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ON REDUCIBILITY OF REPRESENTATIONS INDUCED FROM THE ESSENTIALLY SPEH REPRESENTATIONS AND DISCRETE SERIES

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Dedicated to Marko Tadić on the occasion of his 70th birthday.

Abstract. Let π stand for an essentially Speh representation of the form $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]))$, where ρ is an irreducible cuspidal representation of the general linear group over a non-archimedean local field or its separable quadratic extension, $a \leq 0$, $2a + k > 0$, and $n \geq 1$. Let σ denote a discrete series representation of either symplectic, special odd-orthogonal, or unitary group. We determine when the induced representation $\pi \rtimes \sigma$ reduces.

1. Introduction

Let ρ stand for an irreducible cuspidal representation of the general linear group over a non-archimedean local field or its separable quadratic extension, and let $\delta([\nu^a\rho, \nu^b\rho])$ stand for an irreducible essentially square-integrable representation attached to the segment $[\nu^a\rho, \nu^b\rho]$. The induced representation

$$\delta([\nu^a\rho, \nu^{a+k}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]) \times \cdots \times \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]),$$

where a is a real number, while k and n are non-negative integers such that $n \geq 1$, contains a unique irreducible subrepresentation, which is denoted by $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]))$ and called the essentially Speh representation or shifted Speh representation. We note that, by [20, Theorem 7.5], the essentially Speh representations have a crucial role in the classification of the unitary dual of the general linear group.

We denote by σ a discrete series representation of either symplectic, special odd-orthogonal, or unitary group, and remind the reader that by the

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Mœglin-Tadić classification σ is completely described by the so-called admissible triple, consisting of the Jordan block, the partial cuspidal support, and the ϵ-function.

We describe the reducibility of the induced representation

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

under the assumptions $a \leq 0$ and $2a + k > 0$.

In some restrictive cases an analogous reducibility problem is completely solved. Taking $n = 1$ we obtain the generalized principal series case, for which the reducibility has been described for general a and k in the symplectic and special odd-orthogonal group case in [18], and in the unitary group case in [13]. We note that all the results and proofs appearing in [13] can be used without any change in the symplectic and special odd-orthogonal case. If we additionally restrict ourselves to the case of the strongly positive σ, then the complete composition series of the generalized principal series follow from [17] and [9].

Taking $k = 0$ and general a and n, we obtain the case of representations induced by the Zelevinsky segment representation and the discrete series, which has been initially studied in [10], where the reducibility criterion has been deduced. The complete composition series of the induced representation of such a form have been described in [11], under an additional assumption that a is half-integral.

In the case of cuspidal σ and positive a, a complete description of the composition series of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ has recently been obtained in [3]. We also note that for cuspidal σ reducibility of the representations induced from the essentially Speh ones appears as a particular case of a much more general and important contribution [8].

Finally, reducibility of the induced representations of the form

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \pi_A,$$

for general a, k and n, where π_A stands for an irreducible representation of Arthur type, is described in [2].

Our main result states that the induced representation $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$, under the mentioned conditions on a, k and n, is irreducible if and only if $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \ldots, n-1\}$.

One can use the results of [18] and [12] to describe the reducibility in terms of ρ, k, n, and the admissible triple corresponding to σ.

To obtain the reducibility criterion, we adopt the strategy introduced in [18] and further developed in [19], [10], [12], and [13].

First, if there is an $i \in \{0, 1, \ldots, n-1\}$ such that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ reduces, we note that ρ is a self-dual representation and that the irreducible non-tempered representation

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma)$$

is a subquotient of $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$.

Then, taking the minimal $i \in \{0, 1, \ldots, n-1\}$ such that

$$\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$$

reduces and starting from particular irreducible subquotients constructed in the reducibility proofs appearing in [12], we provide an inductive construction of an irreducible non-tempered subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

which is non-isomorphic to

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma).$$

We note that the mentioned inductive procedure is mostly based on the combination of the intertwining operators method and the calculation of the Jacquet modules using the structural formula.

If $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \ldots, n-1\}$, using the reducibility criterion from [18] and [12], together with [22, Section 7], we deduce a precise description of the Jacquet modules of σ. Such a description enables us to determine the general form of possible irreducible subquotients of $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$. Since such a form is obtained by the calculation of the Jacquet modules, some rather complicated cases, as the one appearing in Proposition 5.5, need to be handled. We note that such a case presents one the main obstructions in this study. The obtained general description of possible irreducible subquotients leads us, after a rather involved calculation, to irreducibility of induced representation of the studied form.

Let us now describe the content of the paper in more detail. In the second section we introduce the notation and some preliminaries, while in the third section we state and prove several technical results which are used afterwards in the paper. In the fourth section we prove the reducibility of $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ when there exists an $i \in \{0, 1, \ldots, n-1\}$ such that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ reduces, and several cases are studied separately. In the fifth section we state and prove the irreducibility results.

2. Preliminaries

Through the paper, we denote by F a non-archimedean local field. We fix one of the following series $\{G_n\}$ of classical groups over F.

In the odd orthogonal group case, we fix an anisotropic orthogonal vector space Y_0 over F of odd dimension and consider the Witt tower based on Y_0. For n such that $2n+1 \geq \dim Y_0$, there is exactly one space V_n in the tower of dimension $2n+1$. Let G_n stand for the special orthogonal group of this space. If V_n stands for the symplectic space of dimension $2n$ in the corresponding Witt tower, we denote by G_n the symplectic group of this space. We also consider the unitary groups $U(n, F'/F)$, where F' stands for a separable quadratic extension of F. There is also an anisotropic unitary space Y_0 over F', and the Witt tower of unitary spaces V_n based on Y_0. We denote by G_n the unitary group of the space V_n of dimension either $2n+1$ or $2n$.

We fix a minimal parabolic subgroup in G_n and consider only the standard parabolic subgroups with respect to this fixed minimal parabolic subgroup. When working with the unitary groups, we let F' denote a separable quadratic extension of F, otherwise let F' denote F. For representations δ_i of $GL(n_i, F')$, $i = 1, 2, \ldots, k$, and representation τ of $G_{n'}$, we denote by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$. We use a similar notation to denote a parabolically induced representation of $GL(m, F')$.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n \geq 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n, F'))$, $R(GL(n, F'))$, and $R(GL) = \oplus_{n \geq 0} R(GL(n, F'))$.

Let n' stand for the Witt index of V_n if V_n is symplectic or even-unitary group, and $n' = n - \frac{1}{2}(\dim_{F'}(Y_0) - 1)$ otherwise. For $\sigma \in \mathrm{Irr}(G_n)$ and $0 \leq k \leq n'$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k, F') \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F')) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n'} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

For $\pi \in \mathrm{Irr}(GL(n, F'))$ we define $m^*(\pi) = \sum_{k=0}^{n} r_{(k)}(\pi) \in R(GL) \otimes R(GL)$, where $r_{(k)}(\pi)$ denotes the normalized Jacquet module of π with respect to the standard parabolic subgroup having the Levi factor equal to $GL(k, F') \times GL(n-k, F')$. We identify $r_{(k)}(\pi)$ with its semisimplification, and then extend m^* linearly to the whole of $R(GL)$.

Let ν stand for a composition of the determinant mapping with the normalized absolute value on F. Let $\rho \in R(GL)$ denote an irreducible supercuspidal representation. By a segment we mean a set of the form $[\rho, \nu^m \rho] := \{\rho, \nu\rho, \ldots, \nu^m \rho\}$, where m is a non-negative integer. The induced representation $\nu^m \rho \times \nu^{m-1} \rho \times \cdots \times \rho$ has a unique irreducible subrepresentation ([23]), denoted by $\delta([\rho, \nu^m \rho])$. Representation $\delta([\rho, \nu^m \rho])$ is essentially square-integrable, and by [23] every irreducible essentially square-integrable representation in $R(GL)$ can be obtained in this way.

The essentially Speh representations are irreducible representations of the form $L(\delta_1, \delta_2, \ldots, \delta_n)$, where $\delta_i \cong \delta([\nu^{a+i-1}\rho, \nu^{a+k+i-1}\rho])$, for $i = 1, 2, \ldots, n$, a real number a, a non-negative integer k, and an irreducible cuspidal representation ρ from $R(GL)$.

For an irreducible smooth representation $\pi \in R(GL)$, let $\widetilde{\pi}$ stand for the contragredient representation of π. If $F = F'$, we say that π is F'/F-selfdual if $\pi \cong \widetilde{\pi}$. If $F \neq F'$, we denote by θ the non-trivial F-automorphism of F', let $\widehat{\pi}$ denote the representation $g \mapsto \widetilde{\pi}(\theta(g))$, and say that the representation π is F'/F-selfdual if $\pi \cong \widehat{\pi}$.

One of the main ingredients in our Jacquet module calculations is the following formula ([7, Theorem 2.1]):

THEOREM 2.1. *Let $\rho \in \mathrm{Irr}(GL(m, F))$ be a supercuspidal representation. Suppose that $a_1, b_1, \ldots, a_m, b_m$ are real numbers such that $b_i - a_i$ is a non-negative integer for $i = 1, 2, \ldots, m$, and that for $i = 1, 2, \ldots, m-1$ we have $a_i < a_{i+1}$ and $b_i < b_{i+1}$. Then the following holds:*

$$\begin{aligned} &m^*(L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{b_m}\rho]))) = \\ &\sum_{(c_1,\ldots,c_m)\in Lad} L(\delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{c_m+1}\rho, \nu^{b_m}\rho]))\otimes \\ &\qquad L(\delta([\nu^{a_1}\rho, \nu^{c_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{c_m}\rho])), \end{aligned}$$

where Lad stands for the set of all ordered m-tuples $(c_1, \ldots, c_m)$ such that $c_1 < \cdots < c_m$, $a_i - 1 \leq c_i \leq b_i$ and $c_i - a_i$ is an integer for all $i = 1, \ldots, m$. We omit $\delta([\nu^x\rho, \nu^y\rho])$ if $x > y$.

We also frequently use the following structural formulas, which follow from [21], [16, Section 15], and Theorem 2.1:

THEOREM 2.2. *Let $\rho \in \mathrm{Irr}(GL(m, F))$ be a supercuspidal representation. If $F = F'$, let $\rho_1 = \widetilde{\rho}$, otherwise let $\rho_1 = \widehat{\rho}$. Let $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$, and let σ denote an admissible representation of finite length of G_n. Write $\mu^*(\sigma) = \sum_{\delta,\sigma'} \delta \otimes \sigma'$. Then we have*

$$\begin{aligned} \mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\delta,\sigma'} &\delta([\nu^{-i}\rho_1, \nu^k\rho_1]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \delta \\ &\otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'. \end{aligned}$$

Suppose that $a_1, b_1, \ldots, a_m, b_m$ are real numbers such that $b_i - a_i$ is a non-negative integer for $i = 1, 2, \ldots, m$, and that for $i = 1, 2, \ldots, m-1$ we have

$a_i < a_{i+1}$ and $b_i < b_{i+1}$. Then we have

$$\mu^*(L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \dots, \delta([\nu^{a_m}\rho, \nu^{b_m}\rho]) \rtimes \sigma)) =$$

$$\sum_{(c_1,\dots,c_m,d_1,\dots,d_m)\in Lad'} \sum_{\delta,\sigma'} L(\delta([\nu^{d_1+1}\rho, \nu^{b_1}\rho]), \dots, \delta([\nu^{d_m+1}\rho, \nu^{b_m}\rho]))\times$$

$$L(\delta([\nu^{-c_m}\rho_1, \nu^{-a_m}\rho_1]), \dots, \delta([\nu^{-c_1}\rho_1, \nu^{-a_1}\rho_1])) \times \delta$$

$$\otimes L(\delta([\nu^{c_1+1}\rho, \nu^{d_1}\rho]), \dots, \delta([\nu^{c_m+1}\rho, \nu^{d_m}\rho])) \rtimes \sigma',$$

where Lad' stands for the set of all ordered $2m$-tuples $(c_1,\dots,c_m,d_1,\dots,d_m)$ such that $c_1 < \dots < c_m$, $d_1 < \dots < d_m$, $a_i - 1 \leq c_i \leq d_i \leq b_i$ and $c_i - a_i$, $d_i - a_i$ are integers for all $i = 1,\dots,m$. We omit $\delta([\nu^x\rho', \nu^y\rho'])$ if $x > y$.

We briefly recall the subrepresentation version of the Langlands classification for general linear groups.

For every essentially square-integrable representation $\delta \in Irr(R(GL))$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Note that $e(\delta([\nu^a\rho, \nu^b\rho])) = (a+b)/2$. Suppose that $\delta_1, \delta_2, \dots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \dots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \dots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \dots \times \delta_k$. Every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \dots, \delta_k)$ and, for a given π, the representations $\delta_1, \delta_2, \dots, \delta_k$ are unique up to a permutation.

We also use the subrepresentation version of the Langlands classification for classical groups, since it is more appropriate for our Jacquet module considerations. We realize a non-tempered irreducible representation π of G_n as a unique irreducible (Langlands) subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \dots \times \delta_k \rtimes \tau$, where τ is an irreducible tempered representation of some G_t, and $\delta_1, \delta_2, \dots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \dots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1, \delta_2, \dots, \delta_k; \tau)$.

We will use the following result ([5, Lemma 5.5]) several times.

Lemma 2.3. *Suppose that $\pi \in R(G_n)$ is an irreducible representation, λ an irreducible representation of the Levi subgroup M of G_n, and π is a subrepresentation of $Ind_M^{G_n}(\lambda)$. If $L > M$, then there is an irreducible subquotient ρ of $Ind_M^L(\lambda)$ such that π is a subrepresentation of $Ind_L^{G_n}(\rho)$.*

By the Mœglin-Tadić classification of discrete series [14, 16], which holds unconditionally, due to [1], [15, Théorème 3.1.1] and [4, Theorem 7.8], a discrete series $\sigma \in G_n$ corresponds to an admissible triple which consists of the Jordan block, the partial cuspidal support, and the ϵ-function. More details on this invariants can be found in [16, 18] and in [6], where slightly different approach, which also covers the classical group case, has been used.

Note that, if a twist by a character of the form ν^x, with $x \in \mathbb{R}$, of some irreducible unitarizable cuspidal representation $\rho \in R(GL)$ appears in the cuspidal support of a discrete series $\sigma \in R(G)$, then ρ is an F'/F-selfdual representation.

Through the paper, we fix a discrete series σ and we denote the corresponding admissible triple by $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$. For an irreducible F'/F-selfdual cuspidal representation ρ_1 of $GL(n_1, F')$, let $\mathrm{Jord}_{\rho_1}(\sigma) = \{x : (x, \rho_1) \in \mathrm{Jord}(\sigma)\}$. If $\mathrm{Jord}_{\rho_1}(\sigma) \neq \emptyset$ and $x \in \mathrm{Jord}_{\rho_1}(\sigma)$, denote $x\_ = \max\{y \in \mathrm{Jord}_{\rho_1}(\sigma) : y < x\}$, if it exists.

3. Some technical results

In this section we state and prove several technical results which are often used in the paper.

Through this section we fix an irreducible F'/F-selfdual cuspidal representation $\rho \in R(GL)$.

Lemma 3.1. *Let a, b denote real numbers such that $b-a$ is a non-negative integer, and let n stand for a positive integer. Suppose that π is an irreducible subquotient of*

$$L(\delta([\nu^a\rho, \nu^b\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho])) \times \delta([\nu^{a+n}\rho, \nu^{b+n}\rho]).$$

Then either

$$\pi \cong L(\delta([\nu^a\rho, \nu^b\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho]), \delta([\nu^{a+n}\rho, \nu^{b+n}\rho]))$$

or

$$\pi \cong L(\delta([\nu^a\rho, \nu^b\rho]), \ldots, \delta([\nu^{a+n-2}\rho, \nu^{b+n-2}\rho]),$$
$$\delta([\nu^{a+n}\rho, \nu^{b+n-1}\rho]), \delta([\nu^{a+n-1}\rho, \nu^{b+n}\rho])).$$

Proof. Using the Langlands classification, we write π as a unique irreducible subrepresentation of

$$\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]) \times \cdots \times \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]).$$

If $x_i > x_{i+1}$ for some $i \in \{1, 2, \ldots, l-1\}$, then $e(\delta([\nu^{x_i}\rho, \nu^{y_i}\rho])) \leq e(\delta([\nu^{x_{i+1}}\rho, \nu^{y_{i+1}}\rho]))$ implies $y_i < y_{i+1}$, so $\delta([\nu^{x_i}\rho, \nu^{y_i}\rho]) \times \delta([\nu^{x_{i+1}}\rho, \nu^{y_{i+1}}\rho])$ is irreducible and isomorphic to $\delta([\nu^{x_{i+1}}\rho, \nu^{y_{i+1}}\rho]) \times \delta([\nu^{x_i}\rho, \nu^{y_i}\rho])$. Thus, we can assume $x_1 \leq x_2 \leq \cdots \leq x_l$. It follows that $x_1 = a$ and $m^*(\pi) \geq \delta([\nu^a\rho, \nu^{y_1}\rho]) \otimes L(\delta([\nu^{x_2}\rho, \nu^{y_2}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$, so $y_1 \in \{b, b+n\}$.

Let us suppose that $y_1 = b+n$. Using Theorem 2.1 we obtain that $a+n \leq b+1$ and $L(\delta([\nu^{x_2}\rho, \nu^{y_2}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$ is an irreducible subquotient of $L(\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho])) \times \delta([\nu^{a+n}\rho, \nu^b\rho])$.

Now we have $x_2 = a+1$ and $m^*(L(\delta([\nu^{x_2}\rho, \nu^{y_2}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))) \geq \delta([\nu^{a+1}\rho, \nu^{y_2}\rho]) \otimes L(\delta([\nu^{x_3}\rho, \nu^{y_3}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$. It directly follows that

$y_2 = b+1$, which gives

$$\pi \cong L(\delta([\nu^a\rho, \nu^{b+n}\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]), \delta([\nu^{x_3}\rho, \nu^{y_3}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$$
$$\hookrightarrow \delta([\nu^a\rho, \nu^{b+n}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times L(\delta([\nu^{x_3}\rho, \nu^{y_3}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$$
$$\cong \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^a\rho, \nu^{b+n}\rho]) \times L(\delta([\nu^{x_3}\rho, \nu^{y_3}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho])),$$

and the Frobenius reciprocity implies that $m^*(\pi)$ contains an irreducible constituent of the form $\nu^{b+1}\rho \otimes \pi'$, which is impossible for $n \geq 2$.

Consequently, $y_1 = b$ and $L(\delta([\nu^{x_2}\rho, \nu^{y_2}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]))$ is an irreducible subquotient of

$$L(\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho])) \times \delta([\nu^{a+n}\rho, \nu^{b+n}\rho]).$$

Continuing in the same way, we obtain $x_i = a+i-1$ and $y_i = b+i-1$ for $i = 1, 2, \dots, n-1$, and

$$L(\delta([\nu^{x_n}\rho, \nu^{y_n}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho])) \leq$$
$$\delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho]) \times \delta([\nu^{a+n}\rho, \nu^{b+n}\rho]).$$

From the composition series follows

$$L(\delta([\nu^{x_n}\rho, \nu^{y_n}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho])) \in$$
$$\{L(\delta([\nu^{a+n-1}\rho, \nu^{b+n-1}\rho]), \delta([\nu^{a+n}\rho, \nu^{b+n}\rho])),$$
$$\delta([\nu^{a+n}\rho, \nu^{b+n-1}\rho]) \times \delta([\nu^{a+n-1}\rho, \nu^{b+n}\rho])\},$$

and hence $l = n+1$, so the lemma is proved. □

LEMMA 3.2. *Suppose that x and y are such that $x+y$ is a negative integer. Suppose that $L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$ is an irreducible subquotient of $L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \dots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \rtimes \tau'$, where τ and τ' are irreducible tempered representations, and for all $i = 1, \dots, l$ we have $z_i + w_i > 0$, $w_i < -x$. Also, suppose that $x + y < x_1 + y_1$. Then*

$$L(\delta([\nu^x\rho, \nu^y\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$$

is an irreducible subquotient of

$$L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \dots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \rtimes L(\delta([\nu^x\rho, \nu^y\rho]); \tau').$$

PROOF. In $R(G)$ we have

$$L(\delta([\nu^x\rho, \nu^y\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$$
$$\leq \delta([\nu^x\rho, \nu^y\rho]) \rtimes L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$$
$$\leq \delta([\nu^x\rho, \nu^y\rho]) \times L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \dots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \rtimes \tau'$$
$$= L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \dots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \times \delta([\nu^x\rho, \nu^y\rho]) \rtimes \tau'.$$

From the semi-simplification in $R(G)$ follows that there is an irreducible subquotient π of $\delta([\nu^x\rho, \nu^y\rho]) \rtimes \tau'$ such that

$$L(\delta([\nu^x\rho, \nu^y\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \dots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$$

is a subquotient of $L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \ldots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \rtimes \pi$.

Since $L(\delta([\nu^x\rho, \nu^y\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$ is a subrepresentation of

$$\delta([\nu^x\rho, \nu^y\rho]) \rtimes L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau),$$

using the Frobenius reciprocity we conclude that

$$\mu^*(L(\delta([\nu^x\rho, \nu^y\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau))$$

contains

$$\delta([\nu^x\rho, \nu^y\rho]) \otimes L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau).$$

Consequently, $\mu^*(L(\delta([\nu^{z_1}\rho, \nu^{w_1}\rho]), \ldots, \delta([\nu^{z_l}\rho, \nu^{w_l}\rho])) \rtimes \pi)$ contains an irreducible constituent of the form $\delta([\nu^x\rho, \nu^y\rho]) \otimes \pi'$. Since we have $z_i + w_i > 0$ and $w_i < -x$ for all $i = 1, \ldots, l$, it follows that $\mu^*(\pi)$ contains an irreducible constituent of the form $\delta([\nu^x\rho, \nu^{y'}\rho]) \otimes \pi'$, for $x \leq y' \leq y$. Using temperedness of τ' and Theorem 2.2, one can easily see that only irreducible constituent of such a form appearing in $\mu^*(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \tau')$ is $\delta([\nu^x\rho, \nu^y\rho]) \otimes \tau'$, which appears with multiplicity one and is contained in $\mu^*(L(\delta([\nu^x\rho, \nu^y\rho]); \tau'))$. Thus, $\pi \cong L(\delta([\nu^x\rho, \nu^y\rho]); \tau')$ and the lemma is proved. □

LEMMA 3.3. *Suppose that* $L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$ *is an irreducible subquotient of*

$$L(\delta([\nu^c\rho, \nu^d\rho]), \delta([\nu^{c+1}\rho, \nu^{d+1}\rho]), \ldots, \delta([\nu^{c+l-1}\rho, \nu^{d+l-1}\rho])) \rtimes \tau',$$

where τ *and* τ' *are irreducible tempered,* $l \geq 1$, *and* $c + d > 0$. *If* $-d-c-2l < x_1 + y_1$, *then* $L(\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$ *is an irreducible subquotient of*

$$L(\delta([\nu^c\rho, \nu^d\rho]), \delta([\nu^{c+1}\rho, \nu^{d+1}\rho]), \ldots, \delta([\nu^{c+l}\rho, \nu^{d+l}\rho])) \rtimes \tau'.$$

PROOF. In $R(G)$ we have

$$\begin{aligned}
&L(\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau) \\
&\leq \delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]) \rtimes L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau) \\
&\leq \delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]) \times L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-1}\rho, \nu^{d+l-1}\rho])) \rtimes \tau' \\
&= L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-1}\rho, \nu^{d+l-1}\rho])) \times \delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]) \rtimes \tau' \\
&= L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-1}\rho, \nu^{d+l-1}\rho])) \times \delta([\nu^{c+l}\rho, \nu^{d+l}\rho]) \rtimes \tau'.
\end{aligned}$$

Thus, there is an irreducible subquotient π of

$$L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-1}\rho, \nu^{d+l-1}\rho])) \times \delta([\nu^{c+l}\rho, \nu^{d+l}\rho])$$

such that $L(\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau)$ is a subquotient of $\pi \rtimes \tau'$. By Lemma 3.1, either

$$\pi \cong L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l}\rho, \nu^{d+l}\rho]))$$

or

$$\pi \cong L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-2}\rho, \nu^{d+l-2}\rho]), \delta([\nu^{c+l}\rho, \nu^{d+l-1}\rho]), \delta([\nu^{c+l-1}\rho, \nu^{d+l}\rho]))$$

If

$$\pi \cong L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-2}\rho, \nu^{d+l-2}\rho]), \delta([\nu^{c+l}\rho, \nu^{d+l-1}\rho]), \delta([\nu^{c+l-1}\rho, \nu^{d+l}\rho])),$$

we have

$$L(\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau) \leq L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-2}\rho, \nu^{d+l-2}\rho]))\times \delta([\nu^{c+l}\rho, \nu^{d+l-1}\rho]) \times \delta([\nu^{c+l-1}\rho, \nu^{d+l}\rho]) \rtimes \tau'.$$

Using the Frobenius reciprocity, we obtain that

$$\mu^*(L(\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]), \delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau))$$

contains

$$\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho]) \otimes L(\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_l}\rho, \nu^{y_l}\rho]); \tau).$$

Since $-d-l < c$, using Theorem 2.2, together with temperedness of τ', one can conclude that

$$\mu^*(L(\delta([\nu^c\rho, \nu^d\rho]), \ldots, \delta([\nu^{c+l-2}\rho, \nu^{d+l-2}\rho]))\times \delta([\nu^{c+l}\rho, \nu^{d+l-1}\rho]) \times \delta([\nu^{c+l-1}\rho, \nu^{d+l}\rho]) \rtimes \tau')$$

does not contain an irreducible constituent of the form $\delta([\nu^{-d-l}\rho, \nu^{-c-l}\rho])\otimes\pi'$, a contradiction. Thus,

$$\pi \cong L(\delta([\nu^c\rho, \nu^d\rho]), \delta([\nu^{c+1}\rho, \nu^{d+1}\rho]), \ldots, \delta([\nu^{c+l}\rho, \nu^{d+l}\rho]))$$

and the lemma is proved. □

The following two lemmas are direct consequences of [22, Section 8] and the proofs of [12, Lemmas 3.2, 5.2]:

Lemma 3.4. *Let $\sigma_1 \in R(G)$ denote a discrete series such that $\mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$. Suppose that a and b are such that $a+b$ is a positive integer and $a-x$ is an integer for $2x+1 \in \mathrm{Jord}_\rho(\sigma_1)$.*

(1) Suppose that $a \leq 0$ and $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. If $\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and $2x+1 = \max(\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma))$, then $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma_2)$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$, where σ_2 stands for the unique discrete series subrepresentation of $\delta([\nu^{x+1}\rho, \nu^b\rho]) \rtimes \sigma_1$.

(2) Suppose that $a \leq 0$, $-2a+1 \in \mathrm{Jord}_\rho(\sigma)$, *and* $\langle -2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma) = \emptyset$. *Let* σ_2 *stand for a unique discrete series subrepresentation of* $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \sigma_1$, *and let* τ *stand for an irreducible tempered subrepresentation of* $\delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \sigma_2$ *which is a subrepresentation of an induced representation of the form* $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \pi$. *Then* τ *is an irreducible subquotient of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$.

(3) Suppose that $a > 0$ *and* $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. *If* $[2a-1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma) \neq \emptyset$ *and* $2x+1 = \max([2a-1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma))$, *then* $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma_2)$ *is an irreducible subquotient of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$, *where* σ_2 *stands for the unique discrete series subrepresentation of* $\delta([\nu^{x+1}\rho, \nu^b\rho]) \rtimes \sigma_1$.

LEMMA 3.5. *Let* $\sigma_1 \in R(G)$ *denote a discrete series such that* $\mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$. *Suppose that* a *and* b *are such that* $a+b$ *is a positive integer,* $a \leq 0$, *and* $a-x$ *is an integer for* $2x+1 \in \mathrm{Jord}_\rho(\sigma_1)$. *Suppose that* $-2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$, $\langle -2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$, *and let* $2x+1 = \min(\langle -2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma_1))$. *Then there is a unique discrete series* σ_2 *such that* σ_1 *is a subrepresentation of* $\delta([\nu^{a+1}\rho, \nu^x\rho]) \rtimes \sigma_2$.

(1) If $x < b$, *then* $L(\delta([\nu^{-b}\rho, \nu^x\rho]); \sigma_2)$ *is an irreducible subquotient of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$.

(2) Suppose that $x = b$, *and let* τ *stand for an irreducible tempered subrepresentation of* $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_2$ *which is a subrepresentation of an induced representation of the form* $\delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \pi$. *Then* τ *is an irreducible subquotient of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$.

In the following several lemmas we treat the tempered case. Through the rest of this section, $\sigma_1 \in R(G)$ stands for a discrete series such that $\mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$. Let a and b denote the real numbers such that $a+b$ is a positive integer and $a-x$ is an integer for $2x+1 \in \mathrm{Jord}_\rho(\sigma_1)$.

Until said otherwise, let $c > 0$ be such that $c-b$ is a positive integer and $2c+1 \notin \mathrm{Jord}_\rho(\sigma_1)$. Let $2x_m+1$ stand for the maximal element of $\mathrm{Jord}_\rho(\sigma_1)$ such that $x_m < c$. We denote by τ a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_1$ which is not a subrepresentation of an induced representation of the form $\delta([\nu^{x_m+1}\rho, \nu^c\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^c\rho]) \rtimes \pi$.

LEMMA 3.6. *Suppose that* $2b+1 \notin \mathrm{Jord}_\rho(\sigma_1)$, *but* $\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$, *and let* $2y+1 = \max(\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma_1))$. *We denote by* σ_2 *a unique discrete series subrepresentation of* $\delta([\nu^{y+1}\rho, \nu^b\rho]) \rtimes \sigma_1$, *and let* τ' *stand for a unique irreducible subrepresentation of* $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_2$ *which is not a subrepresentation of an induced representation of the form* $\delta([\nu^{z+1}\rho, \nu^c\rho]) \times \delta([\nu^{z+1}\rho, \nu^c\rho]) \rtimes \pi$ *for* $z = \max\{b, x_m\}$. *Then* $L(\delta([\nu^{-y}\rho, \nu^{-a}\rho]); \tau')$ *is an irreducible subquotient of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$.

PROOF. Note that τ' is a subrepresentation of

$$\delta([\nu^{y+1}\rho, \nu^b\rho]) \times \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_1.$$

Since $y > -a$, a simple commuting argument implies that $L(\delta([\nu^{-y}\rho, \nu^{-a}\rho]); \tau')$ is a subrepresentation of

$$\delta([\nu^{y+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-y}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1.$$

Thus, $\mu^*(L(\delta([\nu^{-y}\rho,\ \nu^{-a}\rho]); \tau'))$ contains an irreducible constituent of the form $\delta([\nu^{y+1}\rho, \nu^{b}\rho]) \otimes \pi$.

On the other hand, by Lemma 2.3, there is an irreducible subquotient τ'' of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1$ such that τ' is a subrepresentation of $\delta([\nu^{y+1}\rho, \nu^{b}\rho]) \rtimes \tau''$. It is well-known that $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1$ is a direct sum of two mutually non-isomorphic irreducible tempered representations, and by [22, Section 4], exactly one of them is a subrepresentation of an induced representation of the form

$$\delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \rtimes \pi. \tag{3.1}$$

Suppose that τ'' is a subrepresentation of an induced representation of the form (3.1). Then we have

$$\tau' \hookrightarrow \delta([\nu^{y+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \rtimes \pi.$$

If $x_m > b$, we have $z = x_m$, $x_m + 1 > b + 1$, so

$$\delta([\nu^{y+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \rtimes \pi \cong$$
$$\delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{y+1}\rho, \nu^{b}\rho]) \rtimes \pi,$$

contradicting the definition of τ'.

If $x_m < b$, we have $y = x_m$ and $z = b$, so

$$\delta([\nu^{y+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \rtimes \pi \cong$$
$$\delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{b}\rho]) \rtimes \pi \hookrightarrow$$
$$\delta([\nu^{b+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{b+1}\rho, \nu^{c}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{b}\rho]) \times$$
$$\delta([\nu^{x_m+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x_m+1}\rho, \nu^{b}\rho]) \rtimes \pi,$$

which again contradicts the definition of τ'. Consequently, $\tau'' \cong \tau$. This leads to

$$\begin{aligned} L(\delta([\nu^{-y}\rho, \nu^{-a}\rho]); \tau') &\leq \delta([\nu^{a}\rho, \nu^{y}\rho]) \rtimes \tau' \\ &\leq \delta([\nu^{a}\rho, \nu^{y}\rho]) \times \delta([\nu^{y+1}\rho, \nu^{b}\rho]) \rtimes \tau. \end{aligned}$$

Thus, there is a $\pi \in \{\delta([\nu^{a}\rho, \nu^{b}\rho]), L(\delta([\nu^{a}\rho, \nu^{y}\rho]), \delta([\nu^{y+1}\rho, \nu^{b}\rho]))\}$ such that $L(\delta([\nu^{-y}\rho, \nu^{-a}\rho]); \tau')$ is an irreducible subquotient of $\pi \rtimes \tau$.

Obviously, $m^*(L(\delta([\nu^{a}\rho, \nu^{y}\rho]), \delta([\nu^{y+1}\rho, \nu^{b}\rho])))$ does not contain an irreducible constituent of the form $\delta([\nu^{y+1}\rho, \nu^{t}\rho]) \otimes \pi'$, for $y + 1 \leq t$.

On the other hand, if $\delta([\nu^{y+1}\rho, \nu^{t}\rho]) \otimes \pi'$, for $y + 1 \leq t$, is an irreducible constituent of $\mu^*(\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1)$, using Theorem 2.2, we conclude that either $t = c$ or $2t + 1 \in \mathrm{Jord}_\rho(\sigma_1)$. From the description of $\mathrm{Jord}_\rho(\sigma_1)$ we get $t > b$.

Consequently, $\mu^*(L(\delta([\nu^a\rho,\nu^y\rho]),\delta([\nu^{y+1}\rho,\nu^b\rho]))\rtimes\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{y+1}\rho,\nu^b\rho])\otimes\pi'$.

Since $\mu^*(L(\delta([\nu^{-y}\rho,\ \nu^{-a}\rho]);\tau'))$ contains an irreducible constituent of such a form, we obtain $\pi\cong\delta([\nu^a\rho,\nu^b\rho])$. This ends the proof. □

LEMMA 3.7. *Suppose that $a\le 0$ and $[-2a+1,2b+1]\cap\mathrm{Jord}_\rho(\sigma_1)=\{-2a+1\}$. Let σ_2 be a unique discrete series subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^b\rho])\rtimes\sigma_1$, and let τ' stand for a unique irreducible subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])\times\delta([\nu^{-c}\rho,\nu^c\rho])\rtimes\sigma_2$ which is a subrepresentation of an induced representation of the form $\delta([\nu^{-a+1}\rho,\nu^b\rho])\rtimes\pi$ and is not a subrepresentation of an induced representation of the form $\delta([\nu^{z+1}\rho,\nu^c\rho])\times\delta([\nu^{z+1}\rho,\nu^c\rho])\rtimes\pi$ for $z=\max\{b,x_m\}$. Then τ' is an irreducible subquotient of $\delta([\nu^a\rho,\nu^b\rho])\rtimes\tau$.*

PROOF. Since $-2a+1,2c+1\notin\mathrm{Jord}_\rho(\sigma_2)$, it follows from [19, Section 1] or [13, Theorem 2.2.(i)], that $\delta([\nu^a\rho,\nu^{-a}\rho])\times\delta([\nu^{-c}\rho,\nu^c\rho])\rtimes\sigma_2$ is a direct sum of four mutually non-isomorphic irreducible tempered representations. We denote by τ'' an irreducible tempered subrepresentation of $\delta([\nu^{-c}\rho,\nu^c\rho])\rtimes\sigma_2$ such that τ' is a subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\tau''$.

Using [19, Section 1], or [13, Theorem 2.2.(iii), (iv)] we deduce that $\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\tau''$ is a direct sum of two mutually non-isomorphic irreducible tempered representations.

Note that τ'' is a subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^b\rho])\times\delta([\nu^{-c}\rho,\ \nu^c\rho])\rtimes\sigma_1$. Since τ' is not a subrepresentation of an induced representation of the form $\delta([\nu^{z+1}\rho,\nu^c\rho])\times\delta([\nu^{z+1}\rho,\nu^c\rho])\rtimes\pi$, in the same way as in the proof of the previous lemma we deduce that τ'' is an irreducible subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^b\rho])\rtimes\tau$.

Since $[-2a+1,2b+1]\cap\mathrm{Jord}_\rho(\sigma_1)=\{-2a+1\}$, it follows that $\mu^*(\sigma_1)$ does not contain an irreducible constituent of the form $\nu^y\rho\otimes\pi$ for $y\in\{-a+1,-a+2,\dots,b\}$. Using $c>b$ and the structural formula, we conclude that $\delta([\nu^{-a+1}\rho,\nu^b\rho])\otimes\tau$ is a unique irreducible constituent of $\mu^*(\delta([\nu^{-a+1}\rho,\nu^b\rho])\rtimes\tau)$ of the form $\delta([\nu^{-a+1}\rho,\nu^b\rho])\otimes\pi$, and appears there with multiplicity one.

It follows from [19, Section 1] or [13, Theorem 2.2.(iii)], that the induced representation $\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\tau$ is irreducible, so τ' is a unique irreducible tempered subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\tau''$ which contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^b\rho])\otimes\pi$ in the Jacquet module with respect to the appropriate parabolic subgroup.

From

$$\tau'\hookrightarrow\delta([\nu^a\rho,\nu^{-a}\rho])\times\delta([\nu^{-a+1}\rho,\nu^b\rho])\rtimes\tau$$

we conclude that either

$$\tau'\hookrightarrow\delta([\nu^a\rho,\nu^b\rho])\rtimes\tau$$

or

$$\tau'\hookrightarrow L(\delta([\nu^a\rho,\nu^{-a}\rho]),\delta([\nu^{-a+1}\rho,\nu^b\rho]))\rtimes\tau.$$

Since $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \otimes \pi$, $c > b$, and $[-2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma_1) = \{-2a+1\}$, we obtain $\tau' \leq \delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$. This ends the proof. □

In the rest of the section, suppose that $a \leq 0$, and let $c > 0$ be such that $-a-c$ is a positive integer and $2c+1 \notin \mathrm{Jord}_\rho(\sigma_1)$. Also, let $2x_m+1$ stand for the minimal element of $\mathrm{Jord}_\rho(\sigma_1)$ such that $x_m > c$, and denote by τ a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_1$ such that τ is not a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \rtimes \pi$.

Lemma 3.8. *Suppose that $2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$, but $\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma_1) \neq \emptyset$, and let $2y+1 = \min(\langle -2a+1, 2b+1\rangle \cap \mathrm{Jord}_\rho(\sigma_1))$. Let σ_2 stand for a unique discrete series such that σ_1 is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^y\rho]) \rtimes \sigma_2$, and let τ' stand for a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_2$ which is not a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^z\rho]) \rtimes \pi$ for $2z+1 = \min\{2x+1 \in \mathrm{Jord}_\rho(\sigma_2) : x > c\}$. Then $L(\delta([\nu^{-b}\rho, \nu^y\rho]); \tau')$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$.*

Proof. Note that $z = x_m$ if $x_m < y$, and $z = -a$ otherwise, i.e., if $x_m = y$.

Since $-a > c$, we have the following embeddings and an isomorphism:

$$\begin{aligned}\tau &\hookrightarrow \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_1 \\ &\hookrightarrow \delta([\nu^{-c}\rho, \nu^c\rho]) \times \delta([\nu^{-a+1}\rho, \nu^y\rho]) \rtimes \sigma_2 \\ &\cong \delta([\nu^{-a+1}\rho, \nu^y\rho]) \times \delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_2.\end{aligned}$$

By Lemma 2.3, there is an irreducible subquotient π of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_2$ such that τ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^y\rho]) \rtimes \pi$. Let us prove that $\pi \cong \tau'$. Otherwise, $\pi \cong \tau''$, where τ'' is a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho, \nu^c\rho]) \rtimes \sigma_2$ which is also a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^z\rho]) \rtimes \pi'$. This gives an embedding

$$\tau \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^y\rho]) \times \delta([\nu^{c+1}\rho, \nu^z\rho]) \rtimes \pi'.$$

If $x_m < y$, we have $z = x_m$ and $z < -a$, so

$$\delta([\nu^{-a+1}\rho, \nu^y\rho]) \times \delta([\nu^{c+1}\rho, \nu^z\rho]) \cong \delta([\nu^{c+1}\rho, \nu^z\rho]) \times \delta([\nu^{-a+1}\rho, \nu^y\rho]),$$

and τ is a subrepresentation of $\delta([\nu^{c+1}\rho, \nu^z\rho]) \times \delta([\nu^{-a+1}\rho, \nu^y\rho]) \rtimes \pi'$, a contradiction.

If $x_m = y$, we have $z = -a$ and, by Lemma 2.3, either

$$\tau \hookrightarrow \delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \rtimes \pi'$$

or

$$\tau \hookrightarrow L(\delta([\nu^{c+1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a+1}\rho, \nu^{x_m}\rho])) \rtimes \pi'.$$

If τ is a subrepresentation of $L(\delta([\nu^{c+1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a+1}\rho, \nu^{x_m}\rho]) \rtimes \pi'$, it follows that $\mu^*(\tau)$ contains an irreducible constituent of the form $\nu^{-a}\rho \otimes \pi''$, which is impossible since $c < -a$ and $-2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$. Thus, τ is a subrepresentation of $\delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \rtimes \pi'$, a contradiction.

Consequently, τ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \rtimes \tau'$.

Since τ' is a subrepresentation of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$, $c < -a$, and $[-2a+3, 2y+1] \cap \mathrm{Jord}_\rho(\sigma_2) = \emptyset$, it follows from Theorem 2.2 and [16, Lemma 3.6] that $\mu^*(\tau')$ does not contain an irreducible constituent of the form $\nu^{d}\rho \otimes \pi'$ for $d \in [-a+1, y]$. It is now easy to see, using Theorem 2.2 again, that $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \otimes \tau'$ is a unique irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \otimes \pi'$ appearing in $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \rtimes \tau')$, and it appears there with multiplicity one. Thus, τ is a unique irreducible subquotient of $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \rtimes \tau'$ which contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \otimes \pi'$ in the Jacquet module with respect to the appropriate parabolic subgroup.

We have

$$\begin{aligned} L(\delta([\nu^{-b}\rho, \nu^{y}\rho]); \tau') &\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \tau' \\ &= \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \rtimes \tau', \end{aligned}$$

so the Frobenius reciprocity implies that $\mu^*(L(\delta([\nu^{-b}\rho, \nu^{y}\rho]); \tau'))$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \otimes \pi'$. There is an irreducible subquotient π'' of $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \rtimes \tau'$ such that $L(\delta([\nu^{-b}\rho, \nu^{y}\rho]); \tau') \leq \delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \pi''$. Since $b > y$, Theorem 2.2 implies that $\mu^*(\pi'')$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{y}\rho]) \otimes \pi'$, so $\pi'' \cong \tau$, and the lemma is proved. ∎

LEMMA 3.9. *Suppose that* $[-2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma_1) = \{2b+1\}$. *Let* σ_2 *stand for a unique discrete series such that* σ_1 *is a subrepresentation of* $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \sigma_2$, *and let* τ' *stand for a unique irreducible subrepresentation of* $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \times \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$ *which is a subrepresentation of an induced representation of the form* $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \pi$, *and is not a subrepresentation of an induced representation of the form* $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \pi$ *for* $2z+1 = \min\{2x+1 \in \mathrm{Jord}_\rho(\sigma_2) : x > c\}$. *Then* τ' *is an irreducible subquotient of* $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \tau$.

PROOF. By [22, Theorem 8.2.], there is a discrete series σ_3 such that σ_2 is an irreducible subrepresentation of $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \sigma_3$. Note that $2c+1 \in \mathrm{Jord}_\rho(\sigma_3)$, so $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_3$ is irreducible. Since $\mu^*(\sigma_3)$ does not contain an irreducible constituent of the form $\nu^{y}\rho \otimes \pi$ for $y \in \{c+1, c+2, \ldots, z\}$, Theorem 2.2 implies that $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \otimes \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_3$ is a unique irreducible constituent of $\mu^*(\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2)$ of the form $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \otimes \pi$ and appears there with multiplicity one. Thus, there is a unique irreducible tempered subrepresentation of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$ which contains an irreducible constituent of the form $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup. It can be seen in the same way as in the proof of [22, Lemma 4.1] that such an irreducible tempered subrepresentation is also a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \pi$.

Since τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \times \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$, by Lemma 2.3 there is an irreducible tempered subrepresentation τ'' of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$ such that τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau''$.

If τ'' is a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \pi$, using $z \leq b$ and an easy commuting argument we obtain that τ' is also a subrepresentation of an induced representation of such a form, contrary to our assumption. Thus, τ'' is a unique irreducible tempered subrepresentation of $\delta([\nu^{-c}\rho,\ \nu^{c}\rho]) \rtimes \sigma_2$ which is not a subrepresentation of an induced representation of the form $\delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \pi$.

We have

$$\tau' \leq \delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau'' \leq \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \tau''$$

and there is an irreducible subquotient π_1 of $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \tau''$ such that τ' is contained in $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \pi_1$. Since $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \otimes \pi$, we deduce that $\mu^*(\pi_1)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \otimes \pi$.

On the other hand, we have the following embeddings and an isomorphism:

$$\begin{aligned} \tau &\hookrightarrow \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1 \\ &\hookrightarrow \delta([\nu^{-c}\rho, \nu^{c}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \sigma_2 \\ &\cong \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2, \end{aligned}$$

and there is an irreducible subquotient π_2 of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_2$ such that τ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \pi_2$.

Suppose that $\pi_2 \not\cong \tau''$. Then we have

$$\tau \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{c+1}\rho, \nu^{z}\rho]) \rtimes \pi,$$

for some irreducible representation π.

If $z < -a$, we have $z = x_m$ and

$$\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \cong \delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{b}\rho]),$$

which contradicts the definition of τ.

If $z = -a$, we have $x_m = b$ and either

$$\tau \hookrightarrow L(\delta([\nu^{c+1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a+1}\rho, \nu^{x_m}\rho])) \rtimes \pi$$

or

$$\tau \hookrightarrow \delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \rtimes \pi.$$

Since τ is a subrepresentation of $\delta([\nu^{-c}\rho, \nu^{c}\rho]) \rtimes \sigma_1$, $c < -a$, and $-2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$, it follows that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^{-a}\rho \otimes \pi'$. This leads to $\tau \hookrightarrow \delta([\nu^{c+1}\rho, \nu^{x_m}\rho]) \rtimes \pi$, contradicting the definition of τ. Consequently, τ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \tau''$.

Since $[-2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma_2) = \emptyset$, we obtain that $\mu^*(\tau'')$ does not contain an irreducible constituent of the form $\nu^{y}\rho \otimes \pi'$ for $y \in \{-a+1, -a+2, \ldots, b\}$. An easy application of Theorem 2.2 implies that $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \otimes \tau''$ is a unique irreducible constituent of $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \tau'')$ of the form $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \otimes \pi$, and appears there with multiplicity one.

Using the Frobenius reciprocity we conclude that τ is a unique irreducible subquotient of $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \rtimes \tau''$ which contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{b}\rho]) \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup, so $\pi_1 \cong \tau$ and the lemma is proved. □

4. Reducibility

Through this section we fix an irreducible cuspidal F'/F-selfdual representation $\rho \in R(GL)$, and a discrete series $\sigma \in R(G)$. Let $\alpha \geq 0$ be such that $\nu^{\alpha}\rho \rtimes \sigma_{cusp}$ reduces. We also fix a negative real number a such that $a - \alpha$ is an integer, and positive integers k and n such that $a + k > -a$.

Note that in $R(G)$ we have

$$\begin{gathered} L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma) \hookrightarrow \\ L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho])) \rtimes \sigma = \\ L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma. \end{gathered}$$

In this section we prove that the induced representation

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

reduces when there is an $i \in \{0, 1, \ldots, n-1\}$ such that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ reduces, by showing that in such a case there is an irreducible subquotient of $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ different than

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma).$$

Thus, suppose that there is an $i \in \{0, 1, \ldots, n-1\}$ such that the induced representation $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ reduces and let us denote the minimal such i by m.

Let us first consider the case $m = 0$. This is handled in the following proposition:

Proposition 4.1. *Suppose that $\delta([\nu^{a}\rho, \nu^{a+k}\rho]) \rtimes \sigma$ reduces. Then the induced representation $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ also reduces.*

PROOF. We construct an irreducible subquotient of

$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma,$$

different than

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k}\rho,\nu^{-a}\rho]);\sigma),$$

using a case-by-case consideration.

- Suppose that $[-2a+1,2(a+k)+1]\cap \mathrm{Jord}_\rho(\sigma)=\emptyset$.

By the classification of discrete series, $\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma$ contains two mutually non-isomorphic discrete series, which we denote by σ_1 and σ_2. Let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a-1}\rho]);\sigma_1).$$

- Suppose that $[-2a+1,2(a+k)+1]\cap \mathrm{Jord}_\rho(\sigma)\neq\emptyset$ and $-2a+1\notin \mathrm{Jord}_\rho(\sigma)$.

Let $2x_m+1=\min([-2a+1,2(a+k)+1]\cap \mathrm{Jord}_\rho(\sigma))$, and let σ' denote a discrete series such that σ is a unique irreducible subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^{x_m}\rho])\rtimes\sigma'$. If $x_m<a+k$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-a-k}\rho,\nu^{x_m}\rho]);\sigma').$$

By the proof of [12, Lemma 3.2], $L(\delta([\nu^{-a-k}\rho,\nu^{x_m}\rho]);\sigma')\leq\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma$.

If $x_m=a+k$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a-1}\rho]);\tau),$$

where τ is an irreducible subrepresentation of $\delta([\nu^{-a-k}\rho,\nu^{a+k}\rho])\rtimes\sigma'$ such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\times\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\otimes\pi'$. By the proof of [12, Lemma 3.2], $\tau\leq\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma$.

- Suppose that $[-2a+1,2(a+k)+1]\cap \mathrm{Jord}_\rho(\sigma)\neq\emptyset$ and $-2(a+k)+1\notin \mathrm{Jord}_\rho(\sigma)$.

Let $2x_M+1=\max([-2a+1,2(a+k)+1]\cap \mathrm{Jord}_\rho(\sigma))$, and let σ' stand for a unique discrete series subrepresentation of $\delta([\nu^{x_M+1}\rho,\nu^{a+k}\rho])\rtimes\sigma$. If $x_M>-a$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x_M}\rho,\nu^{-a}\rho]);\sigma').$$

By the proof of [12, Lemma 3.2], $L(\delta([\nu^{-x_M}\rho,\nu^{-a}\rho]);\sigma')\leq\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma$.

If $x_M=-a$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a-1}\rho]);\tau),$$

where τ is an irreducible tempered subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])\rtimes\sigma'$ such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\otimes\pi'$. By the proof of [12, Lemma 3.2], $\tau\leq\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma$.

- Suppose that $\{-2a+1, 2(a+k)+1\} \subseteq \mathrm{Jord}_\rho(\sigma)$ and there is $2x+1 \in \mathrm{Jord}_\rho(\sigma) \cap \langle -2a+1, 2(a+k)+1]$ such that $(2x+1)\_$ is defined and $\epsilon_\sigma(((2x+1)\_, \rho), (2x+1, \rho)) = 1$.

We denote by $2y+1$ the minimal element of $\mathrm{Jord}_\rho(\sigma) \cap \langle -2a+1, 2(a+k)+1]$ such that $(2y+1)\_$ is defined and $\epsilon_\sigma(((2y+1)\_, \rho), (2y+1, \rho)) = 1$.

If $(2y+1)\_ = -2a+1$, let τ denote an irreducible tempered representation such that σ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^y\rho]) \rtimes \tau$. If $y < a+k$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-a-k}\rho, \nu^y\rho]); \tau),$$

By the proof of [12, Lemma 3.4], $L(\delta([\nu^{-a-k}\rho, \nu^y\rho]); \tau) \leq \delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$.

If $(2y+1)\_ = -2a+1$ and $y = a+k$, let τ denote an irreducible tempered representation such that σ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \rtimes \tau$ and let τ' stand for a unique irreducible tempered subrepresentation of $\delta([\nu^{-a-k}\rho, \nu^{a+k}\rho]) \rtimes \tau$ such that $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \otimes \pi'$. We define

$$\pi = L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a-1}\rho]); \tau').$$

By the proof of [12, Lemma 3.4], we have $\tau' \leq \delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$.

Let us now suppose that $(2y+1)\_ > -2a+1$ and let $(2y+1)\_ = 2x+1$. We denote by σ' a discrete series such that σ is a subrepresentation of $\delta([\nu^{-x}\rho, \nu^y\rho]) \rtimes \sigma'$. If $y < a+k$, let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-a-k}\rho, \nu^y\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma'),$$

By the proof of [12, Lemma 3.4], $L(\delta([\nu^{-a-k}\rho, \nu^y\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma') \leq \delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$.

If $y = a+k$, let τ denote a unique irreducible tempered subrepresentation of $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma$, given by [12, Lemma 3.3], and let π stand for

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau),$$

By the proof of [12, Lemma 3.4], $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau) \leq \delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$.

It can be seen, using a repeated application of Lemma 3.3, that π is an irreducible subquotient of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$, details being left to the reader. ☐

Let us now consider the more interesting case $m \geq 1$. It follows that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$, since otherwise $\delta([\nu^a\rho, \nu^{a+k}\rho]) \rtimes \sigma$ reduces by the classification of discrete series.

Using [12, Theorems 3.5, 4.6, 5.4], we deduce the following:

(1) $2(a+k+i)+1 \in \mathrm{Jord}_\rho(\sigma)$ for $i = 0, 1, \ldots, m-1$ and $-2(a+i)+1 \in \mathrm{Jord}_\rho(\sigma)$ for $i = 0, 1, \ldots, m-1$ such that $a+i \leq 0$,

(2) $\epsilon_\sigma(2(a+k+i)+1, 2(a+k+i)+3) = -1$ for $i = 0, 1, \ldots, m-2$,

(3) $\epsilon_\sigma(-2(a+i)+1, -2(a+i)+3) = -1$ for $i = 0, 1, \ldots, m-2$ such that $a+i \leq 0$,
(4) $\epsilon_\sigma((2(a+k)+1)\_, 2(a+k)+1) = -1$,
(5) $\epsilon_\sigma(-2a+1, 2x+1) = -1$ for $2x+1 \in \mathrm{Jord}_\rho(\sigma)$ such that $(2x+1)\_ = -2a+1$.

Since $\delta([\nu^{a+m}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma$ reduces, some of the following also holds:
(1) $2(a+k+m)+1 \notin \mathrm{Jord}_\rho(\sigma)$,
(2) $-2(a+m)+1 \notin \mathrm{Jord}_\rho(\sigma)$ and $a+m \leq 0$,
(3) $2(a+k+m)+1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma(2(a+k+m)-1, 2(a+k+m)+1) = 1$,
(4) $-2(a+m)+1 \in \mathrm{Jord}_\rho(\sigma)$, $a+m \leq 0$, and $\epsilon_\sigma(-2(a+m)+1, -2(a+m)+3) = 1$.

In the sequel, we prove that the induced representation

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

reduces considering each of the cases (1) – (4) separately.

Proposition 4.2. *Suppose that* $2(a+k+m)+1 \notin \mathrm{Jord}_\rho(\sigma)$. *Then* $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ *reduces.*

Proof. Let us denote $(2(a+k)+1)\_$ by $2x+1$. We consider two possibilities separately.

First we assume that $2x+1 > -2a+1$.

We define $\sigma^{(0)} = \sigma$ and, for $i = 1, 2, \ldots, m$ let $\sigma^{(i)}$ stand for a unique discrete series subrepresentation of $\nu^{a+k+m-i+1}\rho \rtimes \sigma^{(i-1)}$, which can be obtained using the proof of [13, Lemma 3.2], based on [22, Theorem 8.2]. Also, let $\sigma^{(m+1)}$ denote a unique discrete series subrepresentation of $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m)}$, also given by the proof of [13, Lemma 3.2].

It follows that $\sigma^{(i)}$ is a subrepresentation of

$$\nu^{a+k+m-i+1}\rho \times \cdots \times \nu^{a+k+m}\rho \rtimes \sigma$$

for $i = 1, 2, \ldots, m$, and that $\sigma^{(m+1)}$ is a subrepresentation of

$$\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \times \nu^{a+k+1}\rho \times \cdots \times \nu^{a+k+m}\rho \rtimes \sigma.$$

Using $\epsilon_\sigma(2(a+k+i)+1, 2(a+k+i)+3) = -1$ for $i = 0, 1, \ldots, m-2$, $\epsilon_\sigma((2x+1, \rho), (2(a+k)+1, \rho)) = -1$, together with [22, Theorem 8.2, Proposition 7.2], we obtain that for $j = 1, 2, \ldots, m$, $\mu^*(\sigma^{(j)})$ contains neither an irreducible constituent of the form $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \pi$ nor an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y \in \{a+k+1, a+k+2, \ldots, a+k+m-j\}$.

We prove that

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-m-1}\rho, \nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m+1}\rho, \nu^{-a-m}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

First, it follows from the first part of Lemma 3.4 that $L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\sigma^{(m+1)})$ is an irreducible subquotient of $\delta([\nu^{a}\rho,\nu^{a+k}\rho])\rtimes\sigma^{(m)}$. We proceed inductively, and suppose that for every $l\in\{0,1,\ldots,m-1\}$ we have

$$L(\delta([\nu^{-a-k-l+1}\rho,\nu^{-a-l}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\sigma^{(m+1)}) \\ \leq L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\delta([\nu^{a+1}\rho,\nu^{a+k+1}\rho]),\ldots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]))\rtimes\sigma^{(m-l)}.$$

Using Lemma 3.2 we obtain

$$L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\sigma^{(m+1)}) \\ \leq L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]))\rtimes \\ L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho]);\sigma^{(m-l)}).$$

Since $\sigma^{(m-l)}$ is a subrepresentation of $\nu^{a+k+l+1}\rho\rtimes\sigma^{(m-l-1)}$, it follows from parts (1) and (2) of Lemma 3.4 that $L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho]);\sigma^{(m-l)})$ is an irreducible subquotient of $\delta([\nu^{a+l+1}\rho,\nu^{a+k+l+1}\rho])\rtimes\sigma^{(m-l-1)}$.

Thus,

$$L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\sigma^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]))\times\delta([\nu^{a+l+1}\rho,\nu^{a+k+l+1}\rho])\rtimes\sigma^{(m-l-1)}.$$

Consequently, there is an irreducible subquotient π of

$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]))\times\delta([\nu^{a+l+1}\rho,\nu^{a+k+l+1}\rho])$$

such that

(4.1)

$$L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\sigma^{(m+1)})$$

is an irreducible subquotient of $\pi\rtimes\sigma^{(m-l-1)}$, and Lemma 3.1 implies that we have either

$$\pi\cong L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]),\delta([\nu^{a+l+1}\rho,\nu^{a+k+l+1}\rho]))$$

or

$$\pi\cong L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+l-1}\rho,\nu^{a+k+l-1}\rho]), \\ \delta([\nu^{a+l}\rho,\nu^{a+k+l+1}\rho]),\delta([\nu^{a+l+1}\rho,\nu^{a+k+l}\rho])).$$

Since $\sigma^{(m+1)}$ is a subrepresentation of an induced representation of the form $\delta([\nu^{x+1}\rho,\nu^{a+k}\rho])\times\nu^{a+k+1}\rho\times\cdots\times\nu^{a+k+m}\rho\rtimes\pi'$ and $x+1>-a+1$, using standard commuting argument and the Frobenius reciprocity we conclude that the Jacquet module of the representation (4.1) with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\delta([\nu^{-a-k-l}\rho,\nu^{-a-l-1}\rho])\otimes\delta([\nu^{x+1}\rho,\nu^{a+k}\rho])\otimes\nu^{a+k+1}\rho\otimes\cdots\otimes\nu^{a+k+m}\rho\otimes\pi'.$$

Suppose that

$$\pi \cong L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]), \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]), \delta([\nu^{a+l+1}\rho, \nu^{a+k+l}\rho])) \leq L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]) \times \delta([\nu^{a+l+1}\rho, \nu^{a+k+l}\rho])$$

Then the Jacquet module of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]) \times \delta([\nu^{a+l+1}\rho, \nu^{a+k+l}\rho]) \rtimes \sigma^{(m-l-1)}$$

with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\delta([\nu^{-a-k-l}\rho, \nu^{-a-l-1}\rho]) \otimes \delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \nu^{a+k+1}\rho \otimes \cdots \otimes \nu^{a+k+m}\rho \otimes \pi'.$$

Applying the structural formulas appearing in Theorem 2.2, together with the square-integrability of $\sigma^{(m-l-1)}$, on the induced representation

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]) \times \delta([\nu^{a+l+1}\rho, \nu^{a+k+l}\rho]) \rtimes \sigma^{(m-l-1)}$$

we deduce that an irreducible representation of the form

$$\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \nu^{a+k+1}\rho \otimes \cdots \otimes \nu^{a+k+m}\rho \otimes \pi'$$

is contained in the Jacquet module of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]) \rtimes \sigma^{(m-l-1)}$$

with respect to the appropriate parabolic subgroup. Since $-a < a + k$, $\mu^*(\sigma^{(m-l-1)})$ contains neither an irreducible constituent of the form $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho])\otimes\pi''$ nor of the form $\nu^y\rho\otimes\pi'$ for $y \in \{a+k+1, a+k+2, \ldots, a+k+l+1\}$, and a repeated application of Theorem 2.2 implies that an irreducible representation of the form

$$\nu^{a+k+l}\rho \otimes \cdots \otimes \nu^{a+k+m}\rho \otimes \pi'$$

is contained in the Jacquet module of

$$L(\delta([\nu^a\rho, \nu^{x-1}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]) \rtimes \sigma^{(m-l-1)},$$

with respect to the appropriate parabolic subgroup, which is impossible. Thus,

$$\pi \cong L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho]), \delta([\nu^{a+l+1}\rho, \nu^{a+k+l+1}\rho])).$$

It follows that

$$L(\delta([\nu^{-a-k-m+1}\rho, \nu^{-a-m}\rho]), \dots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \sigma^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+m}\rho, \nu^{a+k+m}\rho])) \rtimes \sigma,$$

and the claim of the proposition can be obtained using a repeated application of Lemma 3.3.

Now we consider the second possibility, $2x+1 = -2a+1$, i.e., $x = -a$. Again we define $\sigma^{(0)} = \sigma$ and, for $i = 1, 2, \dots, m$ let $\sigma^{(i)}$ stand for a unique discrete series subrepresentation of $\nu^{a+k+m-i+1}\rho \rtimes \sigma^{(i-1)}$. Let $\sigma^{(m+1)}$ denote a unique discrete series subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m)}$ and let τ be an irreducible tempered subrepresentation of $\delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \sigma^{(m+1)}$ such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \otimes \pi$. Then we have $\tau \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \times \delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \sigma^{(m)}$.

From the part (2) of Lemma 3.4 follows that τ is an irreducible subquotient of $\delta([\nu^{a}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m)}$ and it can be proved in the same way as in the case $2x+1 > -2a+1$ that

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \dots, \delta([\nu^{-a-k-m-1}\rho, \nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m+1}\rho, \nu^{-a-m}\rho]), \dots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]); \tau)$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

□

Proposition 4.3. *Suppose that $a+m \leq 0$ and $-2(a+m)+1 \notin \mathrm{Jord}_\rho(\sigma)$. Then $L(\delta([\nu^{a}\rho,\ \nu^{a+k}\rho]),\ \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ reduces.*

Proof. We denote $(2(a+k)+1)\_$ by $2x+1$ and consider two possibilities separately.

First we assume that $2x+1 > -2a+1$.

Let us define $\sigma^{(0)} = \sigma$ and, for $i = 1, 2, \dots, m$, let $\sigma^{(i)}$ denote a unique discrete series such that $\sigma^{(i-1)}$ is a subrepresentation of $\nu^{-a-m+i}\rho \rtimes \sigma^{(i)}$, which can be easily obtained using the proof of [13, Lemma 3.2]. Also, let $\sigma^{(m+1)}$ denote a discrete series such that $\sigma^{(m)}$ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{x}\rho]) \rtimes \sigma^{(m+1)}$, which can also be obtained using the proof of [13, Lemma 3.2].

Using $\epsilon_\sigma(-2(a+i)+1, -2(a+i)+3) = -1$ for $i = 0, 1, \dots, m-2$, $\epsilon_\sigma((-2a+1, \rho), (2x+1, \rho)) = -1$, together with [22, Theorem 8.2, Proposition 7.2], we obtain that for $i = 0, 1, 2, \dots, m-1$, $\mu^*(\sigma^{(i)})$ contains neither an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{x}\rho]) \otimes \pi$ nor an irreducible constituent of the form $\nu^{y}\rho \otimes \pi$ for $y \in \{-a-m+i+2, -a-m+i+3, \dots, -a\}$.

We prove that

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-m-1}\rho, \nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m}\rho, \nu^{-a-m+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

First, it follows from the first part of Lemma 3.5 that $L(\delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$ is an irreducible subquotient of $\delta([\nu^{a}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m)}$.

We proceed inductively, and suppose that for every $l \in \{0, 1, \ldots, m-1\}$ we have

$$L(\delta([\nu^{-a-k-l}\rho, \nu^{-a-l+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$$
$$\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho])) \rtimes \sigma^{(m-l)}.$$

Using Lemma 3.2 we obtain

$$L(\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$$
$$\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho]))$$
$$\rtimes L(\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]); \sigma^{(m-l)}),$$

and, using the first part of Lemma 3.5,

$$L(\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$$
$$\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho])) \times$$
$$\delta([\nu^{a+l+1}\rho, \nu^{a+k+l+1}\rho]) \rtimes \sigma^{(m-l-1)}.$$

Using Lemma 3.1, we deduce that there is a

$$\pi \in \{L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l+1}\rho, \nu^{a+k+l+1}\rho])),$$
$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]),$$
$$\delta([\nu^{a+l+1}\rho, \nu^{a+k+l}\rho]), \delta([\nu^{a+l}\rho, \nu^{a+k+l+1}\rho]))\}$$

such that

(4.2)
$$L(\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \sigma^{(m+1)})$$

is an irreducible subquotient of $\pi \rtimes \sigma^{(m-l-1)}$.

Using Frobenius reciprocity and a standard commuting argument, we obtain that the Jacquet module of the representation (4.2) with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]) \otimes \nu^{-a-l+1}\rho \otimes \nu^{-a-l+2}\rho \otimes \cdots \otimes \nu^{-a}\rho \otimes$$
$$\delta([\nu^{-a+1}\rho, \nu^{x}\rho]) \otimes \pi'.$$

Suppose that

$$\pi \cong L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+l-1}\rho,\nu^{a+k+l-1}\rho]), \delta([\nu^{a+l+1}\rho,\nu^{a+k+l}\rho]),\delta([\nu^{a+l}\rho,\nu^{a+k+l+1}\rho])) \leq L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+l-1}\rho,\nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l+1}\rho,\nu^{a+k+l}\rho])\times\delta([\nu^{a+l}\rho,\nu^{a+k+l+1}\rho]).$$

Using Theorem 2.2, the square-integrability of $\sigma^{(m-l-1)}$, and the fact that $\mu^*(\sigma^{(m-l-1)})$ does not contain an irreducible constituent of the form $\nu^y\rho\otimes\pi''$ for $y\in\{-a-l+1,-a-l+2,\dots,-a\}$, we conclude that

$$\mu^*(L(\delta([\nu^{a+1}\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+l}\rho,\nu^{a+k+l-1}\rho]))\times \delta([\nu^{a+l+1}\rho,\nu^{a+k+l}\rho])\rtimes\sigma^{(m-l-1)})$$

contains an irreducible representation of the form $\delta([\nu^{-a+1}\rho,\nu^x\rho])\otimes\pi'$, which is impossible since $-a-l-1<-a+1$, $x<a+k$, and $\mu^*(\sigma^{(m-l-1)})$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^x\rho])\otimes\pi''$.

Consequently, $\pi\cong L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+l+1}\rho,\nu^{a+k+l+1}\rho]))$, and it follows that

$$L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]), \delta([\nu^{-a-k}\rho,\nu^x\rho]);\sigma^{(m+1)})$$

is an irreducible subquotient of $L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho]))\rtimes\sigma$. Now a repeated application of Lemma 3.3 can be used to prove our claim.

Let us comment on the case $2x+1=-2a+1$. Again we define $\sigma^{(0)}=\sigma$ and, for $i=1,2,\dots,m$, let $\sigma^{(i)}$ denote a unique discrete series such that $\sigma^{(i-1)}$ is a subrepresentation of $\nu^{-a-m+i}\rho\rtimes\sigma^{(i)}$.

Let $\sigma^{(m+1)}$ denote a discrete series such that $\sigma^{(m)}$ is a subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\rtimes\sigma^{(m+1)}$, and let τ denote an irreducible tempered subrepresentation of $\delta([\nu^{-a-k}\rho,\nu^{a+k}\rho])\rtimes\sigma^{(m+1)}$ such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\times\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\otimes\pi$. By [22, Corollary 4.2], this is equivalent to the fact that τ embeds into an induced representation of the form $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\times\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\rtimes\pi$.

We prove that

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\dots,\delta([\nu^{-a-k-m-1}\rho,\nu^{-a-m-1}\rho]), \delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]);\tau)$$

is an irreducible subquotient of

$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma.$$

By the second part of Lemma 3.5, τ is an irreducible subquotient of $\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\sigma^{(m)}$. Note that, for $i=0,1,2,\dots,m-1$, $\mu^*(\sigma^{(i)})$ contains neither an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{a+k}\rho])\otimes\pi$ nor an irreducible constituent of the form $\nu^y\rho\otimes\pi$, for $y \in \{-a-m+i+2, -a-m+i+3, \ldots, -a\}$. Now the rest of the proof can be obtained following the same lines as in the proof of the previously considered case $2x+1 > -2a+1$, using the fact that the Jacquet module of

$$L(\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]); \tau)$$

with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\delta([\nu^{-a-k-l-1}\rho, \nu^{-a-l}\rho]) \otimes \nu^{-a-l+1}\rho \otimes \cdots \otimes \nu^{-a}\rho \otimes \\ \delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \otimes \pi'.$$

□

PROPOSITION 4.4. *Suppose that* $2(a+k+m)+1 \in \mathrm{Jord}_\rho(\sigma)$ *and* $\epsilon_\sigma((2(a+k+m)-1, \rho), (2(a+k+m)+1, \rho)) = 1$. *Then* $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ *reduces.*

PROOF. We denote $(2(a+k)+1)\_$ by $2x+1$ and consider two possibilities separately. First we consider that case $2x+1 > -2a+1$.

Let $\sigma^{(1)}$ stand for a discrete series such that σ is a subrepresentation of $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$.

If $m = 1$, we denote by $\tau^{(1)}$ a unique irreducible tempered subrepresentation of $\delta([\nu^{-a-k-1}\rho, \nu^{a+k+1}\rho]) \rtimes \sigma^{(1)}$ such that $\tau^{(1)}$ is not a subrepresentation of an induced representation of the form $\delta([\nu^{x+1}\rho, \nu^{a+k+1}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{a+k+1}\rho]) \rtimes \pi$.

If $m \geq 2$, we denote by $\tau^{(1)}$ a unique irreducible tempered subrepresentation of $\delta([\nu^{-a-k-m}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$ such that $\tau^{(1)}$ is not a subrepresentation of an induced representation of the form $\delta([\nu^{a+k+m-1}\rho, \nu^{a+k+m}\rho]) \times \delta([\nu^{a+k+m-1}\rho, \nu^{a+k+m}\rho]) \rtimes \pi$.

We note that uniqueness of $\tau^{(1)}$ can be obtained using [22, Section 4].

By [12, Theorem 5.4], the induced representation $\nu^{a+k+m}\rho \rtimes \sigma^{(1)}$ is irreducible, so we have

$$\begin{aligned} \tau^{(1)} &\hookrightarrow \delta([\nu^{-a-k-m}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)} \\ &\hookrightarrow \delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \times \nu^{-a-k-m}\rho \rtimes \sigma^{(1)} \\ &\cong \delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \times \nu^{a+k+m}\rho \rtimes \sigma^{(1)} \\ &\cong \nu^{a+k+m}\rho \times \delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}, \end{aligned}$$

so there is an irreducible subquotient π of $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$ such that $\tau^{(1)}$ is contained in $\nu^{a+k+m}\rho \rtimes \pi$. Since $\tau^{(1)}$ is also a subrepresentation of $\nu^{a+k+m}\rho \times \nu^{a+k+m}\rho \times \delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m-1}\rho]) \rtimes \sigma^{(1)}$, it follows that $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{a+k+m}\rho \otimes \pi'$.

Since $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m-1}\rho]) \rtimes \sigma^{(1)}$ is a length two representation, it easily follows that $\mu^*(\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)})$ contains exactly two irreducible constituents of the form $\nu^{a+k+m}\rho \otimes \pi'$. By the classification of discrete series, $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$ has two irreducible subrepresentations, which are both in discrete series. Frobenius reciprocity implies that π is a discrete series subrepresentation of $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$.

Also, by the classification of discrete series, if σ' is a discrete series subrepresentation of $\delta([\nu^{-a-k-m+1}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(1)}$ such that $\epsilon_{\sigma'}(((2(a+k+m)-1)\_, \rho), (2(a+k+m)-1, \rho)) = -1$, we have $\sigma' \cong \sigma$.

Using a definition of $\tau^{(1)}$, following the same lines as in the proof of [13, Lemma 3.5], we conclude that $\epsilon_{\pi}(((2(a+k+m)-1)\_, \rho), (2(a+k+m)-1, \rho)) = -1$. Consequently, $\pi \cong \sigma$, i.e., $\tau^{(1)}$ is an irreducible subquotient of $\nu^{a+k+m}\rho \rtimes \sigma$.

For $m \geq 2$ and $i = 2, 3, \ldots, m$, we let $\sigma^{(i)}$ denote a unique discrete series subrepresentation of $\nu^{a+k+m-i+1}\rho \rtimes \sigma^{(i-1)}$ and let $\tau^{(i)}$ denote an irreducible tempered subrepresentation of $\delta([\nu^{-a-k-m}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(i)}$ such that $\tau^{(i)}$ is not a subrepresentation of an induced representation of the form $\nu^{a+k+m}\rho \times \nu^{a+k+m}\rho \rtimes \pi$.

Let $\sigma^{(m+1)}$ stand for a unique discrete series subrepresentation of $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m)}$, and let $\tau^{(m+1)}$ denote an irreducible tempered subrepresentation of $\delta([\nu^{-a-k-m}\rho, \nu^{a+k+m}\rho]) \rtimes \sigma^{(m+1)}$ such that $\tau^{(m+1)}$ is not a subrepresentation of an induced representation of the form $\nu^{a+k+m}\rho \times \nu^{a+k+m}\rho \rtimes \pi$.

It can be seen in the same way as in the proof of Proposition 4.2 that, for $j = 1, 2, \ldots, m$, $\mu^*(\tau^{(j)})$ contains neither an irreducible constituent of the form $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \pi$ nor an irreducible constituent of the form $\nu^{y}\rho \otimes \pi$, for $y \in \{a+k+1, a+k+2, \ldots, a+k+m-j\}$.

We prove that

$$L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-m-1}\rho, \nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m+1}\rho, \nu^{-a-m}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

First, by Lemma 3.6 we have $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)}) \leq \delta([\nu^{a}\rho, \nu^{a+k}\rho]) \rtimes \tau^{(m)}$.

Note that $\tau^{(m+1)}$ is a subrepresentation of an induced representation of the form

$$\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \times \nu^{a+k+1}\rho \times \nu^{a+k+2}\rho \times \cdots \times \nu^{a+k+m-1}\rho \rtimes \pi.$$

Following the same lines as in the proof of Proposition 4.2, just using Lemma 3.6 instead of Lemma 3.4, we deduce that for $l = 0, 1, \ldots, m-1$ we have

$$L(\delta([\nu^{-a-k-l+1}\rho, \nu^{-a-l}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)}) \leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho])) \rtimes \tau^{(m-l)}.$$

In particular, for $l = m-1$ we get

$$\begin{aligned} &L(\delta([\nu^{-a-k-m+2}\rho, \nu^{-a-m+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)}) \\ &\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho])) \rtimes \tau^{(1)} \\ &\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho])) \times \nu^{a+k+m}\rho \rtimes \sigma, \end{aligned}$$

so there is an irreducible subquotient π of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho])) \times \nu^{a+k+m}\rho$$

such that

(4.3)
$$L(\delta([\nu^{-a-k-m+2}\rho, \nu^{-a-m+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)})$$

is an irreducible subquotient of $\pi \rtimes \sigma$. Obviously, either

$$\pi \cong L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho]), \nu^{a+k+m}\rho)$$

or

$$\pi \cong L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-2}\rho, \nu^{a+k+m-2}\rho]), \delta([\nu^{a+m-1}\rho, \nu^{a+k+m}\rho])).$$

Since $\mu^*(\sigma)$ contains neither an irreducible constituent of the form $\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \pi'$ nor an irreducible constituent of the form $\nu^{y}\rho \otimes \pi'$, for $y \in \{a+k+1, a+k+2, \ldots, a+k+m-1\}$, a repeated application of Theorem 2.2 can be used to see that the Jacquet module of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-2}\rho, \nu^{a+k+m-2}\rho]), \delta([\nu^{a+m-1}\rho, \nu^{a+k+m}\rho])) \rtimes \sigma$$

with respect to the appropriate parabolic subgroup does not contain an irreducible representation of the form

$$\delta([\nu^{x+1}\rho, \nu^{a+k}\rho]) \otimes \nu^{a+k+1}\rho \otimes \nu^{a+k+2}\rho \otimes \cdots \otimes \nu^{a+k+m-1}\rho \otimes \pi'.$$

Consequently, $\pi \cong L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho]), \nu^{a+k+m}\rho)$.

This leads to

$$\begin{aligned} &L(\delta([\nu^{-a-k-m+1}\rho, \nu^{-a-m}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]); \tau^{(m+1)}) \\ &\leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho]), \nu^{a+k+m}\rho) \times \\ &\qquad \delta([\nu^{a+m}\rho, \nu^{a+k+m-1}\rho]) \rtimes \sigma, \end{aligned}$$

and there is an irreducible subquotient π' of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m-1}\rho, \nu^{a+k+m-1}\rho]), \nu^{a+k+m}\rho) \times \delta([\nu^{a+m}\rho, \nu^{a+k+m-1}\rho])$$

such that
(4.4)
$$L(\delta([\nu^{-a-k-m+1}\rho,\nu^{-a-m}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]),\delta([\nu^{-x}\rho,\nu^{-a}\rho]);\tau^{(m+1)})$$

is contained in $\pi' \rtimes \sigma$. Since the Jacquet module of the representation (4.4) with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\delta([\nu^{-a-k-m+1}\rho,\nu^{-a-m}\rho]) \otimes \pi''$, and π' is contained in

$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]),\nu^{a+k+m}\rho)\times$$
$$\delta([\nu^{a+m}\rho,\nu^{a+k+m-1}\rho]) \leq L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\times$$
$$\nu^{a+k+m}\rho\times\delta([\nu^{a+m}\rho,\nu^{a+k+m-1}\rho]) =$$
$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\times\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho])+$$
$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\times$$
$$L(\delta([\nu^{a+m}\rho,\nu^{a+k+m-1}\rho]),\nu^{a+k+m}\rho),$$

using Theorem 2.2 we obtain

$$\pi' \leq L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\times\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho]).$$

Using the same reasoning as before, we conclude

$$\pi' \cong L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]),\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho])).$$

Now the rest of the proof in the case $2x+1 > -2a+1$ follows by a repeated application of Lemma 3.3.

Let us shortly comment on the case $2x+1 = -2a+1$. We define $\sigma^{(1)},\sigma^{(2)},\ldots,\sigma^{(m+1)}$ and $\tau^{(1)},\tau^{(2)},\ldots,\tau^{(m)}$ in the same way as in the previously considered case $2x+1 > -2a+1$.

We denote by $\tau^{(m+1)}$ an irreducible tempered subrepresentation of

$$\delta([\nu^{-a-k-m}\rho,\nu^{a+k+m}\rho])\times\delta([\nu^{a}\rho,\nu^{-a}\rho])\rtimes\sigma^{(m+1)}$$

which is a subrepresentation of an induced representation of the form $\delta([\nu^{-a+1}\rho,\ \nu^{a+k}\rho]) \rtimes \pi$ and is not a subrepresentation of an induced representation of the form $\nu^{a+k+m}\rho\times\nu^{a+k+m}\rho\rtimes\pi$.

By Lemma 3.7, $\tau^{(m+1)}$ is an irreducible subquotient of $\delta([\nu^{a}\rho,\nu^{a+k}\rho]) \rtimes \tau^{(m)}$. It can now be proved in the same way as in the previously considered case $2x+1 > -2a+1$ that

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\ldots,\delta([\nu^{-a-k-m-1}\rho,\nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m+1}\rho,\nu^{-a-m}\rho]),\ldots,\delta([\nu^{-a-k}\rho,\nu^{-a-1}\rho]);\tau^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma.$$

This ends the proof. ∎

Proposition 4.5. *Suppose that* $a+m \leq 0$, $-2(a+m)+1 \in \mathrm{Jord}_\rho(\sigma)$, *and* $\epsilon_\sigma((-2(a+m)+1,\rho),(-2(a+m)+3,\rho)) = 1$. *Then* $L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\ldots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho])) \rtimes \sigma$ *reduces.*

Proof. Let us denote by $2x+1$ an element of $\mathrm{Jord}_\rho(\sigma)$ such that $(2x+1)\_ = -2a+1$. Then $2x+1 \leq 2(a+k)+1$. First we discuss the case $2x+1 < 2(a+k)+1$.

From $\epsilon_\sigma(-2(a+m)+1,-2(a+m)+3) = 1$, by the classification of discrete series follows that there is an irreducible tempered representation $\tau^{(1)}$ such that σ is a subrepresentation of $\nu^{-a-m+1}\rho \rtimes \tau^{(1)}$. Then $\tau^{(1)}$ is a subrepresentation of $\delta([\nu^{a+m}\rho,\nu^{-a-m}\rho]) \rtimes \sigma^{(1)}$, for a discrete series $\sigma^{(1)}$ such that σ is a subrepresentation of $\delta([\nu^{a+m}\rho,\nu^{-a-m+1}\rho]) \rtimes \sigma^{(1)}$. Since $-2(a+m)+3 \notin \mathrm{Jord}_\rho(\sigma^{(1)})$, it follows that $\nu^{-a-m+1}\rho \otimes \tau^{(1)}$ is a unique irreducible constituent of $\mu^*(\nu^{-a-m+1}\rho \rtimes \tau^{(1)})$ of the form $\nu^{-a-m+1}\rho \otimes \pi$. Thus, σ is a unique irreducible subquotient of $\nu^{-a-m+1}\rho \rtimes \tau^{(1)}$ which contains an irreducible constituent of the form $\nu^{-a-m+1}\rho \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup.

Let $2y+1$ denote an element of $\mathrm{Jord}_\rho(\sigma)$ such that $(2y+1)\_ = -2(a+m)+3$. Note that $y = x$ if $m = 1$, and $y = -a-m+2$ otherwise. Then $\tau^{(1)}$ is an irreducible tempered subrepresentation of $\delta([\nu^{a+m}\rho,\nu^{-a-m}\rho]) \rtimes \sigma^{(1)}$ which is not a subrepresentation of an induced representation of the form $\delta([\nu^{-a-m+1}\rho,\nu^{y}\rho]) \rtimes \pi$, since otherwise we would have

$$\begin{aligned}\sigma &\hookrightarrow \nu^{-a-m+1}\rho \times \delta([\nu^{-a-m+1}\rho,\nu^{y}\rho]) \rtimes \pi \\ &\cong \delta([\nu^{-a-m+1}\rho,\nu^{y}\rho]) \times \nu^{-a-m+1}\rho \rtimes \pi,\end{aligned}$$

leading to $\epsilon_\sigma((-2(a+m)+3,\rho),(2y+1,\rho)) = 1$, a contradiction.

For $i = 2,3,\ldots,m$, we define $\sigma^{(i)}$ as a unique discrete series such that $\sigma^{(i-1)}$ is a subrepresentation of $\nu^{-a-m+i}\rho \rtimes \sigma^{(i)}$. Also, let $\sigma^{(m+1)}$ denote a discrete series such that $\sigma^{(m)}$ is a subrepresentation of $\delta([\nu^{-a+1}\rho,\nu^{x}\rho]) \rtimes \sigma^{(m+1)}$.

Note that for $i = 2,3,\ldots,m+1$ we have $-2(a+m)+3 \in \mathrm{Jord}_\rho(\sigma^{(i)})$, and let $\tau^{(i)}$, for $i = 2,3,\ldots,m+1$, denote an irreducible tempered subrepresentation of $\delta([\nu^{a+m}\rho,\nu^{-a-m}\rho]) \rtimes \sigma^{(i)}$ which is not a subrepresentation of an induced representation of the form $\nu^{-a-m+1}\rho \rtimes \pi$.

It can be observed directly from the definitions of σ and $\sigma^{(i)}$ that, for $i = 1,2,\ldots,m-1$, $\mu^*(\sigma^{(i)})$ contains neither an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{x}\rho]) \otimes \pi$ nor an irreducible constituent of the form $\nu^{y}\rho \otimes \pi$, for $y \in \{-a-m+i+2,-a-m+i+3,\ldots,-a\}$.

We prove that

$$L(\delta([\nu^{-a-k-n+1}\rho,\nu^{-a-n+1}\rho]),\ldots,\delta([\nu^{-a-k-m-1}\rho,\nu^{-a-m-1}\rho]),$$
$$\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\ldots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),\delta([\nu^{-a-k}\rho,\nu^{x}\rho]);\tau^{(m+1)})$$

is an irreducible subquotient of

$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma.$$

First, by Lemma 3.8 we have

$$L(\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})\leq\delta([\nu^a\rho,\nu^{a+k}\rho])\rtimes\tau^{(m)}.$$

Following the same lines as in the proof of Proposition 4.3, just using Lemma 3.8 instead of Lemma 3.5, we deduce that for $l=1,2,\dots,m-1$ we have

$$L(\delta([\nu^{-a-k-l}\rho,\nu^{-a-l+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})$$
$$\leq L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+l}\rho,\nu^{a+k+l}\rho]))\rtimes\tau^{(m-l)}.$$

In particular, for $l=m-1\geq 1$ we have

$$L(\delta([\nu^{-a-k-m+1}\rho,\nu^{-a-m+2}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),$$
$$\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})\leq$$
$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\rtimes\tau^{(1)}.$$

Consequently, using Lemma 3.2 we deduce

$$L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),$$
$$\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})\leq$$
$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))$$
$$\rtimes L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]);\tau^{(1)}).$$

Note that $L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]);\tau^{(1)})$ is an irreducible subquotient of

$$\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho])\times\nu^{-a-m+1}\rho\rtimes\tau^{(1)}.$$

Thus, there is an irreducible subquotient π_1 of $\nu^{-a-m+1}\rho\rtimes\tau^{(1)}$ such that $L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]);\tau^{(1)})$ is an irreducible subquotient of $\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho])\rtimes\pi_1$.

Since $\mu^*(L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]);\tau^{(1)}))$ contains an irreducible constituent of the form $\nu^{-a-m+1}\rho\otimes\pi$, it follows that $\mu^*(\pi_1)$ also contains an irreducible constituent of such a form. Thus, $\pi_1\cong\sigma$ and

$$L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),$$
$$\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})\leq$$
$$L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+m-1}\rho,\nu^{a+k+m-1}\rho]))\times\delta([\nu^{a+m}\rho,\nu^{a+k+m}\rho])\rtimes\sigma.$$

Since $\mu^*(\sigma)$ contains neither an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^x\rho])\otimes\pi$ nor an irreducible constituent of the form $\nu^y\rho\otimes\pi$, for $-a-m+2\leq y\leq -a$, and the Jacquet module of

$$L(\delta([\nu^{-a-k-m}\rho,\nu^{-a-m+1}\rho]),\dots,\delta([\nu^{-a-k-1}\rho,\nu^{-a}\rho]),\delta([\nu^{-a-k}\rho,\nu^x\rho]);\tau^{(m+1)})$$

with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\delta([\nu^{-a-k-m}\rho, \nu^{-a-m+1}\rho]) \otimes \nu^{-a-m+2}\rho \otimes \cdots \otimes \nu^{-a}\rho \otimes \delta([\nu^{-a+1}\rho, \nu^{x}\rho]) \otimes \pi,$$

it can be concluded in the same way as in the proof of Proposition 4.3 that

$$\begin{aligned} &L(\delta([\nu^{-a-k-m}\rho, \nu^{-a-m+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]), \delta([\nu^{-a-k}\rho, \nu^{x}\rho]); \tau^{(m+1)}) \\ &\quad \leq L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+m}\rho, \nu^{a+k+m}\rho])) \rtimes \sigma. \end{aligned}$$

The rest of the proof in the case $2x + 1 < 2(a + k) + 1$ follows by a repeated application of Lemma 3.3.

Let us comment on the case $2x + 1 = 2(a + k) + 1$. We define discrete series representations $\sigma^{(1)}, \ldots, \sigma^{(m)}$ and irreducible tempered representations $\tau^{(1)}, \ldots, \tau^{(m)}$ in the same way as in the case $2x + 1 < 2(a + k) + 1$.

Let $\sigma^{(m+1)}$ denote a discrete series such that $\sigma^{(m)}$ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m+1)}$, and let $\tau^{(m+1)}$ stand for an irreducible tempered subrepresentation of

$$\delta([\nu^{a+m}\rho, \nu^{-a-m}\rho]) \times \delta([\nu^{-a-k}\rho, \nu^{a+k}\rho]) \rtimes \sigma^{(m+1)}$$

which is a subrepresentation of an induced representation of the form

$$\delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{a+k}\rho]) \rtimes \pi$$

and is not a subrepresentation of an induced representation of the form $\nu^{-a-m+1}\rho \rtimes \pi$.

By Lemma 3.9, $\tau^{(m+1)}$ is an irreducible subquotient of $\delta([\nu^{a}\rho, \nu^{a+k}\rho]) \rtimes \tau^{(m)}$. It can now be proved following the same lines as in the case $2x + 1 < 2(a + k) + 1$ that

$$\begin{aligned} &L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k-m-1}\rho, \nu^{-a-m-1}\rho]), \\ &\quad \delta([\nu^{-a-k-m}\rho, \nu^{-a-m+1}\rho]), \ldots, \delta([\nu^{-a-k-1}\rho, \nu^{-a}\rho]); \tau^{(m+1)}) \end{aligned}$$

is an irreducible subquotient of

$$L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

This ends the proof. □

From Propositions 4.1 - 4.5 we deduce the main result of this section.

THEOREM 4.6. *If there is an $i \in \{0, 1, \ldots, n-1\}$ such that the induced representation $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ reduces, then the induced representation $L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ reduces.*

5. Irreducibility

We fix an irreducible cuspidal representation $\rho \in R(GL)$, a negative real number a, and positive integers k and n such that $a + k > -a$.

The aim of this section is to prove that

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

is irreducible if $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \dots, n-1\}$.

We start with the following lemma:

Lemma 5.1. *Suppose that $\tau \in R(G)$ is an irreducible tempered representation and let δ_i stand for $\delta([\nu^{x_i}\rho_i, \nu^{y_i}\rho_i])$, where $\rho_i \in R(GL)$ is an irreducible cuspidal representation and $x_i + y_i < 0$, for $i = 1, 2, \dots, m$. If $\delta_1 \times \dots \times \delta_m \rtimes \tau$ has an irreducible subrepresentation π, which is also an irreducible subquotient of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$, then for $i = 1, 2, \dots, m$ we have $\rho_i \cong \widetilde{\rho}$ if $F' = F$, and $\rho_i \cong \widehat{\rho}$ otherwise.*

Proof. We comment only the case $F' = F$, since the other case can be treated in a completely analogous manner. Suppose that there is an $i \in \{1, 2, \dots, m\}$ such that $\rho_i \not\cong \widetilde{\rho}$, and let us denote the minimal such i by $i_{\min}$. Then for $j \in \{1, \dots, i_{\min} - 1\}$ we have $\delta_j \times \delta_{i_{\min}} \cong \delta_{i_{\min}} \times \delta_j$, and using a commuting argument, together with the Frobenius reciprocity, we get that $\mu^*(L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma)$ contains an irreducible constituent of the form $\delta([\nu^{x_{i_{\min}}}\rho_{i_{\min}}, \nu^{y_{i_{\min}}}\rho_{i_{\min}}]) \otimes \pi'$. Since $x_{i_{\min}} + y_{i_{\min}} < 0$, using the square-integrability of σ and Theorem 2.2, we obtain $\rho_{i_{\min}} \cong \widetilde{\rho}$, a contradiction. □

The following lemma is a direct consequence of the square-integrability criterion, [12, Theorems 3.5, 4.6, 5.4] and [22, Proposition 7.2]:

Lemma 5.2. *Suppose that ρ is F'/F-selfdual and that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \dots, n-1\}$. If $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^c\rho, \nu^d\rho]) \otimes \pi$, then $c + d > 0$, $2d + 1 \in \mathrm{Jord}_\rho(\sigma)$, and one of the following holds:*

(1) $d > a + k + n - 1$,
(2) $d \le -a - n + 1$,
(3) $-a + 2 \le d \le a + k$, $c \ge \frac{(2d+1)\_ + 3}{2}$.

First we consider the more complicated case and, until said otherwise, we assume that ρ is F'/F-selfdual and $a - \alpha \in \mathbb{Z}$ for a unique non-negative α such that $\nu^\alpha\rho \rtimes \sigma_{cusp}$ reduces.

Proposition 5.3. *Suppose that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \dots, n-1\}$. If $\delta([\nu^x\rho, \nu^y\rho]) \otimes \pi$ is an irreducible constituent of $\mu^*(L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma)$, where $x + y < 0$, then one of the following holds:*

(1) $y = -a - n + 1$, $-x \leq a + k + n - 1$,

$$\pi \leq L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-2}\rho, \nu^{a+k+n-2}\rho]), \delta([\nu^{-x+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

(2) $y = a + k$, $a + n - 1 \leq -x \leq a + k + n - 1$,

$$\pi \leq L(\delta([\nu^a\rho, \nu^{-a-n+1}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-2}\rho, \nu^{a+k+n-2}\rho]), \delta([\nu^{-x+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

Proof. Since the Jacquet module of $\delta([\nu^x\rho, \nu^y\rho])$ with respect to the appropriate parabolic subgroup contains $\nu^y\rho \otimes \nu^{y-1}\rho \otimes \cdots \otimes \nu^x\rho$, it follows directly from Theorem 2.2 that there are c and d such that $a + n - 2 \leq c \leq a + k + n - 1$, $a - 1 \leq d \leq a + k$, and an irreducible constituent $\delta' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^x\rho, \nu^y\rho]) \leq \delta([\nu^{-c}\rho, \nu^{-a-n+1}\rho]) \times \delta([\nu^{d+1}\rho, \nu^{a+k}\rho]) \times \delta'$$

and

$$\pi \leq L(\delta([\nu^a\rho, \nu^d\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-2}\rho, \nu^{a+k+n-2}\rho]), \delta([\nu^{c+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma'.$$

Since, by the square-integrability criterion, there is no constituent of $\mu^*(\sigma)$ of the form $\nu^t\rho \otimes \pi'$ for $t \leq 0$, we obtain $c > a+n-2$, so $x = -c \geq -a-k-n+1$ and $y \geq -a - n + 1$.

If $y > a + k$, it follows that $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^{x'}\rho, \nu^y\rho]) \otimes \pi'$. From Lemma 5.2 follows $y > a+k+n-1$, which is impossible since $x + y < 0$.

If $-a - n + 1 < y < a + k$, we have $d = a + k$, so $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^{-a-n+2}\rho, \nu^y\rho]) \otimes \pi'$, contradicting the third part of Lemma 5.2. Thus, $y \in \{-a - n + 1, a + k\}$.

If $y = -a-n+1$, we have $d = a+k$ and $\sigma' \cong \sigma$. If $y = a+k$, then $d < a+k$ and, using the third part of Lemma 5.2 again, we deduce $d = -a - n + 2$ and $\sigma' \cong \sigma$. This ends the proof. □

Proposition 5.4. *Suppose that* $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ *is irreducible for all* $i \in \{0, 1, \ldots, n - 1\}$. *Suppose that* $\delta([\nu^x\rho, \nu^y\rho]) \otimes \pi$, $x + y < 0$, *is an irreducible constituent of*

$$\mu^*(L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-t-1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \ldots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma),$$

where $1 \leq t < n$, $y \geq -a - n + t$, $-a - k - n + i \leq x_i < a + n - i$ *for* $i = 1, 2, \ldots, t$, *and* $x_i < x_{i+1}$ *for* $i = 1, 2, \ldots, t - 1$. *Then one of the following holds:*

(1) $y = -a-n+t+1$, $x_t < a+n-t-2$, $x_t < x$,

$$\pi \leq L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \dots, \delta([\nu^{a+n-t-2}\rho, \nu^{a+k+n-t-2}\rho]), \delta([\nu^{-x+1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \dots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma,$$

(2) $y = a+k$, *there is an* $i \in \{1,2,\dots,t\}$ *such that* $a-1 \leq x_i - 1 < a+k$, $x_{i-1} < -a-k-1$ *if* $i \geq 2$, $x_{i-1} < x$,

$$\pi \leq L(\delta([\nu^a\rho, \nu^{x_i-1}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \dots, \delta([\nu^{a+n-t-1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \dots, \delta([\nu^{-x_{i+1}+1}\rho, \nu^{a+k+n-i-1}\rho]), \delta([\nu^{-x+1}\rho, \nu^{a+k+n-i}\rho]), \delta([\nu^{-x_{i-1}+1}\rho, \nu^{a+k+n-i+1}\rho]), \dots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma,$$

(3) $y = a+k$, $a-1 \leq -a-n+t+1 < a+k$, $-x_t - 1 > a+k$, $x_t < x$,

$$\pi \leq L(\delta([\nu^a\rho, \nu^{-a-n+t}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \dots, \delta([\nu^{a+n-t-2}\rho, \nu^{a+k+n-t-2}\rho]), \delta([\nu^{-x+1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \dots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

Proof. Since the Jacquet module of $\delta([\nu^x\rho, \nu^y\rho])$ with respect to the appropriate parabolic subgroup contains $\nu^y\rho \otimes \cdots \otimes \nu^x\rho$, using Theorems 2.1 and 2.2 we deduce that there is an irreducible representation π_1 of the form

$$L(\delta([\nu^{y_1}\rho, \nu^{x_1-1}\rho]), \dots, \delta([\nu^{y_t}\rho, \nu^{x_t-1}\rho]), \delta([\nu^{y_{t+1}}\rho, \nu^{-a-n+t+1}\rho])),$$

where $y_i < y_{i+1}$ for $i = 1,2,\dots,t$, a d such that $a-1 \leq d \leq a+k$, and an irreducible constituent $\delta' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^x\rho, \nu^y\rho]) \leq \pi_1 \times \delta([\nu^{d+1}\rho, \nu^{a+k}\rho]) \times \delta'.$$

Since $x+y<0$ and σ is square-integrable, some of the segments

$$[\nu^{y_1}\rho, \nu^{x_1-1}\rho], \dots, [\nu^{y_t}\rho, \nu^{x_t-1}\rho], [\nu^{y_{t+1}}\rho, \nu^{-a-n+t+1}\rho]$$

have to be non-empty. Lemma 5.2 implies that $\mu^*(\sigma)$ contains neither an irreducible constituent of the form $\delta([\nu^{x_i}\rho, \nu^y\rho]) \otimes \sigma''$ for $x_i \leq y \leq x_{i+1}-2$, $i \in \{1,2,\dots,t-1\}$, nor an irreducible constituent of the form $\delta([\nu^{x_t}\rho, \nu^y\rho]) \otimes \sigma''$ for $x_t \leq y \leq -a-n+t$. Consequently, exactly one of the segments $[\nu^{y_1}\rho, \nu^{x_1-1}\rho]$, ..., $[\nu^{y_t}\rho, \nu^{x_t-1}\rho]$, $[\nu^{y_{t+1}}\rho, \nu^{-a-n+t+1}\rho]$ is non-empty, since otherwise $m^*(\pi_1 \times \delta([\nu^{d+1}\rho, \nu^{a+k}\rho]) \times \delta')$ does not contain $\nu^y\rho \otimes \cdots \otimes \nu^x\rho$, by Theorem 2.1 and multiplicativity of m^*.

Thus, either there are c and d such that $a+n-t-1 \leq c \leq -x_t - 1$ and $a-1 \leq d \leq a+k$, and an irreducible constituent $\delta' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^x\rho, \nu^y\rho]) \leq \delta([\nu^{-c}\rho, \nu^{-a-n+t+1}\rho]) \times \delta([\nu^{d+1}\rho, \nu^{a+k}\rho]) \times \delta' \tag{5.1}$$

and

$$\pi \leq L(\delta([\nu^a\rho, \nu^d\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-t-2}\rho, \nu^{a+k+n-t-2}\rho]), \delta([\nu^{c+1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \ldots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma',$$

or there is an $i \in \{1, 2, \ldots, t\}$, c and d such that $-x_i - 1 \leq c \leq -x_{i-1} - 1$, with $x_0 = -a-k-n$, $a-1 \leq d \leq a+k$, and an irreducible constituent $\delta' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^x\rho, \nu^y\rho]) \leq \delta([\nu^{-c}\rho, \nu^{x_i-1}\rho]) \times \delta([\nu^{d+1}\rho, \nu^{a+k}\rho]) \times \delta' \tag{5.2}$$

and

$$\pi \leq L(\delta([\nu^a\rho, \nu^d\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-t-1}\rho, \nu^{a+k+n-t-1}\rho]), \delta([\nu^{-x_t+1}\rho, \nu^{a+k+n-t}\rho]), \ldots, \delta([\nu^{-x_{i+1}+1}\rho, \nu^{a+k+n-i-1}\rho]), \delta([\nu^{c+1}\rho, \nu^{a+k+n-i}\rho]), \delta([\nu^{-x_{i-1}+1}\rho, \nu^{a+k+n-i+1}\rho]), \ldots, \delta([\nu^{-x_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma'.$$

In the same way as in the proof of Proposition 5.3 we deduce $-a-n+t+1 \leq y \leq a+k$. If $y = -a-n+t+1$, we conclude at once that $d = a+k$ and $\sigma' \cong \sigma$. If $-a-n+t+1 < y < a+k$, then $d = a+k$ and $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^z\rho, \nu^y\rho]) \otimes \sigma''$ for $z \leq -a-n+t+2$, which contradicts the third part of Lemma 5.2.

It remains to consider the case $y = a+k$. Using Lemma 5.2 we deduce $d < a+k$.

If (5.1) holds, $d > -a-n+t+1$ implies $\delta' \cong \delta([\nu^{-a-n+t+2}\rho, \nu^d\rho])$, which is impossible by the third part of Lemma 5.2. Thus $d = -a-n+t+1$ and $\sigma' \cong \sigma$. Note that this case appears when $x_t < -a-k-1$.

Suppose that (5.2) holds and note that there is a unique $i \in \{1, 2, \ldots, t\}$ such that $x_i - 1 \geq a-1$ and $a+k < -x_{i-1} - 1$. Let us suppose that $d > x_i - 1$. This implies $\delta \cong \delta([\nu^{x_i}\rho, \nu^d\rho])$, so $2d+1 \in \mathrm{Jord}_\rho(\sigma)$. Since σ is square-integrable, we have $d > -x_i$.

If $a+n-1 > 0$, we have $2z+1 \notin \mathrm{Jord}_\rho(\sigma)$ for $z \leq -a$ and, by Lemma 5.2 and [22, Proposition 7.2], $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\delta([\nu^{x_i}\rho, \nu^d\rho]) \otimes \sigma'$ for $x_i \leq 0$ and $d \leq a+k$.

On the other hand, if $a+n-1 < 0$, we have $d > -a-n+i$, and $2d+1 \in \mathrm{Jord}_\rho(\sigma)$ implies $d > -a$. Since $d < a+k$ and $x_i \leq 0$, this is impossible by the third part of Lemma 5.2. Thus, $d = x_i - 1$ and $\sigma' \cong \sigma$. □

PROPOSITION 5.5. *Suppose that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \ldots, n-1\}$. Suppose that $\delta([\nu^x\rho, \nu^y\rho]) \otimes \pi$, $x+y < 0$, is an* *irreducible constituent of*

$$\mu^*(L(\delta([\nu^a\rho,\nu^{x_1}\rho]),\dots,\delta([\nu^{a+t-1}\rho,\nu^{x_t}\rho]),\delta([\nu^{a+t}\rho,\nu^{a+k+t}\rho]),\dots,$$
$$\delta([\nu^{a+n-t-r-s-1}\rho,\nu^{a+k+n-t-r-s-1}\rho]),\delta([\nu^{-y_s+1}\rho,\nu^{a+k+n-t-r-s}\rho]),\dots,$$
$$\delta([\nu^{-y_1+1}\rho,\nu^{a+k+n-t-r-1}\rho]),\delta([\nu^{-z_t+1}\rho,\nu^{a+k+n-t-r}\rho]),\dots,$$
$$\delta([\nu^{-z_1+1}\rho,\nu^{a+k+n-r-1}\rho]),\delta([\nu^{-w_r+1}\rho,\nu^{a+k+n-r}\rho]),\dots,$$
$$\delta([\nu^{-w_1+1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma),$$

where $y\geq a+k+t-1$, r,s,t *are non-negative integers such that* $t\geq 1$ *and* $2t+s+r<n$, *and*

(1) $a+i-2\leq x_i<a+k+i-1$ *for* $i=1,2,\dots,t$, $x_i<x_{i+1}$ *for* $i=1,2,\dots,t-1$, *if* $x_i\geq 0$ *then* $x_i=-a-n+r+i$, *and if* $x_i<0$ *then* $x_i<a+n-r-i-1$,
(2) $-a-k-n+t+r+i\leq y_i<a+n-t-r-i$ *for* $i=1,2,\dots,s$, $y_i<y_{i+1}$ *for* $i=1,2,\dots,s-1$,
(3) $-a-k-n+r+i\leq z_i<-a-k-i+1$ *for* $i=1,2,\dots,t$, $z_i<z_{i+1}$ *for* $i=1,2,\dots,t-1$, $z_i\leq x_i$ *for* $i=1,2,\dots,t$,
(4) $-a-k-n+i\leq w_i<a+n-i$ *for* $i=1,2,\dots,r$, $w_i<w_{i+1}$ *for* $i=1,2,\dots,r-1$,
(5) $z_1>w_r$ *if* $r\geq 1$, *and* $y_1>x_t+1$ *if* $s\geq 1$.

Then $y=a+k+t$, *and one of the following holds:*

- $s=0$, $a+t-1\leq -a-n+t+r+1<a+k+t$, $z_t<-a-n+t+r+2$, *and*

$$\pi\leq L(\delta([\nu^a\rho,\nu^{x_1-1}\rho]),\dots,\delta([\nu^{a+t-1}\rho,\nu^{x_t-1}\rho]),\delta([\nu^{a+t}\rho,\nu^{-a-n+t+r+1}\rho]),$$
$$\delta([\nu^{a+t+1}\rho,\nu^{a+k+t+1}\rho]),\dots,\delta([\nu^{a+n-t-r-2}\rho,\nu^{a+k+n-t-r-2}\rho]),$$
$$\delta([\nu^{-x+1}\rho,\nu^{a+k+n-t-r-1}\rho]),\delta([\nu^{-z_t+1}\rho,\nu^{a+k+n-t-r}\rho]),\dots,$$
$$\delta([\nu^{-z_1+1}\rho,\nu^{a+k+n-r-1}\rho]),\delta([\nu^{-w_r+1}\rho,\nu^{a+k+n-r}\rho]),\dots,$$
$$\delta([\nu^{-w_1+1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma,$$

- $s\geq 1$, $a+t\leq y_1<a+k+t$, *and*

$$\pi\leq L(\delta([\nu^a\rho,\nu^{x_1-1}\rho]),\dots,\delta([\nu^{a+t-1}\rho,\nu^{x_t-1}\rho]),\delta([\nu^{a+t}\rho,\nu^{y_1-1}\rho]),$$
$$\delta([\nu^{a+t+1}\rho,\nu^{a+k+t+1}\rho]),\dots,\delta([\nu^{a+n-t-r-s-1}\rho,\nu^{a+k+n-t-r-s-1}\rho]),$$
$$\delta([\nu^{-y_s+1}\rho,\nu^{a+k+n-t-r-s}\rho]),\dots,\delta([\nu^{-y_2+1}\rho,\nu^{a+k+n-t-r-2}\rho]),$$
$$\delta([\nu^{-x+1}\rho,\nu^{a+k+n-t-r-1}\rho]),\delta([\nu^{-z_t+1}\rho,\nu^{a+k+n-t-r}\rho]),\dots,$$
$$\delta([\nu^{-z_1+1}\rho,\nu^{a+k+n-r-1}\rho]),\delta([\nu^{-w_r+1}\rho,\nu^{a+k+n-r}\rho]),\dots,$$
$$\delta([\nu^{-w_1+1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma.$$

Proof. In the same way as in the proofs of previous two propositions we conclude $y = a + k + t$. Using Theorems 2.1 and 2.2, we obtain that there is an irreducible representation π_1 of the form

$$L(\delta([\nu^{b_1}\rho, \nu^{x_1}\rho]), \ldots, \delta([\nu^{b_t}\rho, \nu^{x_t}\rho]), \delta([\nu^{b_{t+1}}\rho, \nu^{a+k+t}\rho])),$$

where $b_i < b_{i+1}$ for $i = 1, 2, \ldots, t$, an irreducible representation π_2 of the form

$$L(\delta([\nu^{c_1}\rho, \nu^{w_1-1}\rho]), \ldots, \delta([\nu^{c_{r+t+s}}\rho, \nu^{y_s-1}\rho]), \delta([\nu^{c_{r+s+t+1}}\rho, \nu^{-a-n+t+r+s+1}\rho])),$$

where $c_i < c_{i+1}$ for $i = 1, 2, \ldots, r + s + t + 1$, and an irreducible constituent $\delta' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^x\rho, \nu^{a+k+t}\rho]) \leq \pi_1 \times \pi_2 \times \delta'.$$

The segment $[\nu^{b_{t+1}}\rho, \nu^{a+k+t}\rho]$ is obviously non-empty, and since $x + a + k + t < 0$ and σ is square-integrable it follows that at least one of the segments $[\nu^{c_1}\rho, \nu^{w_1-1}\rho], \ldots, [\nu^{c_{r+t+s}}\rho, \nu^{y_s-1}\rho], [\nu^{c_{r+s+t+1}}\rho, \nu^{-a-n+t+r+s+1}\rho]$ is also non-empty.

Conditions (1) – (5), together with Lemma 5.2, imply that if $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^{d_1}\rho, \nu^{d_2}\rho]) \otimes \sigma''$ for $d_1 \leq d_2$ and $d_1 \in \{w_1, \ldots, w_r, z_1, \ldots, z_t, y_1, \ldots, y_s\}$, then $d_2 > a + k + t$. Using the condition (1) and Lemma 5.2 we conclude that $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\delta([\nu^{x_i+1}\rho, \nu^d\rho]) \otimes \sigma''$ for $x_i + 1 \leq d \leq a + k + t$ and $i \in \{1, 2, \ldots, t\}$. Consequently, $\sigma' \cong \sigma$.

Also, for all $i \in \{1, 2, \ldots, t\}$ we have $z_i < -a - k$ and $x_i \geq a - 1 > -a - k - 1$. Thus, all the segments $[\nu^{c_1}\rho, \nu^{w_1-1}\rho], \ldots, [\nu^{c_{r+t}}\rho, \nu^{z_t-1}\rho]$ are also empty.

If $s \geq 1$, then conditions (1) and (5) imply $y_1 - 1 \geq a + t - 1$ and $b_i > x_i$ for $i = 1, 2, \ldots, t$. Consequently,

$$\delta([\nu^x\rho, \nu^{a+k+t}\rho]) \leq L(\delta([\nu^{c_{r+t+1}}\rho, \nu^{y_1-1}\rho]), \ldots, \delta([\nu^{c_{r+t+s}}\rho, \nu^{y_s-1}\rho]),$$
$$\delta([\nu^{c_{r+s+t+1}}\rho, \nu^{-a-n+t+r+s+1}\rho])) \times \delta([\nu^{b_{t+1}}\rho, \nu^{a+k+t}\rho]),$$

and the claim directly follows.

If $s = 0$, we have at once

$$\delta([\nu^x\rho, \nu^{a+k+t}\rho]) \leq \delta([\nu^{c_{r+t+1}}\rho, \nu^{-a-n+t+r+1}\rho]) \times \delta([\nu^{b_{t+1}}\rho, \nu^{a+k+t}\rho]),$$

and it is easy to see that the claim also holds in this case. ☐

Theorem 5.6. *Suppose that $\delta([\nu^{a+i}\rho, \nu^{a+k+i}\rho]) \rtimes \sigma$ is irreducible for all $i \in \{0, 1, \ldots, n-1\}$. Then $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ is irreducible.*

Proof. Suppose that $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho])$ and $\delta([\nu^{a_2}\rho, \nu^{b_2}\rho])$ are irreducible representations such that $a_1 + b_1 \leq a_2 + b_2$ and $b_1 \geq b_2$. Then we have $a_1 \leq a_2$, so

$$\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \cong \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]).$$

Let us denote an irreducible subquotient of $L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma$ by π. If π is tempered, using the cuspidal support considerations and the fact that $2(a+k+n-1)+1\in\mathrm{Jord}_\rho(\sigma)$, we conclude that π is a subrepresentation of an induced representation of the form $\delta([\nu^{-a-k-n+1}\rho,\nu^{a+k+n-1}\rho])\rtimes\tau$, which is impossible since

$$\mu^*(L(\delta([\nu^a\rho,\nu^{a+k}\rho]),\dots,\delta([\nu^{a+n-1}\rho,\nu^{a+k+n-1}\rho]))\rtimes\sigma)$$

does not contain an irreducible constituent of the form $\nu^{a+k+n-1}\rho\otimes\pi'$.

Using the Langlands classification, together with Lemma 5.1 and the discussion from the beginning of the proof, we conclude that there are $\delta_1,\delta_2,\dots,\delta_m$, $\delta_i\cong\delta([\nu^{c_i}\rho,\nu^{d_i}\rho])$ such that $c_i+d_i<0$ for $i=1,2,\dots,m$, $d_i\le d_{i+1}$ for $i=1,2,\dots,m-1$, and an irreducible tempered representation $\tau\in R(G)$ such that π is a unique irreducible subrepresentation of $\delta_1\times\delta_2\times\cdots\times\delta_m\rtimes\tau$.

Let $\pi^{(m)}\cong\tau$. Using the Frobenius reciprocity, together with the transitivity of Jacquet modules, we deduce that there exist irreducible representations $\pi^{(1)},\dots,\pi^{(m-1)}\in R(G)$ such that $\mu^*(\pi)$ contains $\delta_1\otimes\pi^{(1)}$, $\mu^*(\pi^{(i)})$ contains $\delta_{i+1}\otimes\pi^{(i+1)}$, for $i=1,\dots,m-2$, and the Jacquet module of $\pi^{(i)}$ with respect to the appropriate parabolic subgroup contains $\delta_{i+1}\otimes\cdots\otimes\delta_m\otimes\tau$, for $i=1,\dots,m-1$.

Using Proposition 5.3, we obtain an irreducible representation $L_1\in R(GL)$ such that $\pi^{(1)}\le L_1\rtimes\sigma$. If $m\ge 2$, using either Proposition 5.4 or Proposition 5.5, we obtain an irreducible representation $L_2\in R(GL)$ such that $\pi^{(2)}\le L_2\rtimes\sigma$.

Continuing in the same way, using a repeated application of Propositions 5.4 and 5.5, we conclude that τ is an irreducible subquotient of the induced representation $L_m\rtimes\sigma$, where L_m is a representation of the form

$$L(\delta([\nu^a\rho,\nu^{x_1}\rho]),\dots,\delta([\nu^{a+t-1}\rho,\nu^{x_t}\rho]),\delta([\nu^{a+t}\rho,\nu^{a+k+t}\rho]),\dots,\\ \delta([\nu^{a+n-s-1}\rho,\nu^{a+k+n-s-1}\rho]),\delta([\nu^{-y_s+1}\rho,\nu^{a+k+n-s}\rho]),\dots,\\ \delta([\nu^{-y_1+1}\rho,\nu^{a+k+n-1}\rho])),$$

where s and t are non-negative integers such that $s+t\ge n$, $a+t-1\le 0$, $a-i-2\le x_i\le a+k+i-1$, and if $x_i<a+k+i-1$ then $x_i<-a+i+2$ for $i=1,2,\dots,t$, $x_i<x_{i+1}$, for $i=1,2,\dots,t-1$, $-a-k-n+i+1\le y_i\le -a-n+i+2$, for $i=1,2,\dots,s$, $y_i<y_{i+1}$ for $i=1,2,\dots,s-1$.

Suppose that $\tau\not\cong\sigma$ and let x be such that $\nu^x\rho$ appears in the cuspidal support of L_m and for every $\nu^y\rho$ appearing in the cuspidal support of L_m we have $|y|\le|x|$. Then $|x|\in\{a+k,,\dots,a+k+n-1,-a,\dots,-a-n+1\}$, so $2|x|+1\in\mathrm{Jord}_\rho(\sigma)$. Using the cuspidal support considerations we conclude that τ can be written as a subrepresentation of an induced representation of the form $\delta([\nu^{-|x|}\rho,\nu^{|x|}\rho])\rtimes\tau'$. Using Theorem 2.2, the square-integrability of σ, and Theorem 2.1, we conclude at once that $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^z\rho,\nu^{|x|}\rho])\otimes\pi'$, for $-|x|<z\le|x|$. By Lemma 5.2

this is possible only if $|x| = a+k$, and Theorem 2.2 implies $z \leq -a+1$, which contradicts the part (3) of Lemma 5.2. Thus, $\tau \cong \sigma$.

If $c_i = -a-n+i$ for $i = 1,2,\ldots,m$, using a repeated application of the first part of Proposition 5.4 we get

$$\pi \cong L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma).$$

If there is an $i \in \{1,2,\ldots,m\}$ such that $c_i \neq -a-n+i$, using the second and the third part of Proposition 5.4, and Proposition 5.5 repeatedly, we deduce that $d_i < d_{i+1}$ for $i = 1,2,\ldots,m-1$, and there are non-negative integers t_1, t_2, $0 \leq t_1 < t_2 \leq n-1$ such that $c_m = -a-k-t_2$ and $d_m = a+k+t_1$. By [12, Theorem 3.2], the induced representation $\delta([\nu^{-a-k-t_2}\rho, \nu^{a+k+t_1}\rho]) \rtimes \sigma$ is irreducible so

$$\delta([\nu^{-a-k-t_2}\rho, \nu^{a+k+t_1}\rho]) \rtimes \sigma \cong \delta([\nu^{-a-k-t_1}\rho, \nu^{a+k+t_2}\rho]) \rtimes \sigma.$$

Since $d_i < a+k+t_2 \leq a+k+t_1-1$, an easy commuting argument shows that π is a subrepresentation of an induced representation of the form $\nu^{a+k+t_2}\rho \rtimes \pi''$, but it follows directly from Theorem 2.2 and Lemma 5.2 that $\mu^*(L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu^{a+k+t_2}\rho \otimes \pi''$.

Consequently, every irreducible subquotient of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

is isomorphic to $L(\delta([\nu^{-a-k-n+1}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-a-k}\rho, \nu^{-a}\rho]); \sigma)$, and it is an easy consequence of Theorem 2.2 that it appears in the composition series of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ with multiplicity one. Thus, $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ is irreducible. □

In the rest of this section we shortly comment the remaining cases. We begin by recalling a well-known result.

LEMMA 5.7. *(1) If $\mu^*(\sigma)$ contains an irreducible constituent of the form $\nu^x\rho_1 \otimes \pi$, for an irreducible cuspidal representation $\rho_1 \in R(GL)$, then ρ_1 is F'/F-selfdual.*

(2) Suppose that $\rho_1 \in R(GL)$ is an irreducible cuspidal F'/F-selfdual representation and let α be such that $\nu^\alpha\rho_1 \rtimes \sigma_{cusp}$ reduces. If $\mu^*(\sigma)$ contains an irreducible constituent of the form $\nu^x\rho_1 \otimes \pi$ then $x - \alpha \in \mathbb{Z}$.

THEOREM 5.8. *Suppose that ρ is not F'/F-selfdual. Then*

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

is irreducible.

PROOF. Inspecting the cuspidal support of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

we conclude that it does not contain an irreducible tempered subquotient.

Using Lemma 5.1, in the same way as in the proof of Theorem 5.6 we write an irreducible non-tempered subquotient π of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ as a unique irreducible subrepresentation of $\delta_1 \times \delta_2 \times \cdots \times \delta_m \rtimes \tau$, for $\delta_1, \delta_2, \ldots, \delta_m$, $\delta_i \cong \delta([\nu^{c_i}\rho_1, \nu^{d_i}\rho_1])$ such that $c_i + d_i < 0$ for $i = 1, 2, \ldots, m$, $d_i \leq d_{i+1}$ for $i = 1, 2, \ldots, m-1$, and an irreducible tempered representation $\tau \in R(G)$. Here $\rho_1 = \widetilde{\rho}$ if $F = F'$, and $\rho_1 = \widehat{\rho}$ otherwise.

Using the first part of Lemma 5.7, in the same way as in the proof of Propositions 5.3 and 5.4 we deduce that for $i = 1, 2, \ldots, m$ we have $d_i = -a - n + i$ and $c_i < c_{i+1}$ for $i = 1, 2, \ldots, m-1$. Thus, $m \leq n$ and τ is an irreducible subquotient of

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-m-1}\rho, \nu^{a+k+n-m-1}\rho]),$$
$$\delta([\nu^{-c_m+1}\rho, \nu^{a+k+n-m}\rho]), \ldots, \delta([\nu^{-c_1+1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma.$$

Using the cuspidal support considerations again, we conclude $m = n$ and $c_i = -a - k - n + i$ for $i = 1, 2, \ldots, n$. Now irreducibility of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ can be obtained in the same way as in the proof of Theorem 5.6. ∎

THEOREM 5.9. *Suppose that ρ is F'/F-selfdual and for α such that $\nu^\alpha\rho \rtimes \sigma_{cusp}$ reduces we have $a - \alpha \notin \mathbb{Z}$. Then*

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$$

is irreducible.

PROOF. If $2a \notin \mathbb{Z}$, the proof follows in the same way as the one of Theorem 5.8.

Suppose that $2a \in \mathbb{Z}$. Then the induced representation $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ does not contain a discrete series subquotient.

Since $\nu^{a+k+n-1}\rho$ appears in the cuspidal support of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ exactly once, and for $x \neq a + k + n - 1$ such that $\nu^x\rho$ appears in the cuspidal support of $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]))$ we have $|x| < a+k+n-1$, it follows that $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma$ does not contain an irreducible tempered subquotient.

We note that in the considered case the statements of Propositions 5.3, 5.4 and 5.5 also hold, and can be proved in an analogous way, just more easily, using the second part of Lemma 5.7 instead of Lemma 5.2. Now the rest of the proof follows in the same way as in the one of Theorem 5.6, details being left to the reader. ∎

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O reducibilnosti reprezentacija induciranih iz esencijalno Spehinih reprezentacija i diskretnih serija

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Sažetak. Neka je π esencijalno Spehina reprezentacija oblika $L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]))$, pri čemu je ρ ireducibilna kuspidalna reprezentacija opće linearne grupe nad nearhimedskim lokalnim poljem ili njegovim separabilnim kvadratnim proširenjem, $a \leq 0$, $2a + k > 0$ te $n \geq 1$. Neka je σ diskretna serija simplektičke grupe, specijalne neparno ortogonalne grupe ili unitarne grupe. Proučavamo kada se inducirana reprezentacija $\pi \rtimes \sigma$ reducira.

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