K^2 \cong \mathbb{F}_2: 1 + \varpi x \mapsto \bar{x} \quad (x \in \mathcal{O}_K),$$

where \bar{x} denotes the modulo \mathfrak{p}_K reduction, we transform χ_1 into the non-trivial character of \mathbb{F}_2 . Moreover, since we have $\det(\sigma) \cong \delta_{K/\mathbb{Q}_2} \cdot (\chi|_{\mathbb{Q}_2^\times})$, where $\delta_{K/\mathbb{Q}_2} := \det(\mathrm{Ind}_{K/\mathbb{Q}_2} \mathbb{1})$, the triviality of $\det(\sigma)$ implies that $\chi|_{\mathbb{Q}_2^\times} = \delta_{K/\mathbb{Q}_2}$ (see [9, 29.2]). By [7, (10.1.6) Proposition], δ_{K/\mathbb{Q}_2} is unramified and takes value the Jacobi symbol $(\frac{2}{7})$ at 2. But $(\frac{2}{7}) = 1$ since $3^2 \equiv 2 \pmod{7}$. Hence $\chi|_{\mathbb{Q}_2^\times}$ is trivial. In particular, $\chi(\varpi)$ must be 1. Conversely, $\chi|_{\mathbb{Q}_2^\times}$ is trivial when $\chi(\varpi) = 1$.

On the other hand, [26] introduces the Galois closure L of K , which is the splitting field of K/\mathbb{Q}_2 . Then the 21-dimensional representation ρ of $W_{\mathbb{Q}_2}$ is introduced, which is given by $\text{Ind}_{L/\mathbb{Q}_2}(\eta)$, where η is a quadratic character of L^\times taking value 1 at the uniformizer ϖ and giving on restriction to U_L^1 any non-trivial character trivial on U_L^2 (there are 7 of them, any of which induces to ρ). Then ρ splits into the sum of three irreducible representations of dimension 7, only one of which is self-dual.

But we can take $\eta = \chi \circ N_{L/K}$, which shows that σ is the self-dual irreducible component of ρ . That proves what we claimed. \square

The above arguments can be generalized to any 2-adic field F as a base field.

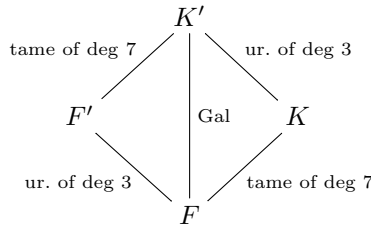
PROPOSITION 5.2. *Let π be a simple supercuspidal representation of $\text{Sp}_6(F)$ and σ be the corresponding representation of W_F , which is an orthogonal irreducible representation of dimension 7. Then σ takes values (up to conjugation) in the subgroup I of $G_2(\mathbb{C})$, and its image is then either I or its subgroup of index 3.*

PROOF. Again by [22] (cf. [10]), σ is given by $\text{Ind}_{K/F}(\chi)$, where K/F is a totally ramified extension of degree 7 and χ is a quadratic character of K^\times with $\text{Sw}(\chi) = 1$.

Let q_F be the cardinality of k_F . We first consider the case where $7 \nmid (q_F - 1)$. In this case, there is a unique such extension K up to isomorphism, generated by a 7th root Π of a uniformizer of F . Its Galois closure K'/F is KF' where F'/F is the unramified extension of degree 3 obtained by adjoining 7th roots of unity. Then we have

$$\text{Gal}(K'/F) \cong \text{Gal}(K'/F') \rtimes \text{Gal}(K'/K),$$

where the inertia subgroup $\text{Gal}(K'/F')$ is cyclic of order 7. The Frobenius element of $\text{Gal}(K'/K)$ acts on $\text{Gal}(K'/F')$ by taking q_F -th powers, an order 3 automorphism of $\text{Gal}(K'/F')$.



The possible inducing characters of K^\times , to get by induction to W_F an irreducible special orthogonal representation with Swan exponent 1, are the quadratic characters χ non-trivial on U_F^1 but trivial on U_F^2 , and taking the value 1 on the uniformizer Π . There are $q_F - 1$ of them, as it should. Consider

the character $\chi' = \chi \circ \text{Nr}_{K'/K}$ of K'^{\times} , which induces to $W_{F'}$ the restriction σ' of σ . We identify the residue field $k_{K'}$ of K' with $U_{K'}^1/U_{K'}^2$, by the map

$$U_{K'}^1/U_{K'}^2 \cong k_{K'} : 1 + \Pi x \mapsto \bar{x} \quad (x \in \mathcal{O}_{K'}).$$

The action of $\text{Gal}(K'/F')$ is given by multiplication by 7th roots of 1, reflecting the action on Π , and $\text{Gal}(K'/K)$ acts by its natural action on $k_{K'} = k_{F'}$. The field $k_{F'}$ is an extension of \mathbb{F}_8 , and the action of $\text{Gal}(K'/F')$ by 7th roots of 1 is an action via a morphism into \mathbb{F}_8^{\times} . It follows that the restriction of σ to $W_{K'}$, which is the direct sum of all conjugates of χ' , factors through the quotient of $k_{F'}$ by an \mathbb{F}_8 hyperplane, and in particular its image is a line over \mathbb{F}_8 . It follows again that the image of σ is isomorphic to I .

We next consider the case where $7|(q_F - 1)$. In this case, the situation is simpler, as K/F is cyclic of degree 7. A similar, but simpler, analysis goes through to show that the image of σ is isomorphic to the index 3 subgroup of I . □

REMARK 5.3. Proposition 5.2 guarantees only that the L -parameter of any simple supercuspidal representation of $\text{Sp}_6(F)$ is obtained by lifting an L -parameter of G_2 . Thus, a priori, it is possible that two inequivalent (i.e., not $G_2(\mathbb{C})$ -conjugate) L -parameters of G_2 lift to the L -parameter of a simple supercuspidal representation of $\text{Sp}_6(F)$. However, it does not happen since any two subgroups of $G_2(\mathbb{C})$ which are conjugate in $\text{GL}_7(\mathbb{C})$ are conjugate also in $G_2(\mathbb{C})$ ([17, 1.2 Corollary 1]). We would like to thank the referee for pointing us to the reference [17].

REMARK 5.4. In fact, when $p \neq 2$, at most the half of the L -parameters of Sp_6 corresponding to simple supercuspidal representations come from G_2 . When $p > 3$, we can check that indeed exactly the half of them come from G_2 . Moreover, any such L -parameter comes from G_2 if and only if the central character of a simple supercuspidal representation of Sp_6 contained in the corresponding L -packet is trivial. Some details will appear in an ongoing work of the second author with Yoichi Mieda.

APPENDIX A. SOME BASIC FACTS ON REPRESENTATIONS OF FINITE GROUPS

In this section, we collect some basic facts on representations of finite groups. The facts introduced in this section might be well-known, but we give proofs for the sake of completeness.

A.1. *Symmetric and Exterior squares of induced representations.* Let G be a finite group and H a subgroup of G of index 2. We fix an element γ of $G \setminus H$. Let σ be a representation of H . If we put

$$\begin{cases} \tau(h)(v_1 \otimes v_2) := (\sigma(h)v_1) \otimes (\sigma^\gamma(h)v_2) \\ \tau(\gamma h)(v_1 \otimes v_2) := (\sigma^\gamma(h)v_2) \otimes (\sigma(\gamma^2)\sigma(h)v_1) \end{cases}$$

for any $h \in H$, then it can be easily checked that τ is a representation of G which extends the representation $\sigma \otimes \sigma^\gamma$ of H .

PROPOSITION A.1. *We put $\rho := \text{Ind}_H^G(\sigma)$. Let ω be the non-trivial character of G/H . Then we have*

1. $\wedge^2 \rho \cong \text{Ind}_H^G(\wedge^2 \sigma) \oplus \tau \otimes \omega$.
2. $\text{Sym}^2 \rho \cong \text{Ind}_H^G(\text{Sym}^2 \sigma) \oplus \tau$.

We first show some lemmas.

LEMMA A.2. *Let V_1 and V_2 be finite dimensional \mathbb{C} -vector spaces equipped with isomorphisms $I_1: V_1 \rightarrow V_2$ and $I_2: V_2 \rightarrow V_1$. If we define an automorphism $I_1 \otimes I_2$ of $V_1 \otimes V_2$ by $I_1 \otimes I_2(v_1 \otimes v_2) := I_2(v_2) \otimes I_1(v_1)$, then we have*

$$\text{Tr}(I_1 \otimes I_2 \mid V_1 \otimes V_2) = \text{Tr}(I_2 \circ I_1 \mid V_1).$$

PROOF. We take a \mathbb{C} -basis $\{e_1, \dots, e_n\}$ of V_1 and define a \mathbb{C} -basis $\{e'_1, \dots, e'_n\}$ of V_2 by $e'_i := I_1(e_i)$. Then $\{e_j \otimes e'_j\}_{1 \leq i, j \leq n}$ forms a \mathbb{C} -basis of $V_1 \otimes V_2$. Since we have

$$I_1 \otimes I_2(e_i \otimes e'_j) = I_2(e'_j) \otimes I_1(e_i) = (I_2 \circ I_1)(e_j) \otimes e'_i,$$

$e_i \otimes e'_j$ contributes to the trace of $I_1 \otimes I_2$ only when $i = j$ (see the matrix in the basis $\{e_j \otimes e'_j\}_{1 \leq i, j \leq n}$). Moreover, the contribution is exactly that of e_i to the trace of $I_2 \circ I_1$ (seen in the basis $\{e_i\}_{1 \leq i \leq n}$). \square

- LEMMA A.3. *1. We have $\text{Tr}(\sigma)(g^2) = \text{Tr}(\sigma^\gamma)(g^2)$ for any $g \in G$, where γ is a(ny) element of $G \setminus H$.*
2. We have

$$\text{Tr}(\tau)(g) = \begin{cases} \text{Tr}(\sigma)(g) \cdot \text{Tr}(\sigma^\gamma)(g) & \text{if } g \in H, \\ \text{Tr}(\sigma)(g^2) & \text{if } g \in G \setminus H. \end{cases}$$

PROOF. We show (1). When $g \in G \setminus H$, by writing $g = h\gamma$ with $h \in H$, we get $\gamma g^2 \gamma^{-1} = \gamma h \gamma h = h^{-1} g^2 h$. In particular, $\gamma g^2 \gamma^{-1}$ and g^2 are conjugate in H . Hence we have $\text{Tr}(\sigma^\gamma)(g^2) = \text{Tr}(\sigma)(g^2)$.

We next show (2). As τ is an extension of $\sigma \otimes \sigma^\gamma$, the equality $\text{Tr}(\tau)(g) = \text{Tr}(\sigma)(g) \cdot \text{Tr}(\sigma^\gamma)(g)$ for $g \in H$ is obvious. For the case where $g = \gamma h \in G \setminus H$, by Lemma A.2, we get

$$\text{Tr}(\tau)(g) = \text{Tr}(\sigma(\gamma^2)\sigma(h)\sigma^\gamma(h)) = \text{Tr}(\sigma)(\gamma^2 h \gamma h \gamma^{-1}) = \text{Tr}(\sigma^\gamma)(g^2) = \text{Tr}(\sigma)(g^2),$$

where we used (1) in the last equality. \square

PROOF OF PROPOSITION A.1. By noting that $\rho \otimes \rho \cong \text{Sym}^2 \rho \oplus \wedge^2 \rho$ and $\Psi^2 \rho \cong \text{Sym}^2 \rho - \wedge^2 \rho$ (and that the same is true for σ), it suffices to show that

- (1') $\rho \otimes \rho \cong \text{Ind}_H^G(\sigma \otimes \sigma) \oplus \tau \oplus \tau \otimes \omega$ and
- (2') $\Psi^2 \rho \cong \text{Ind}_H^G(\Psi^2 \sigma) \oplus \tau - \tau \otimes \omega$.

We first note that the Frobenius formula for induced representations implies that

$$\mathrm{Tr}(\rho)(g) = \begin{cases} \mathrm{Tr}(\sigma)(g) + \mathrm{Tr}(\sigma^\gamma)(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H. \end{cases}$$

Thus, we have

$$\mathrm{Tr}(\rho \otimes \rho)(g) = \mathrm{Tr}(\rho)(g)^2 = \begin{cases} (\mathrm{Tr}(\sigma)(g) + \mathrm{Tr}(\sigma^\gamma)(g))^2 & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H, \end{cases}$$

and

$$\mathrm{Tr}(\Psi^2 \rho)(g) = \mathrm{Tr}(\rho)(g^2) = \mathrm{Tr}(\sigma)(g^2) + \mathrm{Tr}(\sigma^\gamma)(g^2)$$

where we note that g^2 always belongs to H in the last equality. Similarly, we have

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}_H^G(\sigma \otimes \sigma))(g) &= \begin{cases} \mathrm{Tr}(\sigma \otimes \sigma)(g) + \mathrm{Tr}((\sigma \otimes \sigma)^\gamma)(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H, \end{cases} \\ &= \begin{cases} \mathrm{Tr}(\sigma)(g)^2 + \mathrm{Tr}(\sigma^\gamma)(g)^2 & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}_H^G(\Psi^2 \sigma))(g) &= \begin{cases} \mathrm{Tr}(\Psi^2 \sigma)(g) + \mathrm{Tr}(\Psi^2 \sigma^\gamma)(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H, \end{cases} \\ &= \begin{cases} \mathrm{Tr}(\sigma)(g^2) + \mathrm{Tr}(\sigma^\gamma)(g^2) & \text{if } g \in H, \\ 0 & \text{if } g \in G \setminus H. \end{cases} \end{aligned}$$

Thus, the identities (1') and (2') follows by using Lemma A.3. \square

A.2. Faithful irreducible representation of a p -group.

LEMMA A.4. *Let G be a p -group. If G has a faithful irreducible representation, then the center $Z(G)$ of G is a non-trivial cyclic group.*

PROOF. Suppose that ρ is a faithful irreducible representation of G . It is well-known that any p -group has a non-trivial center (see, e.g., [31, Theorem 14 in Chapter 8]). Thus $Z(G)$ is a non-trivial finite abelian p -group. If we let ω_ρ be the central character of ρ , then ω_ρ is also faithful. However, $Z(G)$ must be cyclic so that $Z(G)$ has a faithful 1-dimensional character. \square

A.3. *Representations of a certain p -group.* Let p be a prime number. Let H be a finite p -group and $Z(H)$ its center. We assume that H is non-abelian and that there exists a subgroup $Z \subset Z(H)$ satisfying the following conditions:

1. Z is cyclic of order p , and
2. H/Z is an elementary abelian p -group.

We remark that G is said to be an *extra-special* p -group if Z can be taken to be $Z(H)$.

We note that, by the condition (2), the image of the commutator map $H \times H \rightarrow H: (x, y) \mapsto xyx^{-1}y^{-1}$ lies in Z . Thus, by fixing an identification $Z \cong \mathbb{F}_p$, we obtain a symplectic form (i.e., non-degenerate alternating bilinear form) on $H/Z(H)$

$$\langle -, - \rangle: H/Z(H) \times H/Z(H) \rightarrow \mathbb{F}_p.$$

We take a maximal totally isotropic subspace $U \subset H/Z(H)$ with respect to $\langle -, - \rangle$. We let H' be the preimage of U in H , which is abelian.

The following proposition is given in [9, 16.4 Lemma 2] in the case where $Z = Z(H)$ (in fact, it is allowed that $Z = Z(H)$ is any finite cyclic group in *loc. cit.*).

PROPOSITION A.5. *For any character χ of $Z(H)$ whose restriction to Z is non-trivial, there uniquely (up to isomorphism) exists an irreducible representation ρ_χ of H on which $Z(H)$ acts via χ . Moreover, ρ_χ is explicitly given by $\text{Ind}_{H'}^H(\chi')$ for any character χ' of H' extending χ (note that such χ' can be always taken since H' is abelian).*

PROOF. We just reproduce the proof of [9, 16.4 Lemma 2] in our setting.

The center $Z(H)$ obviously acts on $\text{Ind}_{H'}^H(\chi')$ via χ . Thus let us next check that $\text{Ind}_{H'}^H(\chi')$ is irreducible. Since H' is normal in H , it suffices to show that any element $h \in H$ satisfying $(\chi')^h = \chi'$ lies in H' . If $h \in H$ is such an element, we have $\chi'(hh'h^{-1}h'^{-1}) = 1$ for any $h' \in H'$. Since χ' extends the character χ of Z , which is faithful, this implies that $hh'h^{-1}h'^{-1} = 1$ for any $h' \in H'$. As H' is the preimage of the maximal totally isotropic subspace U of $H/Z(H)$ with respect to $\langle -, - \rangle$, we see that h must belong to H' .

Let us finally show the uniqueness of ρ_χ by a counting argument. For this, we let $|H/Z(H)| = p^{2d}$ and $|Z(H)/Z| = p^r$, hence $|H| = p^{2d+r+1}$. Note that

$$\begin{aligned} |H| &= \sum_{\rho: \text{ irr. rep. of } H} (\dim \rho)^2 \\ &= \sum_{\substack{\rho: \text{ irr. rep. of } H \\ \text{trivial on } Z}} (\dim \rho)^2 + \sum_{\substack{\rho: \text{ irr. rep. of } H \\ \text{non-trivial on } Z}} (\dim \rho)^2. \end{aligned}$$

Since H/Z is abelian, the first term is simply given by $|H/Z| = p^{2d+r}$. On the other hand, by noting that $\dim(\text{Ind}_{H'}^H(\chi'))^2 = p^{2d}$ and that the number of characters of $Z(H)$ non-trivial on Z is given by $p^r(p-1)$, we see that the second term is bounded below by $p^{2d+r}(p-1)$. Thus we get $|H| \geq p^{2d+r+1}$. However, we already know that $|H| = p^{2d+r+1}$. Therefore, we conclude that the representations $\text{Ind}_{H'}^H(\chi')$ exhausts all irreducible representations of H whose restrictions to Z are non-trivial. \square

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O Swanovim eksponentima simetričnih i anti-simetričnih kvadrata Galoisovih reprezentacija

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SAŽETAK. Neka je F lokalno nearhimedsko polje i E konačno Galoisovo proširenje polja F , čija je Galoisova grupa G . Ako je ρ reprezentacija grupe G na kompleksnom vektorskom prostoru V , onda možemo komponirati ρ s bilo kojom tenzorskom operacijom R na V . Na taj način dobijemo novu reprezentaciju $R \circ \rho$. U ovom radu proučavamo odnos između Swanovih eksponenta $\text{Sw}(\rho)$ i $\text{Sw}(R \circ \rho)$. Posebno diskutiramo slučaj kada je R simetrični ili antisimetrični kvadrat. Ovi slučajevi su od posebnog interesa u lokalnoj Langlandsovoj korespondenciji rascjepivih klasičnih grupa definiranih nad F , preko slutnje o formalnim stupnjevima. Vezano za to, dajemo eksplicitan opis Langlandsovog parametra jednostavnih kuspidualnih reprezentacija. Za ireducibilnu reprezentaciju ρ , glavni rezultati ovog rada određuju $\text{Sw}(\text{Sym}^2 \rho)$ i $\text{Sw}(\wedge^2 \rho)$ iz $\text{Sw}(\rho)$ kada je rezidualna karakteristika p polja F neparna, a u slučaju $p = 2$ ograničava ih u terminima od $\text{Sw}(\rho)$. U posljednjem slučaju dajemo i slutnju koja sadrži bolju ogradu za koju dajemo neke argumente za istinitost.

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