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LOGARITHMIC VERTEX ALGEBRAS RELATED TO $\mathfrak{sp}(4)$

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Dedicated to Marko Tadić on the occasion of his 70th birthday

Abstract. We present several results and conjectures pertaining to parafermion vertex algebra and related logarithmic vertex algebras. Starting from the tensor product of two copies of the singlet vertex algebra $\mathcal{M}(2)$, we consider various subalgebras that appear in its decomposition including $N_{-1}(sl(2))$ and its $\mathbb{Z}_2$-fixed point algebra, and the S_2-symmetric orbifold of the singlet vertex algebra $\mathcal{M}(2)$. In particular, we show that $N_{-1}(sl(2))$ has an extension to a W-algebra of type $(2,3,4,5,6,7,8)$. Finally we state some conjectures about singlet and triplet type W-algebras of type $\mathfrak{sp}(4)$ and their characters.

1. Introduction

The development of the parafermion conformal field theory was significantly aided by the introduction of parafermion vertex algebras. These algebras are initially defined as subalgebras of the generalized vertex algebras generated by Z-operators that were needed for the vertex-operator-theoretic interpretation of Rogers-Ramanujan partition identities [25],[24].

Over the last ten years, there has been extensive research on the parafermion vertex operator algebras linked to rational affine vertex operator algebras at positive integer levels, resulting in a comprehensive understanding of their structure [10, 15, 16]. However, for other levels, such as generic levels, their structure remains largely unknown.

For $\mathfrak{g} = \mathfrak{sl}(2)$, much more is known due to recent breakthroughs in understanding affine W-algebras at admissible levels. At generic levels, it is known that this vertex algebra is non-freely generated and is of type $(2,3,4,5)$. There are two distinguished generic levels of interest here: $k = -\frac{3}{2}$ and $k = -1$. It can be easily seen that $k = -\frac{3}{2}$ is the only level that admits an embedding of the principal affine W algebra for $\mathfrak{g}$ of type A_2 inside the parafermion algebra. This case was considered in our previous paper with Wang [6] where

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we obtained many structural results including results for related logarithmic algebras.

In this paper we focus on the parafermion algebra at $k = -1$. This case is interesting because $k = -1$ is the sole level that permits embedding of the affine W-algebra associated to $\mathfrak{sp}(4)$ inside the parafermion algebra. The objective of this paper is to repeat the analysis conducted in our previous paper [6], now involving the rank two symplectic Lie algebra.

The paper is structured as follows: Section 2 provides a review of notation and basic facts related to the symplectic fermion vertex algebra and its orbifold subalgebras. Section 3 focuses on the generic parafermion vertex algebra $N_k(\mathfrak{sl}_2)$ of level $k = -1$. The singlet algebra $\mathcal{M}(2)^{\otimes 2}$ is studied in relation to $N_{-1}(\mathfrak{sl}_2)$ including decomposition of the former in terms of irreducible $N_{-1}(\mathfrak{sl}_2)$-modules. Several results are proven, including a new type of character formula. Sections 4 and 5 primarily deal with decompositions of $N_{-1}(\mathfrak{sl}_2)$ and its orbifold subalgebra $N_{-1}(\mathfrak{sl}_2)^+$ in terms of the affine W-algebra $W_{-5/2}(\mathfrak{sp}(4))$. Section 6 focuses on another automorphism of order two of $\mathcal{M}(2)^{\otimes 2}$, the symmetric orbifold. A structure theorem for the orbifold subalgebra is presented in this section. In Section 7, the Virasoro vertex algebra $L(-2,0)$ contained in $\mathcal{M}(2)$ is considered. The decomposition of $L(-2,0)^{\otimes 2}$ in terms of $W_{-5/2}(\mathfrak{sp}(4))$ modules is explored in this section. The last part of the paper contemplates on the existence of a series of singlet and triplet type vertex algebras associated to $\mathfrak{sp}(4)$. Motivated by the $\mathfrak{sl}(3)$ singlet type algebras, conjectural expressions for their q-characters are presented.

2. Preliminaries

In this part we setup some notation and summarize facts we need later.

- Let $\mathfrak{g}$ be the simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$.
- Let $\hat{\mathfrak{g}}$ be the associated affine Lie algebra, and $\hat{\mathfrak{h}}$ be the associated Heisenberg subalgebra.
- Let $V^k(\mathfrak{g})$ be the universal affine vertex operator algebra of level k associated to the simple Lie algebra $\mathfrak{g}$.
- Let $V_k(\mathfrak{g})$ be the simple quotient of $V^k(\mathfrak{g})$.
- Let $W^k(\mathfrak{g})$ be the universal principal affine W-algebra $W^k(\mathfrak{g}, f_{pr})$ of level k.
- Let $W_k(\mathfrak{g})$ be the simple quotient of be the simple quotient of $W^k(\mathfrak{g})$.
- Let $N^k(\mathfrak{g}) = \{v \in V^k(\mathfrak{g}) \mid h(n)v = 0 \quad h \in \mathfrak{h}, n \in \mathbb{Z}_{\geq 0}\}$ be the parafermion subalgebra of $V^k(\mathfrak{g})$.
- Let $N_k(\mathfrak{g}) = \{v \in V_k(\mathfrak{g}) \mid h(n)v = 0 \quad h \in \mathfrak{h}, n \in \mathbb{Z}_{\geq 0}\}$ be the parafermion subalgebra of $V_k(\mathfrak{g})$.

- For a $\lambda \in P_+$, let $V_{\mathfrak{g}}(\lambda)$ be the irreducible finite-dimensional $\mathfrak{g}$–module with the highest weight λ, where P_+ denotes the set of dominant integral weights for $\mathfrak{g}$.
- Let $V^k(\lambda)$ be the generalized Verma module for $\hat{\mathfrak{g}}$–module induced from $\mathfrak{g}$–module $V_{\mathfrak{g}}(\lambda)$. Let $L_k(\lambda)$ be its simple quotient.
- For $\lambda, \mu \in P_+$, let $T^{k+3}_{\lambda,\mu}$ denotes the $\mathcal{W}^k(\mathfrak{g})$–module obtained as $H_{DS}(V^k(\lambda - (k+3)\mu)$ (cf. [9]).
- Let $L(c, 0)$ denotes the simple Virasoro vertex algebra of central charge c.
- For $k = -3 + \frac{1}{p}$ and $\mathfrak{g} = \mathfrak{sp}(4)$, the universal affine vertex algebra $V^k(\mathfrak{sp}(4))$ is simple (cf. [21]), and therefore by [7] $H_{DS}(V^k(\mathfrak{sp}(4)) = W^k(\mathfrak{sp}(4))$ is simple. In particular, $W^{-5/2}(\mathfrak{sp}(4))$ is a simple vertex algebra of central cherge $c = -4$.
- Let $\mathcal{M}(p)$ denotes the singlet vertex algebra of central charge $1 - 6\frac{(p-1)^2}{p}$ (cf. [2]). The singlet vertex algebra $\mathcal{M}(2)$ is isomorphic to the principal affine $\mathcal{W}$–algebra $W_{-3/2}(\mathfrak{sl}(3))$ of central charge $c = -2$ (cf. [30]).

We shall need the following facts which are well-known. Let $\mathfrak{g} = \mathfrak{sl}(2)$ with a Chevalley basis $\{e, f, h\}$ and let $k = -1$. Then we have:

- $V^k(\mathfrak{g}) = V_k(\mathfrak{g})$.
- $V^k(j\omega_1) = L_k(j\omega_1)$, $j \in \mathbb{Z}_{\geq 0}$, where ω_1 is the fundamental dominant weight for $\mathfrak{sl}(2)$.
- $N^k(j) := N^k(j\omega_1) = N_k(j\omega_1)$, $j \in \mathbb{Z}_{\geq 0}$, where $N^k(j\omega_1) = \{v \in V^k(j\omega_1) \mid h(n)v = 0, \ \forall\ n \in \mathbb{Z}_{\geq 0}\}$ and $N_k(j\omega_1) = \{v \in L_k(j\omega_1) \mid h(n)v = 0, \ \forall\ n \in \mathbb{Z}_{\geq 0}\}$.

We denote by $\mathrm{ch}[M](q) := \mathrm{tr}_M q^{L(0)}$ the character of a V-module M; from the context it should be clear what the vertex algebra is. Also, for simplicity we suppressed the conformal anomaly $-\frac{c}{24}$. For a vertex algebra V, the vertex algebra $V \otimes V$ admits an S_2-action permuting two tensor factors. The fixed point subalgebra will be denoted by $(V \otimes V)^{S_2}$. Its character is known to be

$$\mathrm{ch}[(V \otimes V)^{S_2}](q) = \frac{1}{2}\left(\mathrm{ch}[V]^2(q) + \mathrm{ch}[V](q^2)\right). \tag{2.1}$$

We will often use the q-Pochhammer symbol $(a; q)_n := \prod_{i=1}^{n}(1 - aq^{n-1})$ and $(q)_n = (q; q)_n$.

2.1. *The vertex algebra $W^k(\mathfrak{sp}(4))$.* Let $W^k(\mathfrak{sp}(4))$ denotes the principal affine W–algebra $W^k(\mathfrak{sp}(4), f_{pr})$ of level k and central charge

$$c = c_k = -\frac{2(12 + 5k)(13 + 6k)}{3 + k}.$$

It is generated by the Virasoro field $L(z) = \sum_{m\in\mathbb{Z}} L(m)z^{-m-2}$ and another field of conformal weight 4:

$$W(z) = \sum_{m\in\mathbb{Z}} W(m)z^{-m-4}.$$

The bracket relations (i.e, the OPE) are well-known (see for instance [17, Section 3.2]) and not needed here.

Let $L^W(c, h, h_W)$ denotes the irreducible highest weight $W^k(\mathfrak{sp}(4))$–module of the highest weight (h, h_W) with respect to $(L(0), W(0))$.

Note that for $k = -3 + 1/p$ we have $c_k = 86 - \frac{60}{p} - 30p$. We expect that this level is generic and the following conjecture holds:

Conjecture 2.1. *Let $k = -3 + 1/p$ and $c_k = 86 - \frac{60}{p} - 30p$. Then $T^{k+3}_{\lambda,0}$ is an irreducible $W^k(\mathfrak{sp}(4))$–module and $H_{DS}(L_k(\lambda)) = T^{k+3}_{\lambda,0}$ for each dominant integral weights λ for $\mathfrak{sp}(4)$.*

2.2. *Symplectic fermion vertex algebra $\mathcal{SF}(d)$.* The symplectic fermion vertex algebra $\mathcal{SF}(d)$ (see [1] for more details) is the universal vertex superalgebra generated by odd fields/vectors b_i and c_i $(i = 1, \dots, n)$ with the following non-trivial λ–bracket

$$[(b_i)_\lambda c_j] = \delta_{i,j}\lambda.$$

$\mathcal{SF}(d)$ can be realized on the irreducible level one module for the Lie superalgebra with generators

$$\{K, b_i(n), c_i(n), n \in \mathbb{Z}\}$$

and relations

$$\{b_i(n), b_j(m)\} = \{c_i(n), c_j(m)\} = 0, \quad \{b_i(n), c_j(m)\} = n\delta_{i,j}\delta_{n+m,0}K.$$

Here K is central and other super-commutators are trivial. As a vector space,

$$\mathcal{SF}(d) = \bigwedge \operatorname{span}\{b_i(-m), c_i(-m),\ m \in \mathbb{Z}_{>0}, i = 1, \dots, n\}.$$

The fields b_i, c_i can be identified as formal Laurent series acting on $\mathcal{SF}(d)$.

$$b_i(x) = \sum_{n\in\mathbb{Z}} b_i(n)x^{-n-1}, \quad c_i(x) = \sum_{n\in\mathbb{Z}} c_i(n)x^{-n-1}$$

The vertex algebra $\mathcal{SF}(d)$ has the following Virasoro element of central charge $c = -2d$:

$$\omega_{\mathcal{SF}(d)} = \sum_{i=1}^{d} : b_i c_i : .$$

Let $L(z) = Y(\omega_{\mathcal{SF}(d)}, z)$. There is a charge operator $J \in \operatorname{End}(\mathcal{SF}(d))$ such that

$$[J, b_i(n)] = b_i(n), \quad [J, c_i(n)] = -c_i(n)$$

which defines on $\mathcal{SF}(d)$ the $\mathbb{Z}$–gradation:

$$\mathcal{SF}(d) = \sum_{\ell \in \mathbb{Z}} \mathcal{SF}(d)^{(\ell)}, \quad \mathcal{SF}(d)^{(\ell)} = \{v \in \mathcal{SF}(d) \mid Jv = \ell v\}.$$

Let us recall few known facts on the vertex algebra $\mathcal{SF}(d)$:

- The automorphism group of $\mathcal{SF}(d)$ is $Sp(2d,\mathbb{C})$ (cf. [1]) and the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2d)$ acts on $\mathcal{SF}(d)$ by derivations.
- The vertex algebra $\mathcal{SF}(d)^{GL(d,\mathbb{C})}$ is a simple $\mathcal{W}$–algebra of type $W(2,3,\cdots,2d+1)$ (cf. [14], [8], [23]). In particular:
- The vertex algebra $\mathcal{SF}(1)^{GL(1)}$ is isomorphic to the singlet vertex algebra $\mathcal{M}(2)$ of central charge $c=-2$ (cf. [30]).
- The vertex algebra $\mathcal{SF}(2)^{GL(2)}$ is isomorphic to the parafermion vertex algebra $N_{-1}(\mathfrak{sl}(2))$.
- The vertex algebra $\mathcal{SF}(3)^{GL(3)}$ is isomorphic to the parafermion vertex algebra of the Breshadsky-Polykov vertex algebra $W_k(\mathfrak{sl}(3), f_{min})$ at level $k=-5/2$.
- The vertex algebra $\mathcal{SF}(d)^{Sp(2d,\mathbb{C})}$ is isomorphic to the simple principal affine W-algebra $W_k(\mathfrak{sp}(2d), f_{pr})$ at level $k=-d-1/2$ (and central charge $c=-2d$). It is freely generated by fields:

$$W^m(z) = \frac{1}{(m-2)!}\sum_{i=1}^{d}\big(: b_i(z)\partial_z^{m-2}c_i(z) : + : \partial_z^{m-2}b_i(z)c_i(z) :\big),\ m=2,4,\dots,2d.$$

- $\mathcal{SF}(d)$ is a completely reducible as $W_k(\mathfrak{sp}(2d), f_{pr}) \times \mathfrak{sp}(2d)$–modules have

$$\mathcal{SF}(d) \cong \bigoplus_{\mu \in P_+} V_{\mathfrak{sp}(4)}(\mu) \otimes E_\mu \tag{2.2}$$

  where E_μ is an irreducible $W_k(\mathfrak{sp}(4))$–module. The character of E_μ is given in [14]:

$$\mathrm{ch}[E_\mu] = \frac{q^{m^2+\frac{n^2}{2}+mn+m+\frac{n}{2}}(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})(1-q^{2m+n+3})}{(q;q)_\infty^2}, \tag{2.3}$$

  where $\mu = n\omega_1 + m\omega_2 \in P_+$.
- Assume that G is any reductive Lie subgroup of $Sp(2d,\mathbb{C})$. Then the orbifold vertex algebra $\mathcal{SF}(d)^G$ is completely reducible as $W_k(\mathfrak{sp}_{2d}, f_{pr}) \times G$–module:

$$\mathcal{SF}(d)^G \cong \bigoplus_{\mu \in P_+} V_{\mathfrak{sp}(4)}(\mu)^G \otimes E_\mu. \tag{2.4}$$

- Let $G_0 = SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C}) \subset Sp(2d,\mathbb{C})$. Then

$$\mathcal{SF}(d)^{G_0} = L(-2,0)^{\otimes d}.$$

3. Weyl modules for $W_k(\mathfrak{sp}(4))$

Let α_1 and α_2 denote the positive simple roots of $\mathfrak{sp}(4)$ with normalization

$$(\alpha_1,\alpha_1)=1,\ (\alpha_2,\alpha_2)=2,\ (\alpha_1,\alpha_2)=-1.$$

Let P_+ be the set of dominant integral weights for $\mathfrak{g}=\mathfrak{sp}(4)$. Let

$$\omega_1=\alpha_1+\frac{1}{2}\alpha_2, \omega_2=\alpha_1+\alpha_2$$

denote the fundamental dominant weights. For $\mu\in P_+$, let $V(\mu)$ be the irreducible, finite-dimensional $\mathfrak{g}$–module with highest weight μ. Let P_+^{even} be the subset of P_+ give by

$$P_+^{even}=\{r\omega_1+s\omega_2|\ r,s\in 2\mathbb{Z}_{\geq 0}.\}$$

Let $\mathcal{T}_{\mu,0}^{k+3}$ denotes that $W_k(\mathfrak{sp}(4))$–module $H_{DS}(L_k(\mu))$ as defined in [9].

Proposition 3.1. *For $\lambda=n\omega_1+m\omega_2\in P^+$ we have*

$$\mathrm{ch}[T_{\lambda,0}^{k+3}](q)=$$
$$\frac{q^{m^2+\frac{n^2}{2}+mn+m+\frac{n}{2}}(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})(1-q^{2m+n+3})}{(q;q)_\infty^2}.$$

Proof. According to Arakawa-Frenkel [9],

$$\mathrm{ch}[T_{\lambda,0}^{k+3}](q)=\frac{q^{\tilde{\Delta}_{\lambda,0}^{\frac{1}{2}}}}{(q;q)_\infty^2}\sum_{w\in W}(-1)^{\ell(w)}q^{-\langle w(\lambda+\rho),\rho^\vee\rangle}$$

where

$$\tilde{\Delta}_{\lambda,0}^{\frac{1}{2}}=(\lambda,\lambda+2\rho)+\langle\rho,\rho^\vee\rangle$$

Plugging in $\lambda=n\omega_1+m\omega_2$ immediately gives

$$(\lambda,\lambda+2\rho)=m^2+mn+\frac{n^2}{2}+3m+2n,\ \langle\rho,\rho^\vee\rangle=\frac{7}{2}$$
$$\sum_{w\in W}(-1)^{\ell(w)}q^{-\langle w(\lambda+\rho),\rho^\vee\rangle}$$
$$=q^{-\frac{7}{2}-2m-\frac{3}{2}n}(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})(1-q^{2m+n+3}),$$

which proves the formula. ∎

Conjecture 3.2. *Let $\kappa=1/2$. Then $T_{\mu,0}^{\kappa}\cong E_\mu$.*

Proposition 3.1 and the character formula for E_μ from [14] imply that this conjecture is true at the level of characters.

4. Structure of $N_{-1}(\mathfrak{sl}(2))$ and $N_{-1}(\mathfrak{sl}(2))^+$

In this section we present the decomposition of parafermion vertex algebra $N_{-1}(\mathfrak{sl}(2))$ and its $\mathbb{Z}_2$–orbifold $N_{-1}(\mathfrak{sl}(2))^+$ as a $W^{-5/2}(\mathfrak{sp}(4))$–module.

4.1. *Structure of* $N_{-1}(\mathfrak{sl}(2))$. We need the following result which easily follows from [22, Section 3].

Lemma 4.1. *$V(\lambda)^{GL(2)}$ is at most 1-dimensional. Moreover,* $\dim V(\lambda)^G = 1$ *if and only if* $\lambda \in P_+^{even}$.

Proof. We use the branching rules formula for the restriction $GL(n) \subset SP(2n)$ in Subsection 2.3.2 of [22], in the special case $n = 2$. Then for every dominant integral weight λ, the trivial representation appears in the decomposition of $SP(4)$–module $V(\lambda)$ as $GL(2)$–module if and only if $\lambda \in P_+^{even}$ and then multiplicity is one. □

Using Lemma 4.1 and the decomposition (2.4) we get:

Proposition 4.2. $N_{-1}(\mathfrak{sl}(2)) = \bigoplus_{\mu \in P_+^{even}} E_\mu$.

4.2. *Structure of* $N_{-1}(\mathfrak{sl}(2))^+$. $N_k(\mathfrak{sl}(2))$ has an involution θ induced by Chevalley's involution of $\mathfrak{sl}(2)$. It acts on the standard generators as follows: $e \to f$, $f \to e$, and $h \to -h$. This automorphism θ is uniquely determined by $\theta(W_3) = -W_3$, where W_3 is the weight 3 primary generator of $N_k(\mathfrak{sl}(2))$. We focus on the generic case when $V_k(\mathfrak{sl}(2)) = V^k(\mathfrak{sl}(2))$, which holds true for $k = -1$. The fixed point $\mathfrak{sl}_2$-algebra under this automorphism is denoted as $V_k(\mathfrak{sl}(2))^+$, and the fixed point parafermion (sub)algebra is denoted as $N_k(\mathfrak{sl}(2))^+$. Similarly, $V_k(\mathfrak{sl}_2)^-$ and $N_k(\mathfrak{sl}(2))^-$ represent their (-1)-eigenspaces. Additionally, $\hat{\mathfrak{h}}$ refers to the Heisenberg Lie algebra associated with $\mathbb{C}h$, where $\langle h, h \rangle = 2$, and $M(1)$ is the Heisenberg vertex superalgebra contained in $V_k(\mathfrak{sl}(2))$.

Theorem 4.3. *For any generic k, we have the following character formula:*

$$\mathrm{ch}[N_k(\mathfrak{sl}(2))^+](q) = \frac{\sum_{n\geq 1}(-1)^n q^{n(n+1)/2} + \sum_{n\geq 0}(-1)^n q^{n^2}}{(q;q)_\infty^2}.$$

Proof. Denote by $V_k(\mathfrak{sl}(2))(0)$ the zero weight subalgebra of $V_k(\mathfrak{sl}(2))$ under the h-action. Then $V_k(\mathfrak{sl}(2))(0) = M(1) \otimes N_k(\mathfrak{sl}(2))$. The zero weight space is spanned by the monomials:

$$h(-i_1)\cdots h(-i_m)e(-j_1)\cdots e(-j_n)f(-k_1)\cdots f(-k_n)\mathbf{1}.$$

Then writing generating series for the fixed point algebra gives

(4.1)

$$\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] = \frac{1}{(q;q)_\infty}\sum_{n\geq 0}\left(\frac{q^n}{(q)_n}\right)^2 = \mathrm{CT}_x \frac{1}{(q;q)_\infty (qx;q)_\infty (qx^{-1};q)_\infty}$$

We get decompositions:

$$V_k(\mathfrak{sl}(2))(0)^+ = M(1)^+ \otimes N_k(\mathfrak{sl}(2))^+ \oplus M(1)^- \otimes N_k(\mathfrak{sl}(2))^-, \tag{4.2}$$

$$V_k(\mathfrak{sl}(2))(0)^- = M(1)^+ \otimes N_k(\mathfrak{sl}(2))^- \oplus M(1)^- \otimes N_k(\mathfrak{sl}(2))^+. \tag{4.3}$$

We already know

$$\mathrm{ch}[M(1)^{\pm}] = \frac{1}{2}\left(\frac{1}{(q;q)_\infty} \pm \frac{1}{(-q;q)_\infty}\right) \tag{4.4}$$

To compute $V_k(\mathfrak{sl}(2))(0)^+$ we first observe that the trace can be computed on the associated graded algebra $\mathrm{gr}(V_k(\mathfrak{sl}(2))(0))$ which is slightly more convenient due to commutativity. For this computation we use again

$$v := h(-i_1)\cdots h(-i_m)e(-j_1)\cdots e(-j_n)f(-k_1)\cdots f(-k_n)\mathbf{1} \tag{4.5}$$

Observe that the automorphism θ maps v to

$$(-1)^m h(-i_1)\cdots h(-i_m)f(-j_1)\cdots f(-j_n)e(-k_1)\cdots e(-k_n)\mathbf{1}.$$

To compute the character we need

$$\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)^{\pm}] = \frac{1}{2}\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] \pm \frac{1}{2}\mathrm{tr}_{V_k(\mathfrak{sl}(2))(0)}\theta q^{L(0)}. \tag{4.6}$$

Notice that the last trace computed on the set of monomials (4.5) is non-trivial if and only if $i_1 = j_1, \dots, j_n = k_n$. Counting monomials contributing to non-zero trace gives

$$\mathrm{tr}_{V_k(\mathfrak{sl}(2))(0)}\theta q^{L(0)} = \frac{1}{(-q;q)_\infty (q^2;q^2)_\infty}.$$

Using this formula combined with (4.6) and plugging into character formulas for (4.2)-(4.3) gives a 2×2 linear system for $\mathrm{ch}[N_k(\mathfrak{sl}(2))^{\pm}]$. It's easy to see that

$$\mathrm{ch}[N_k(\mathfrak{sl}_2)^+] = \frac{\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)^+]\mathrm{ch}[M(1)^+] - \mathrm{ch}[V_k(\mathfrak{sl}(2))(0)^-]\mathrm{ch}[M(1)^-]}{\mathrm{ch}[M(1)^+]^2 - \mathrm{ch}[M(1)^-]^2}$$

Using again (4.6) and (4.4) quickly gives:

$$\begin{aligned}
&\mathrm{ch}[N_k(\mathfrak{sl}(2))^+] \\
&= (q^2;q^2)_\infty \left(\frac{1}{2}\left(\frac{1}{(q;q)_\infty} + \frac{1}{(-q;q)_\infty}\right)\cdot\left(\frac{1}{2}\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] + \frac{1}{2}\frac{1}{(-q;q)_\infty(q^2;q^2)_\infty}\right)\right. \\
&\left. - \frac{1}{2}\left(\frac{1}{(q;q)_\infty} - \frac{1}{(-q;q)_\infty}\right)\cdot\left(\frac{1}{2}\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] - \frac{1}{2}\frac{1}{(-q;q)_\infty(q^2;q^2)_\infty}\right)\right) \\
&= \frac{1}{2}\frac{(q^2;q^2)_\infty}{(-q;q)_\infty}\mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] + \frac{1}{2}\frac{1}{(q^2;q^2)} \\
&= \frac{1}{2}(q;q)_\infty \mathrm{ch}[V_k(\mathfrak{sl}(2))(0)] + \frac{1}{2}\frac{1}{(q^2;q^2)},
\end{aligned}$$

as claimed. To prove the second formula observe that we can express the character of $N_k(\mathfrak{sl}(2))$ using false theta series in the form

$$\frac{1}{2}\left(\frac{1+2\sum_{n\geq 1}(-1)^n q^{n(n+1)/2}}{(q;q)_\infty^2}+\frac{(q;q^2)_\infty(q;q)_\infty}{(q;q)_\infty^2}\right)$$
$$=\frac{1}{(q;q)_\infty^2}\left(\frac{1}{2}+\sum_{n\geq 1}(-1)^n q^{n(n+1)/2}+\frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)$$
$$=\frac{1}{(q;q)_\infty^2}\left(\sum_{n\geq 1}(-1)^n q^{n(n+1)/2}+\sum_{n\geq 0}(-1)^n q^{n^2}\right),$$

where in the penultimate line we used the Jacobi triple product identity.

□

The previous theorem has the following generalization: let $\mathfrak{g}$ be a simple Lie algebra and $N_k(\mathfrak{g})$ the associated parafermion algebras. Then

$$\mathrm{ch}[N_k(\mathfrak{g})^+]=\frac{1}{2}\mathrm{ch}[N_k(\mathfrak{g})]+\frac{1}{2}\frac{\left(\sum_{n\in\mathbb{Z}}(-1)q^{n^2}\right)^m}{(q;q)_\infty^{2m}}$$

where m is the number of positive roots of $\mathfrak{g}$.

The following result concerning q-series is independent of representation theory.

Proposition 4.4. *We have the following identities:*

$$1+2\sum_{n\geq 1}(-1)^n q^{n(n+1)/2}$$
(4.7)
$$=\sum_{\substack{m\in 2\mathbb{Z}_{\geq 0}\\ n\in 2\mathbb{Z}_{\geq 0}}} q^{m^2+\frac{n^2}{2}+mn+m+\frac{n}{2}}(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})(1-q^{2m+n+3}),$$

$$\sum_{n\geq 1}(-1)^n q^{n(n+1)/2}+\sum_{n\geq 0}(-1)^n q^{n^2}$$
(4.8)
$$=\sum_{\substack{m\in 2\mathbb{Z}_{\geq 0}\\ n\in 4\mathbb{Z}_{\geq 0}}} q^{m^2+\frac{n^2}{2}+mn+m+\frac{n}{2}}(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})(1-q^{2m+n+3}).$$

Proof. We prove the first identity only, the second identity can be derived in a similar manner. To do this, we first perform the substitutions $m\to 2m$ and $n\to 2n$ on the right-hand side of (4.7), which allows us to rewrite the summations as over $n\geq 0$ and $m\geq 0$. Next, we simplify the exponent $4m^2+4mn+2n^2+2m+n$ by completing the square, obtaining $2(n+m)^2+2m^2+(n+m)+m$, and then let $n+m\to n$. Expanding terms in parentheses of (4.7) and completing squares in the exponents gives:

$$\sum_{n\geq 0}\sum_{m=0}^{n} q^{2n^2+2m^2+m+n}(1-q^{m+1})(1-q^{n-m+1})(1-q^{n+2})(1-q^{n+m+3})$$

$$=\sum_{(a,b)\in S_+}\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+a)^2+2(m+b)^2+(n+a)+(m+b)}$$

$$-\sum_{(a,b)\in S_-}\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+a)^2+2(m+b)^2+(n+a)+(m+b)},$$

where $S_+=\{(0,0),(\frac{1}{2},1),(1,-\frac{1}{2}),(\frac{3}{2},\frac{1}{2})\}$ and $S_-=\{(0,\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}),(1,1),(\frac{3}{2},0)\}$ are shift vectors for the summations variables. Next we pair elements from S_+ with S_- so that differences between shift vectors are integral: $(0,0)$ with $(1,1)$, $(1,-\frac{1}{2})$ with $(0,\frac{1}{2})$, $(\frac{3}{2},\frac{1}{2})$ with $(\frac{1}{2},-\frac{1}{2})$, and $(\frac{1}{2},1)$ with $(\frac{3}{2},0)$. This pairing results in cancellation of terms among the corresponding double sums:

$$\sum_{n\geq 0}\sum_{m=0}^{n} q^{2n^2+2m^2+n+m}-\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+1)^2+2(m+1)^2+(n+1)+(m+1)}$$

$$=\sum_{n\geq 0}\sum_{m=0}^{n} q^{2n^2+2m^2+n+m}-\sum_{n\geq 1}\sum_{m=1}^{n+1} q^{2n^2+2m^2+n+m}=\sum_{n\geq 0} q^{2n^2+n}.$$

Similarly,

$$\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+1)^2+2(m-\frac{1}{2})^2+(n+1)+(m-\frac{1}{2})}-\sum_{n\geq 0}\sum_{m=0}^{n} q^{2n^2+2(m+\frac{1}{2})^2+n+(m+\frac{1}{2})}$$

$$=\sum_{n\geq 1} q^{2n^2+n}-\sum_{n\geq 1} q^{n^2},$$

$$\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+\frac{3}{2})^2+2(m+\frac{1}{2})^2+(n+\frac{3}{2})+(m+\frac{1}{2})}-\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+\frac{1}{2})^2+2(m-\frac{1}{2})^2+(n+\frac{1}{2})+(m-\frac{1}{2})}$$

$$=-\sum_{n\geq 0} q^{(2n+1)(n+1)},$$

$$\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+\frac{1}{2})^2+2(m+1)^2+(n+\frac{1}{2})+(m+1)}-\sum_{n\geq 0}\sum_{m=0}^{n} q^{2(n+\frac{3}{2})^2+2m^2+(n+\frac{3}{2})+m}$$

$$=\sum_{n\geq 1} q^{(n+1)^2}-\sum_{n\geq 1} q^{(2n+1)(n+1)}.$$

Adding these four identities proves the assertion (4.7).

□

Based on the previous result and formula (2.3), we get:

THEOREM 4.5. *With* $\kappa = \frac{1}{2}$ *and* $\mu = n\omega_1 + m\omega_2$ *as before, we have*

$$N_{-1}(\mathfrak{sl}(2))^+ = \sum_{\substack{m\in 2\mathbb{Z}_{\geq 0} \\ n\in 4\mathbb{Z}_{\geq 0}}} E_\mu. \tag{4.9}$$

PROOF. Proposition 4.4 implies that

$$\mathrm{ch}[N_{-1}(\mathfrak{sl}(2))^+] = \sum_{\substack{m\in 2\mathbb{Z}_{\geq 0} \\ n\in 4\mathbb{Z}_{\geq 0}}} \mathrm{ch}[E_\mu]. \tag{4.10}$$

Since we know that $N_{-1}(\mathfrak{sl}(2))^+ \subset N_{-1}(\mathfrak{sl}(2)) \subset \mathcal{SF}(2)$ is a completely reducible $W^{-5/2}(\mathfrak{sp}(4))$–module, then Proposition 4.2 implies that

$$N_{-1}(\mathfrak{sl}_2)^+ = \sum_{\substack{m\in 2\mathbb{Z}_{\geq 0} \\ n\in 2\mathbb{Z}_{\geq 0}}} a(n,m) E_\mu,$$

where $a(n,m) \in \{0,1\}$ are multiplicities with $a(0,0) = 1$.

Observe that for $n \equiv 2 \mod 4$ we have

$$\mathrm{ch}[E_\nu] = q^{a_\nu}(1 + O(q)), \qquad \alpha_\nu \in 2\mathbb{N} - 1,$$

while for $n \equiv 0 \mod 4$ we have

$$\mathrm{ch}[E_\mu] = q^{a_\mu}(1 + O(q)), \quad a_\mu \in 2\mathbb{N},$$

so lowest conformal weights have *different* parity if n are incongruent modulo 4.

Let us reparametrize all lowest conformal weights with $n \equiv 0 \mod 4$ as $\{\mu_i\}_{i\geq 0}$ such that $a_{\mu_0} \leq a_{\mu_1} \leq a_{\mu_2} \leq \cdots$. Clearly $a_{\mu_0} = 0$. Similarly we reparametrize all lowest conformal weights with $n \equiv 2 \mod 4$ as $\{\nu_i\}_{i\geq 0}$ such that $a_{\nu_1} \leq a_{\nu_2} \leq a_{\nu_3} \leq \cdots$.

Claim: $a(n,m) = 1$ for $n \equiv 0 \mod 4$ and $a(n,m) = 0$ for $n \equiv 2 \mod 4$.

To prove this claim, we will demonstrate that the right-hand side of (4.10) is the only possible representation of the character of $\mathrm{ch}[N_{-1}(sl_2)^+]$. In other words if

$$\mathrm{ch}[N_{-1}(\mathfrak{sl}(2))^+] = \sum_{\substack{m\in 2\mathbb{Z}_{\geq 0} \\ n\in 2\mathbb{Z}_{\geq 0}}} a(n,m)\mathrm{ch}[E_\mu]. \tag{4.11}$$

we will prove that $a(n,m)$ are as claimed. By checking the initial terms in the q-expansion we se that only E_{μ_i} (with $n \equiv 0 \mod 4$) contribute to the character. Notice also that for every k such that $a_{\mu_k} > a_{\mu_{k-1}}$ from (4.10) we get a congruence

$$\mathrm{ch}[N_{-1}(\mathfrak{sl}(2))^+] \equiv \mathrm{ch}[E_{\mu_0}] + \cdots + \mathrm{ch}[E_{\mu_{k-1}}] \mod (q^{a_{\mu_k}}), \tag{4.12}$$

because all coefficients up to $q^{a_{\mu_k}}$ agree with the character of $N_{-1}(\mathfrak{sl}(2))^+$.

Suppose the aforementioned claim is false. Then we can find a decomposition with the smallest $k > 1$ in which E_{μ_k} does not appear in the character decomposition. We can express it as follows

$$\mathrm{ch}[N_{-1}(\mathfrak{sl}(2))^+] = \mathrm{ch}[E_{\mu_0}] + \cdots + \mathrm{ch}[E_{\mu_{k-1}}] + \widehat{\mathrm{ch}[E_{\mu_k}]} + \mathrm{ch}[E_{\nu_j}] + \cdots,$$

where E_{ν_j} is the first module with $n \equiv 2 \mod 4$ that appears in the decomposition and $\widehat{\cdot}$ indicates that the term is omitted. Assume first that $\mu_k > \mu_{k-1}$. Then we clearly cannot have $a_{\nu_j} < a_{\mu_k}$ because it would contradict (4.12). Therefore we must have $a_{\nu_j} = a_{\mu_k}$. But this would contradict the fact that a_{ν_i} and a_{μ_k} have different parity. If $\mu_{k-1} = \mu_k$ then again we would require $\mu_k = \nu_j$, a contradiction. □

Using the theorem and the fact that $N_{-1}(\mathfrak{sl}(2))^+$ is generated by primaries of degrees $2, 4, 6, 8, 10$ [23], we can determine that the first non-trivial occurrence arises at degree 14.

Remark 4.6. The approach to Theorem 4.3 can be facilitated using a result established with Wang in [6]. In that paper, we demonstrated the following decomposition:

$$N_{-\frac{3}{2}}(\mathfrak{sl}(2)) = \bigoplus_{m\geq 0} L(-10, 2m(m-1), 0) \tag{4.13}$$

where $L(-10, 2m(m-1), 0)$ denotes irreducible (Weyl) modules for the affine W-algebra $W_k(\mathfrak{sl}(3))$, with $k = -\frac{5}{2}$. By employing the explicit construction described in [6], we observe that each module in the aforementioned decomposition remains invariant under θ. Consequently, in order to compute $\mathrm{ch}[N_{-\frac{3}{2}}(\mathfrak{sl}(2))^+]$, it suffices to determine $\mathrm{ch}[L(-10, 2m(m-1), 0)^+]$. To achieve this, we rely on a BGG-type resolution for $L(-10, 2m(m-1), 0)$ as outlined in [9]. We assume the applicability of the approach in [9] to our modules, allowing us to obtain the following BGG-type resolution of the irreducible module $L(-10, 2m(m-1), 0)$ using Verma modules:

$$0 \to M(2m(m+1)) \to M(2m(m+\frac{1}{2}))^{\oplus 2} \to M(2m(m-\frac{1}{2}))^{\oplus 2} \to M(2m(m-1)) \to 0.$$

Considering that θ induces $\alpha_1 \leftrightarrow \alpha_2$ for simple roots, we observe that the two direct summands in the resolution also interchange under θ. Utilizing this fact, we can establish $\mathrm{ch}[L(-10, 2m(m-1), 0)^+] = \frac{q^{2m(m-1)}(1-q^m)^2(1-q^{2m})}{(q;q)_\infty^2}$. By summing over m, we then establish the claim presented in Theorem 4.3.

5. The vertex algebra $\mathcal{M}(2)^{\otimes 2}$ as $N_{-1}(\mathfrak{sl}(2))$–module.

Our first result uses our previous work [4].

THEOREM 5.1. *We have:*

$$\mathcal{M}(2)^{\otimes 2} = \bigoplus_{s=0}^{\infty} N_{-1}(2s).$$

PROOF. By [4], $\mathcal{M}(2)^{\otimes 2}$ is isomorphic to a parafermion algebra of a charge zero component of two copies of $\beta\gamma$ vertex algebra which is exactly algebra denoted by $\mathcal{V}_0^{(1)}$ in [5]. Using decomposition

$$\mathcal{V}_0^{(1)} = \bigoplus_{s=0}^{\infty} L_{-1}(2s\omega_1),$$

and same arguments as in [6], we get $\mathcal{M}(2)^{\otimes 2} = \bigoplus_{s=0}^{\infty} N_{-1}(2s)$. $\square$

The character of $\mathcal{M}(2)^{\otimes 2}$ appears as the constant term of the Schur's index of type (A_1, D_2) (here $D_2 = A_1 \times A_1$).

COROLLARY 5.2. *We have*

$$\mathrm{ch}[\mathcal{M}(2)^{\otimes^2}](q) = \frac{\sum_{n_1 \geq 0, n_2 \in \mathbb{Z}} \mathrm{sgn}(n_2)(-1)^{n_1} q^{\frac{n_1(n_1+1)}{2} + n_1 n_2 + n_2^2 + n_2}}{(q;q)_\infty^2}.$$

PROOF. Note that (see [12] for instance):

$$\mathrm{ch}[\mathcal{M}(p)](q) = \frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n) q^{2n^2+n} = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q)_k^2}. \tag{5.1}$$

We also have:

$$\begin{aligned} \mathrm{ch}[N_{-1}(2s)](q) &= q^{s(s+1)} \mathrm{CT} \frac{(x^{-s} + \cdots + 1 + \cdots + x^s)}{(xq;q)_\infty (x^{-1}q;q)_\infty} \\ &= \frac{q^{s(s+1)} \left(\Phi_0(q) + \Phi_{-1}(q) - 2\Phi_{-s-1}(q)\right)}{(q;q)_\infty^2}. \end{aligned}$$

$$\begin{aligned}
\mathrm{ch}[\mathcal{M}(2)](q)^2 &= \sum_{s=0}^{\infty} q^{s(s+1)} \mathrm{CT}_x \frac{(x^{-s}+\cdots+1+\cdots+x^s)}{(xq;q)_\infty (x^{-1}q;q)_\infty} \\
&= \mathrm{CT}_x \frac{\sum_{s=0}^{\infty} \frac{x^{s+1/2}-x^{-s-1/2}}{x^{1/2}-x^{-1/2}} q^{s(s+1)}}{(xq;q)_\infty (x^{-1}q;q)_\infty} \\
&= \frac{1}{(q;q)_\infty^2} \sum_{m\in\mathbb{Z}} \sum_{s\geq |m|} q^{s(s+1)} \left(\Phi_m(q) - \Phi_{m-1}(q)\right) \\
&= \frac{1}{(q;q)_\infty^2} \Big(\sum_{n_1,n_2\geq 0} (-1)^{n_1} q^{\frac{n_1(n_1+1)}{2}+n_1n_2+n_2^2+n_2} \\
&\quad - \sum_{n_1,n_2\geq 0} (-1)^{n_1} q^{\frac{n_1(n_1+1)}{2}-n_1(n_2+1)+n_2^2+n_2} \Big) \\
&= \frac{\sum_{n_1\geq 0, n_2\in\mathbb{Z}} \mathrm{sgn}(n_2)(-1)^{n_1} q^{\frac{n_1(n_1+1)}{2}+n_1n_2+n_2^2+n_2}}{(q;q)_\infty^2}.
\end{aligned}$$

□

Using the same arguments as in [6] we get:

Corollary 5.3. *$\mathcal{M}(2)^{\otimes 2}$ is generated by $N_{-1}(\mathfrak{sl}_2) + N_{-1}(2\omega_1)$. In other words, it is generated by $N_{-1}(\mathfrak{sl}_2)$ and a primary field W^2 of conformal weight 2.*

Remark 5.4. This decomposition is an analog of the decomposition from [6] :

$$\mathcal{W}^0(2)_{A_2} = N_{-3/2}(\mathfrak{sl}(3)) = \bigoplus_{j=0}^{\infty} N_{-3/2}(2j).$$

6. S_2-permutation orbifold of $\mathcal{M}(2)$

In this part we consider the S_2-permutation orbifold of $W^k(\mathfrak{sl}(3))$ of level k and central charge $c(k) = 2 - 24\frac{(k+2)^2}{k+3}$. We denote the usual generators of $W^k(\mathfrak{sl}(3))$ by T (Virasoro) and W (primary generator of degree 3) satisfying the usual OPE with central charge $c = c(k)$ (we will use parametrization by the central charge). We denote by $W_1 = W \otimes \mathbf{1}$, $W_2 = \mathbf{1} \otimes W$, and $T_p = T\otimes\mathbf{1} + \mathbf{1}\otimes T$, $T_m = T\otimes\mathbf{1} - \mathbf{1}\otimes T$, and we also let $W_p = W\otimes\mathbf{1} + \mathbf{1}\otimes W$ and $W_m = W\otimes\mathbf{1} - \mathbf{1}\otimes W$. Then $\mathcal{W}_c := W_c(sl(3))^{\otimes 2}$ is strongly and freely generated by T_p, T_m, W_p and W_m. For brevity we introduce

$$U(a,b) :=: W_m^{(a)} W_m^{(b)} :$$
$$V(a,b) :=: T_m^{(a)} T_m^{(b} :$$
$$Z(a,b) :=: T_m^{(a)} W_m^{(b)} :$$

elements fixed by the S_2-automorphism switching tensor factors.

Next result was recently proven by M. Penn and one of us.

THEOREM 6.1. *[27] For all but finitely many values of c, which includes $c = -2$, the orbifold subalgebra $\mathcal{W}_c^{S_2}$ is strongly generated by the set*

$$\{T_p, W_p, U(0,0), U(2,0), U(4,0), V(0,0), V(2,0),$$
$$V(4,0), Z(0,0), Z(1,0), Z(2,0), Z(3,0)\}.$$

Moreover, this set is minimal and generators can be modified so that the orbifold is a W-algebra of type $(2,3,4,5,6^3,7,8^3,9)$.

6.1. *The S_2-orbifold of $\mathcal{M}(2)$.* We can further reduce the set of generators in the theorem by taking into account null vectors at central charge $c = -2$. A straightforward computation yields the following expressions (using OPE notation for simplicity):

$$v_{sing1} = -\frac{19}{54} : T'T' : -\frac{14}{27} : T''T : -\frac{16}{27} : TTT : + : WW : + \frac{4T^{(4)}}{81}$$
$$v_{sing2} = -\frac{3}{2} : T'W : + : TW' : -\frac{W^{(3)}}{8}$$

which are two linearly independent null vectors in $\mathcal{M}(2)$. Thus, they belong to the maximal ideal of $W^{-3/2}(\mathfrak{sl}(3))$. Applying the symmetrization map, the S_2-fixed vectors:

$$sv_{sing1} := v_{sing1} + \sigma(v_{sing1}), \quad sv_{sing2} := v_{sing1} + \sigma(v_{sing2}),$$

where σ is the non-trivial element in S_2, are now null vectors in $\mathcal{W}_{c=-2}^{S_2}$. This allows us to eliminate two generators $Z(3,0)$, $V(2,0)$ from the set in Theorem 6.1 and keep only $U(2,0)$. Using descendants of sv_{sing1} and sv_{sing2} allows to eliminate two generators of degree 8 and also generators of degree 9. This keeps us with generators up to degree 8. We can now conclude

THEOREM 6.2. *The vertex algebra $(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2}$ is a W-algebra of type $(2,3,4,5,6,7,8)$ with a minimal strong set of generators*

$$\{T_p, W_p, V(0,0), Z(0,0), V(2,0), Z(2,0), U(2,0)\}.$$

All generating vectors (except T_p) can be modified to be primary.

Of course this vertex algebra is not generated by the degree 3 generator W_p so it cannot be analyzed using universal $W_\infty[\lambda]$ algebra as in [26].

REMARK 6.3. One can prove Theorem 6.2 directly using symplectic fermion construction without going through Theorem 6.1 and singular vectors.

PROPOSITION 6.4. *The vertex algebra $(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2}$ is completely reducible as $N_{-1}(\mathfrak{sl}(2))$–module with the following decomposition:*

$$(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2} = \bigoplus_{j=0}^{\infty} N_{-1}(4j\omega_1). \tag{6.1}$$

PROOF. If we prove

$$\mathrm{ch}[(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2}](q) = \sum_{j \geq 0} \mathrm{ch}[N_{-1}(4j\omega_1)](q). \tag{6.2}$$

it implies that (6.1) holds. This is because $N_{-1}(\mathfrak{sl}(2))$ is fixed under the automorphism group, and thus we have

$$(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2} = \bigoplus_{j=0}^{\infty} a(2j) N_{-1}(2j\omega_1),$$

where $a(m) \in \{0,1\}$, and $a(0) = 1$. Observe $\mathrm{ch}[N_{-1}(2j\omega_1)] = q^{j(j+1)}(1 + O(q))$, so the lowest conformal weight of $N_{-1}(2j\omega_1)]$ is $j(j+1)$. If $a(4j) = 0$ for some j, then because of (6.2), it is easy to see that the character $(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2}$ agree with $\sum_{j \geq 0} a(2j)\mathrm{ch}[N_{-1}(2j\omega_1)]$.

We are left to prove relation (6.2). We require (2.1):

$$\mathrm{ch}[(\mathcal{M}(2) \otimes \mathcal{M}(2))^{S_2}] = \frac{1}{2}\mathrm{ch}[\mathcal{M}(2)](q)^2 + \frac{1}{2}\mathrm{ch}[\mathcal{M}(2)](q^2).$$

We already know that

$$\frac{1}{2}\mathrm{ch}[\mathcal{M}(2)](q)^2 = \frac{1}{(q)_\infty^2} \sum_{s \geq 0} q^{s(s+1)} \left(\frac{1}{2} - \Phi_{-s-1}(q) \right)$$

and it is not hard to see that

$$\frac{1}{2}\mathrm{ch}[\mathcal{M}(2)](q^2) = \frac{1}{2} \frac{\sum_{n \geq 0} (-1)^n q^{n(n+1)}}{(q^2;q^2)_\infty} =$$
$$\frac{1}{2} \frac{\left(\sum_{n \geq 0} (-1)^n q^{n(n+1)}\right) \cdot \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}\right)}{(q)_\infty^2}.$$

The right-hand side of (6.2) equals

$$\frac{1}{(q)_\infty^2} \sum_{s \geq 0} q^{2s(2s+1)} \left(1 - 2\Phi_{-2s-1}(q)\right).$$

To finish the proof we have to show that

$$-\sum_{s \geq 0} q^{2s(2s+1)} \Phi_{-2s-1}(q) + \sum_{s \geq 0} q^{(2s+1)(2s+2)} \Phi_{-2s-2}(q)$$

$$= \left(\sum_{n\geq 0}(-1)^n q^{n(n+1)}\right)\cdot\left(\sum_{n\geq 1}(-1)^n q^{n^2}\right),$$

or equivalently

$$\sum_{s,m\geq 0}(-1)^{s+m+1}q^{s(s+1)-(s+1)m+\frac{m^2+m}{2}} = \left(\sum_{n\geq 0}(-1)^n q^{n(n+1)}\right)\cdot\left(\sum_{n\geq 1}(-1)^n q^{n^2}\right).$$

To prove the last identity we rewrite the left-hand side as

$$\sum_{s,m\geq 0}(-1)^{s+m+1}q^{s(s+1)-(s+1)m+\frac{m^2+m}{2}} = \sum_{s,m\geq 0}(-1)^{s+m+1}q^{\left(\frac{m}{2}\right)^2+(s-\frac{m}{2})(s-\frac{m}{2}+1)},$$

and introduce let $t = s-m$, so that the summation is over m and t. Then it can be easily shown that the resulting double sum agrees with the one on the right-hand side. ∎

COROLLARY 6.5. *Vertex algebra* $(\mathcal{M}(2)\otimes\mathcal{M}(2))^{S_2}$ *is generated by* $N_{-1}(\mathfrak{sl}(2)) + N_{-1}(4\omega_1)$. *In particular,* V *is generated by* $N_{-1}(\mathfrak{sl}(2))$ *and a primary field* W^6 *of conformal weight* 6.

PROOF. From Theorem 6.2 and Proposition 6.4, we see that all generators of $(\mathcal{M}(2)\otimes\mathcal{M}(2))^{S_2}$ are within $N_{-1}(\mathfrak{sl}(2)) + N_{-1}(4\omega_1)$. Through direct computation we see that the S_2-orbifold subalgebra is generated by a primary vector of degree 6. However, any such vector necessarily belongs to $N_{-1}(\mathfrak{sl}(2)) + N_{-1}(4\omega_1)$. Hence, we can conclude our argument. ∎

6.2. *Further W-algebras and Schur's indices.* We use the decomposition from [5] (see also [3]):

$$\mathcal{V}_0^{(p)} = \bigoplus_{s=0}^{\infty} L_{-2+1/p}(2s\omega_1).$$

Let $\mathcal{U}^0(p) := \mathrm{Com}(M(1),\mathcal{V}_0^{(p)})$. The character of $\mathcal{U}^0(p)$ is given by the following result:

PROPOSITION 6.6. *We have:*

$$\mathrm{ch}[\mathcal{U}^0(p)] = \frac{\sum_{n_1\geq 0, n_2\in\mathbb{Z}}\mathrm{sgn}(n_2)(-1)^{n_1}q^{\frac{n_1(n_1+1)}{2}+n_1n_2+p(n_2^2+n_2)}}{(q;q)_\infty^2},$$

where $sgn(n) = 1$ *for* $n\geq 0$ *and* 0 *otherwise.*

It is not hard to see using the same approach as in [12] or [11]

PROPOSITION 6.7.

$$q^{\frac{2p+2}{24}}\mathrm{ch}[\mathcal{U}^0(p)] = \eta(\tau)^2\eta(p\tau)^2\cdot\mathrm{CT}_{\zeta_1,\zeta_2}\frac{\vartheta(z_1;p\tau)\vartheta(z_2;p\tau)\vartheta(z_1+z_2;p\tau)}{\vartheta(z_1;\tau)\vartheta(z_2;\frac{p}{2}\tau)\vartheta(z_1+z_2;\frac{p}{2}\tau)},$$

where

$$\vartheta(z;\tau) = -iq^{1/8}\zeta^{-1/2}(q;q)_\infty(\zeta;q)_\infty(\zeta^{-1}q;q)_\infty,$$

is the Jacobi theta function with $\zeta := e^{2\pi i y}$*, and* CT *denotes the constant term with respect to the Fourier expansion in* ζ_i*.*

We end with a conjecture that generalizes a character identity for $\mathrm{ch}[\mathcal{M}(2)^{\otimes^2}]$ from Section 3. Observe that Corollary 5.2 together with (5.1) gives

$$\frac{\sum_{n_1\geq 0, n_2\in\mathbb{Z}} \mathrm{sgn}(n_2)(-1)^{n_1} q^{\frac{n_1(n_1+1)}{2}+n_1n_2+(n_2^2+n_2)}}{(q;q)_\infty^2} = (q;q)_\infty^2 \sum_{n_1,n_2\geq 0} \frac{q^{n_1+n_2}}{(q)_{n_1}^2 (q)_{n_2}^2}.$$

Some clues from 4d/2d dualities in physics suggest that the following identity should hold (for $k \geq 0$):

$$\frac{\sum_{n_1\geq 0, n_2\in\mathbb{Z}} \mathrm{sgn}(n_2)(-1)^{n_1} q^{\frac{n_1(n_1+1)}{2}+n_1n_2+(k+1)(n_2^2+n_2)}}{(q;q)_\infty^2}$$
$$= (q)_\infty^{2k+2} \sum_{n_1,\dots,n_{2k+2}\geq 0} \frac{q^{\mathbf{n}\cdot A_{D_{2k+2}}\cdot \mathbf{n}^T + n_1+\cdots+n_{2k+2}}}{(q)_{n_1}^2 (q)_{n_2}^2 \cdots (q)_{n_{2k+2}}^2},$$

where $\mathbf{n} = (n_1, ..., n_{2k+2})$ and $A_{D_{2k+2}}$ is the adjacency matrix of the Dynkin diagram of type D_{2k+2} (for $k = 0$, we have $D_2 = A_1 \times A_1$, whose Dynkin diagram consists of two nodes with no edges).

7. The symmetric orbifold $(L(-2,0) \otimes L(-2,0))^{S_2}$

Let $\mathcal{SF}(d)$ be the vertex algebra of symplectic fermion. The symmetric group S_d acts naturally on $\mathcal{SF}(d)$.

Recall that $\mathrm{Aut}(\mathcal{SF}(d)) = Sp(2d, \mathbb{C})$. Let

$$G_0 = SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C}).$$

The action of S_d and G_0 commutes, so we have the of the group G_0 acts on $(\mathcal{SF}(d))^{S_d}$.

Here we consider the case $d = 2$.

We use the decomposition

$$\mathcal{SF}(d) = \bigoplus_{\lambda\in P_+} V_{\mathfrak{sp}(4)}(\lambda) \otimes E_\lambda$$

where $V_{sp(4)}(\lambda)$ is irreducible finite-dimensional $sp(4)$–module, with highest weight λ, and E_λ is irreducible $W^{-5/2}(\mathfrak{sp}(4)) = (\mathcal{SF}(d))^{\mathfrak{sp}(4)}$–module. Next we notice that $\dim V_{sp(4)}(\lambda)^{G_0} \leq 1$, and $\dim V_{sp(4)}(\lambda)^{G_0} = 1$ if and only if $\lambda = m\omega_2$ for certain $m \in \mathbb{Z}_{\geq 0}$. (This result follows from [20] or [22]). This proves the first half of the following result.

Theorem 7.1. *We have*

$$L(-2,0) \otimes L(-2,0) = \bigoplus_{m=0}^{\infty} E_{m\omega_2}$$

$$(L(-2,0)\otimes L(-2,0))^{S_2}=\bigoplus_{m=0}^{\infty}E_{2m\omega_2},$$

PROOF. We only have to prove the second relation. Using (2.1) and the Jacobi triple product identity, it is easy to prove

$$\mathrm{ch}[(L(-2,0)\otimes L(-2,0))^{S_2}]=\frac{1}{2}\left(\frac{1}{(q^2;q)_\infty^2}+\frac{1}{(q^4;q^2)}\right)=\frac{(1-q)\sum_{n\geq 0}(-1)^n q^{n^2}}{(q;q)_\infty^2}$$

holds. Also, using

$$\mathrm{ch}[E_{2m\omega_1}]=\frac{q^{4m^2+2m}(1-q)(1-q^{m+1})(1-q^{m+2})(1-q^{2m+3})}{(q;q)_\infty^2}$$

one easily demonstrates that $\mathrm{ch}[(L(-2,0)\otimes L(-2,0))^{S_2}]=\sum_{m\geq 0}\mathrm{ch}[E_{2m\omega_1}]$. Therefore the second relation in the theorem holds at the level of characters. On the other hand, using the fact that $W^{-5/2}(sp(4))$ is fixed under the non-trivial element of S_2 we obtain decomposition

$$(L(-2,0)\otimes L(-2,0))^{S_2}=\bigoplus_{m=0}^{\infty}a(m)E_{m\omega_2},$$

where the multiplicities $a(m)\in\{0,1\}$, and $a(0)=1$. Therefore, it suffices to show that $a(2m-1)=0$ for all $m\geq 1$. It is easy to see that $a(1)=0$ and $a(2)=1$. Then we observe

$$\mathrm{Coeff}_{q^{4m^2-2m}}(q;q)_\infty^2\left(\bigoplus_{m=0}^{\infty}a(m)\mathrm{ch}[E_{m\omega_2}]\right)=a(2m-1)-a(2m-3).$$

But q^{4m^2-2m}, $m\geq 2$, does not appear as term in $(1-q)\sum_{n\geq 0}(-1)^n q^{n^2}$ because $4m^2-2m=n^2$ and $4m^2-2m=n^2+1$ does not have integral solutions except for $m=n=0$, and $m=n=1$, respectively. We conclude that $a(2m-1)=a(2m-3)$ for every $m\geq 2$, which together with $a(1)=0$ gives $a(2m-1)=0$ for every m and completes the proof. □

REMARK 7.2. *One can easily demonstrate that the aforementioned decomposition is essentially the only case in which a $W^k(\mathfrak{sp}(4))$-algebra can be embedded within the tensor product of two Virasoro vertex operator algebras $L(c_1,0)\otimes L(c_2,0)$. Specifically, aside from the case where $c_1=c_2=-2$, we can also observe two degenerate cases that arise due to the presence of singular vectors of degree 4: $c_1=0$ or $c_2=0$, and, $c_1=-22/5$ or $c_2=-22/5$.*

8. Conjectural singlet and triplet type W-algebras for $\mathfrak{sp}(4)$

In this section, we present a conjecture regarding the existence of a $\mathfrak{sp}(4)$ singlet and triplet type vertex algebra, based on the work of Feigin and Tipunin [18], [29] (see also [13]).

As previously mentioned, $V_{\mathfrak{sp}(4)}(\lambda)$ denoted an irreducible highest weight representation of $\mathfrak{sp}(4)$ of highest weight λ. We observe that for $\lambda \in P_+ \cap Q = 2\mathbb{Z}_{\geq 0}\omega_1 + \mathbb{Z}_{\geq 0}\omega_2$, the zero weight subspace $V_{\mathfrak{sp}(4)}(\lambda)_0$ is non-zero. Moreover, using the Weyl character formula we have:

$$\dim V_{\mathfrak{sp}(4)}(n\omega_1 + m\omega_2)_0 = 1 + \frac{m}{2} + \frac{n}{2} + \frac{mn}{2}, \quad n \in 2\mathbb{Z}_{\geq 0}, m \in 2\mathbb{Z}_{\geq 0}$$

and

$$\dim V_{\mathfrak{sp}(4)}(n\omega_1 + m\omega_2)_0 = \frac{(n+1)(m+1)}{2}, \quad n \in 2\mathbb{Z}_{\geq 0}, m \in 2\mathbb{Z}_{\geq 0} + 1.$$

For an even $p \geq 2$, we define two q-series:

$$S_p(q) := \sum_{m \in \mathbb{Z}_{\geq 0}, n \in 2\mathbb{Z}_{\geq 0}} \dim V_{\mathfrak{sp}(4)}(n\omega_1 + m\omega_2)_0 q^{\frac{p}{2}(m^2 + \frac{n^2}{2} + mn + 3m + 2n) - 2m - \frac{3n}{2}}$$
$$\times (1 - q^{m+1})(1 - q^{n+1})(1 - q^{m+n+2})(1 - q^{2m+n+3}).$$

and

$$T_p(q) := \sum_{m \in \mathbb{Z}_{\geq 0}, n \in 2\mathbb{Z}_{\geq 0}} \dim V_{\mathfrak{sp}(4)}(n\omega_1 + m\omega_2) q^{\frac{p}{2}(m^2 + \frac{n^2}{2} + mn + 3m + 2n) - 2m - \frac{3n}{2}}$$
$$\times (1 - q^{m+1})(1 - q^{n+1})(1 - q^{m+n+2})(1 - q^{2m+n+3}),$$

where

$$\dim V_{\mathfrak{sp}(4)}(n\omega_1 + m\omega_2) = \frac{(m+1)(n+1)(m+n+2)(2m+n+3)}{6}. \tag{8.1}$$

CONJECTURE 8.1. *There exist simple vertex algebras $W_{C_2}(p)^0$ and $W_{C_2}(p)$, of central charge $c = 86 - \frac{60}{p} - 30p$ whose characters are given by*

$$\frac{S_p(q)}{(q;q)_\infty^2} \quad \text{and} \quad \frac{T_p(q)}{(q;q)_\infty^2},$$

respectively.

Using [14, 19], it is not hard to see that for $p = 2$ two vertex algebras in question are $\mathcal{M}(2)^{\otimes^2} \hookrightarrow \mathcal{W}(2)^{\otimes^2}$ where $\mathcal{W}(2)$ is the $(1,2)$-triplet vertex algebra. For $p = 4$, we anticipate the existence of vertex algebras with central charge $c = -49$ and of types $(2, 4, 6, 9^2, 11)$ and $(2, 4, 6^5, 9^{10}, 11^5)$, respectively.

Regarding the modular properties of the proposed characters (after adding the $q^{-c/24}$ term) it is worth noting that by utilizing the properties of the Weyl group as described in [13], the T_p-series can be expressed as a summation over a full lattice, taking into account the number of Weyl chambers. Thus, we can represent $T_p(q) = \frac{1}{8}\sum_{m \in \mathbb{Z}, n \in 2\mathbb{Z}}(\cdot)$ (observe that (8.1) makes sense for all n and m). This summation corresponds to a collection of derivatives of rank two theta functions, whose modular properties have been thoroughly studied (see [13]). Consequently, this observation suggests that

$W_{C_2}(p)$ is expected to be *lisse*. (Non)-modular properties of $S_p(q)$ series are more complicated and can be accessed using the methods developed in [11].

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References

[1] T. Abe, *A $\mathbb{Z}_2$–orbifold model of the symplectic fermionic vertex operator superalgebra*, Math. Z. **255** (2007), 755–792

[2] D. Adamović, *Classification of irreducible modules of certain subalgebras of free boson vertex algebra*, J. Algebra **270** (2003), 115–132

[3] D. Adamović, *A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$*, Transform. Groups **21** (2016), 299–327.

[4] D. Adamović and A. Milas, *On some vertex algebras related to $V_{-1}(sl(n))$ and their characters*, Transform. Groups **26** (2021), 1–30.

[5] D. Adamović, T. Creutzig, N. Genra and J. Yang, *The vertex algebras $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$*, Comm. Math. Phys. **383** (2021), 1207–1241.

[6] D. Adamović, A. Milas and Q. Wang, *On parafermion vertex algebras of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ at level -3/2*, Commun. Contemp. Math. **24** (2022), no. 1, Paper No. 2050086, 23 pp.

[7] T. Arakawa, *Representation theory of W-algebras*, Invent. Math. **169** (2007), 219–320.

[8] T. Arakawa, T. Creutzig and A. Linshaw, *Cosets of Bershadsky-Polyakov algebras and rational $\mathcal{W}$-algebras of type A*, Selecta Math. (N.S.) **23** (2017), 2369–2395.

[9] T. Arakawa and E. Frenkel, *Quantum Langlands duality of representations of $\mathcal{W}$-algebras*, Compos. Math. **155** (2019), 2235–2262.

[10] T. Arakawa, C. H. Lam and H. Yamada, *Zhu's algebra, C_2-algebra and C_2-cofiniteness of parafermion vertex operator algebras*, Adv. Math. **264** (2014), 261–295.

[11] K. Bringmann, J. Kaszian, A. Milas and C. Nazaroglu, *Integral representations of rank two false theta functions and their modularity properties*, Res. Math. Sci. **8** (2021), no. 4, Paper No. 54, 31 pp.

[12] K. Bringmann, J. Kaszian, A. Milas and S. Zwegers, *Rank two false theta functions and Jacobi forms of negative definite matrix index*, Adv. in Appl. Math. **112** (2020), 101946, 19 pp.

[13] K. Bringmann and A. Milas, *W-algebras, higher rank false theta functions, and quantum dimensions*, Selecta Math. (N.S.) **23** (2017), 1249–1278.

[14] T. Creutzig and A. Linshaw, *Orbifolds of symplectic fermion algebras*, Trans. Am. Math. Soc. **369** (2017), 467–494.

[15] C. Dong, C. H. Lam, Q. Wang and H. Yamada, *The structure of parafermion vertex operator algebras*, J. Algebra **323** (2010), 371–381.

[16] C. Dong, C. H. Lam and H. Yamada, *W-algebras related to parafermion algebras*, J. Algebra **322** (2009), 2366–2403.

[17] J. Fasquel, *OPEs of rank two W-algebras*, `arXiv:2210.15728`.

[18] B. Feigin and I. Yu Tipunin, *Logarithmic CFTs connected with simple Lie algebras*, Preprint, `arXiv:1002.5047` (2010).

[19] I. Flandoli and S. Lentner, *Logarithmic conformal field theories of type B_n, $\ell = 4$ and symplectic fermions*, J. Math. Phys. **59** (2018), no. 7, 071701, 35 pp.

[20] R. Goodman and N. R. Wallach, Symmetry, representations, and invariants, Grad. Texts in Math. **255**, Springer, Dordrecht, 2009.

[21] M. Gorelik and V. Kac, *On simplicity of vacuum modules*, Adv. Math. **211** (2007), 621–677.

[22] R. Howe, E.-C. Tan and J. F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), 1601–1626.

[23] S. Kanade and A. Linshaw, *Universal two-parameter even spin W_∞-algebra*, Adv. Math. **355** (2019), 106774, 58pp.

[24] J. Lepowsky and M. Primc, Structure of the Standard Modules for the Affine Lie Algebra $A_1^{(1)}$, Contemp. Math., Vol. **46**, American Mathematical Soc., Providence, RI, 1985.

[25] J. Lepowsky and R. L. Wilson, *The structure of standard modules, I. Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. **77** (1984), 199–290.

[26] A. Linshaw, *Universal two-parameter W_∞–algebra and vertex algebras of type $W(2,3,\dots,N)$*, Compos. Math. **157** (2021), 12–82.

[27] A. Milas and M. Penn, *2-permutation orbifolds of W-algebras*, submitted.

[28] A. Milas, M. Penn and C. Sadowski, *Permutation orbifolds of Virasoro vertex algebras and W-algebras*, J. Algebra **570** (2021), 267–296.

[29] S. Sugimoto, *On the Feigin-Tipunin conjecture*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 86, 43 pp.

[30] W. Wang, *$W_{1+\infty}$–algebra, W_3–algebra, and Friedan-Martinec-Shenker bosonization*, Comm. Math. Phys. **195** (1998), 95–111.

Logaritamske verteks algebre vezane za $\mathfrak{sp}(4)$

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Sažetak. U članku navodimo nekoliko rezultata i slutnji vezanih za parafermionske verteks-algebre i s njima povezane logaritamske verteks-algebre. Krećemo od tenzorskog produkta dvije kopije singlet verteks-algebre $\mathcal{M}(2)$, te promatramo različite podalgebre koje se pojavljuju u njezinoj dekompoziciji, uključujući $N_{-1}(sl(2))$ i pripadni $\mathbb{Z}_2$-orbifold $N_{-1}(sl(2))^+$, te S_2-simetrični orbifold singlet verteks-algebre $\mathcal{M}(2)$. Posebno, pokazujemo da se $N_{-1}(sl(2))$ može proširiti do W-algebre tipa $(2,3,4,5,6,7,8)$. Na kraju iznosimo neke slutnje o egzistenciji W–algebri koje generaliziraju triplet i singlet algebru, te analiziramo njihove potencijalne karaktere.

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