RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 28 = 558 (2024): 327-337
DOI: https://doi.org/10.21857/yq32oh8v89

SHARPENING THE DIRAC INEQUALITY

Pavle Pandžić and Ana Prlić

Dedicated to Marko Tadić

Abstract. We explain an idea towards a possible proof of a conjecture of Salamanca-Riba and Vogan. This conjecture, also called the Convex hull conjecture, sharpens the well known Dirac inequality of Partahasarathy, which has been useful in several partial classifications of unitary representations of real reductive groups. The idea we present originates from collaboration with David Renard.

1. Introduction

Let G be a connected real reductive Lie group with Cartan involution Θ and let $K = G^{\Theta}$ be the corresponding maximal compact subgroup of G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of the Lie algebra of G corresponding to Θ and let $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{p}_0$. Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$; for example, we can take the Killing form extended over the center, or the trace form.

The main problem in the representation theory of real reductive Lie groups is the so-called unitary dual problem: determining the set $\hat{G}$ of equivalence classes of the irreducible unitary representations of G. This problem is still considered unsolved in general, although there has been a lot of progress over the past decades. For example, let us mention that there is an algorithm implemented by the atlas software; see [1].

In [14] Salamanca Riba and Vogan conjectured that the study of $\hat{G}$ can be reduced to the study of $(\mathfrak{g}, K)$–modules with unitarily small lowest K-types. The Salamanca-Vogan conjecture was recently proved for the case $G = U(p, q)$ (see [17]). After introducing some basic definitions and results, we will explain our idea that could be helpful for proving the Salamanca-Vogan conjecture in general. Our approach is completely different from the approach used in [17].

2020 *Mathematics Subject Classification.* 22E47.
Key words and phrases. Real reductive group, representation, Harish-Chandra module, Dirac operator, Dirac inequality.

The algebraic Dirac operator D is an element of the tensor product of the universal enveloping algebra $U(\mathfrak{g})$ and the Clifford algebra $C(\mathfrak{p})$ of $\mathfrak{p}$ with respect to B. It is defined as

$$D = \sum_i b_i \otimes d_i \;\; \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where b_i is any basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B. Then D is independent of the choice of b_i, and K-invariant for the adjoint action on both factors. The (geometric version of the) Dirac operator was introduced by Parthasarathy [12]. It was used for the construction of the discrete series representations as sections of certain spinor bundles on the homogeneous space G/K.

The Dirac operator is very useful in representation theory; see for example [3], [4]. One of the main uses of the Dirac operator is Parthasarathy's Dirac inequality [13] which we explain below.

Let M be a unitary $(\mathfrak{g}, K)$–module, with an invariant inner product $\langle \cdot \,|\, \cdot \rangle_M$, and let S be a spin module for $C(\mathfrak{p})$. Then the Dirac operator acts on $M \otimes S$. There is a standard inner product $\langle \cdot \,|\, \cdot \rangle_S$ on S such that elements of $\mathfrak{p}_0$ are skew self-adjoint (see [4, 2.3.9] for more details).

Since the elements of $\mathfrak{p}_0$ are skew self-adjoint with respect to $\langle \cdot \,|\, \cdot \rangle_M$ and with respect to $\langle \cdot \,|\, \cdot \rangle_S$, it follows that D is self-adjoint with respect to the inner product on $M \otimes S$ defined by

$$\langle m_1 \otimes s_1 \,|\, m_2 \otimes s_2 \rangle = \langle m_1 \,|\, m_2 \rangle_M \, \langle s_1 \,|\, s_2 \rangle_S, \qquad m_1, m_2 \in M, s_1, s_2 \in S. \tag{1.1}$$

In particular, we have $D^2 \geq 0$ (Parthasarathy's Dirac inequality). This inequality can be written more explicitly. To do that, we recall a formula for D^2 due to Parthasarathy.

Let $\mathrm{Cas}_{\mathfrak{g}}$, $\mathrm{Cas}_{\mathfrak{k}_\Delta}$ denote the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$. Here $\mathfrak{k}_\Delta = \Delta(\mathfrak{k})$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X), \qquad X \in \mathfrak{k},$$

where the map $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ is given by

$$\mathfrak{k} \xrightarrow{\mathrm{ad}} \mathfrak{so}(\mathfrak{p}) \cong \textstyle\bigwedge^2 \mathfrak{p} \overset{q}{\hookrightarrow} C(\mathfrak{p}).$$

(The map q is the Chevalley map, i.e., the skew symmetrization. It is also called the quantization map.)

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. We choose compatible systems of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$, and denote by ρ respectively $\rho_{\mathfrak{k}}$ the corresponding half sums of positive roots. The announced formula for D squared is

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}\|^2).$$

Suppose that M has infinitesimal character corresponding to $\Lambda \in \mathfrak{h}^*$ under the Harish-Chandra isomorphism, and let τ be the highest weight of a $\tilde{K}$-type E_τ appearing in $M \otimes S$. ($\tilde{K}$ is the spin double cover of K.) Then by the relation between Casimir actions and infinitesimal characters, the Dirac inequality $D^2 \geq 0$ on E_τ can be rewritten as

$$\|\Lambda\|^2 \leq \|\tau + \rho_{\mathfrak{k}}\|^2.$$

Dirac inequality was crucial for several partial classifications of unitary modules. For example, Vogan and Zuckerman [16] used the Dirac inequality to classify unitary $(\mathfrak{g}, K)$-modules with nonzero $(\mathfrak{g}, K)$-cohomology. Furthermore, Enright-Howe-Wallach [2] and independently Jakobsen [6] used the Dirac inequality to classify unitary highest weight $(\mathfrak{g}, K)$-modules. More recently, together with Souček, Tuček and Savin we reproved the classification of [2] and [6] in a more elementary way, using the Dirac inequality to full extent [9], [10], [11], [8].

As explained above, the Dirac inequality is a very useful necessary condition for unitarity, but it is by no means a sufficient condition. We mention that a sufficient condition in terms of the Dirac operator is described in [7]. Namely, let us assume that G is simple noncompact (or semisimple with no compact factors). Let $\langle \cdot \,|\, \cdot \rangle_M$ be an inner product on M, not necessarily invariant. We extend $\langle \cdot \,|\, \cdot \rangle_M$ to an inner product $\langle \cdot \,|\, \cdot \rangle$ on $M \otimes S$ in the same way as above, i.e., by (1.1). Suppose that the Dirac operator D is self-adjoint with respect to $\langle \cdot \,|\, \cdot \rangle$. Then the inner product $\langle \cdot \,|\, \cdot \rangle_M$ is $\mathfrak{g}$-invariant, so M is unitary. The reason for this is the fact that D and $1 \otimes \mathfrak{p}$ generate all of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

As we already said, if M is unitary with infinitesimal character corresponding to $\Lambda \in \mathfrak{h}^*$, and if $M \otimes S$ contains a K-type E_τ, then the Dirac inequality holds, and on E_τ it can be written as

$$\|\Lambda\|^2 \leq \|\tau + \rho_{\mathfrak{k}}\|^2.$$

If Λ is real, then $\|\Lambda\|$ is the Euclidean norm, so Λ is in a ball of radius $\|\tau + \rho_{\mathfrak{k}}\|$. Let us recall the well known Vogan's conjecture that was formulated in [15] and proved by Huang and Pandžić in [3].

The Dirac cohomology of M is defined as

$$H_D(M) = \ker D / \ker D \cap \mathrm{im} D.$$

Since D is K-invariant, $H_D(M)$ is a module for $\tilde{K}$.

Theorem 1.1 (Vogan's conjecture; [15], [3]). *Let M be a $(\mathfrak{g}, K)$-module with infinitesimal character corresponding to $\Lambda \in \mathfrak{h}^*$ and suppose that $H_D(M)$ contains a $\tilde{K}$ module E_τ of highest weight $\tau \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$. Then Λ is conjugate to $\tau + \rho_{\mathfrak{k}}$ under the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.*

If M is unitary, then $H_D(M) = \ker D = \ker D^2$. This and Vogan's conjecture imply that the only unitary points on the sphere where Dirac inequality becomes an equality, have Λ in the Weyl group orbit of $\tau + \rho_{\mathfrak{k}}$. This can be interpreted as a sharpening of the Dirac inequality. Further sharpening has been conjectured by Salamanca-Riba and Vogan:

CONJECTURE 1.2 (Salamanca-Vogan convex hull conjecture; [14], Conjecture 5.7.). *Let M be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character corresponding to $\Lambda \in \mathfrak{h}^*$ and suppose that $M \otimes S$ contains a $\tilde{K}$-module E_τ of highest weight $\tau \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$. Then Λ is in the convex hull of the Weyl group orbit of $\tau + \rho_{\mathfrak{k}}$.*

For example, if M is spherical (i.e., contains the trivial K–type), then $\tau = \rho_n = \rho - \rho_{\mathfrak{k}}$ appears in $M \otimes S$. So the Dirac inequality says that for unitary M with infinitesimal character Λ,

$$\|\Lambda\|^2 \leq \|\rho\|^2.$$

In this case, the Salamanca-Vogan conjecture says that Λ is in the convex hull of the Weyl group orbit of ρ.

We mention that the Salamanca-Vogan conjecture has recently been replaced by even sharper conjectural inequalities of similar nature; see e.g. [17], where the sharper conjecture is proved for $G = U(p, q)$.

We acknowledge the use of the software system Macaulay2 [5] for the explicit computations of the invariant (2.2) as well as the examples in Sect. 3.

2. SOME IDEAS TOWARDS A POSSIBLE PROOF

The convex hull is defined by linear inequalities on $\mathfrak{h}^*_{\mathbb{R}}$. The set of these inequalities is invariant under the Weyl group $W_{\mathfrak{g}}$, but individual inequalities are not $W_{\mathfrak{g}}$-invariant. Therefore we need the following lemma, which originates from collaboration with David Renard:

LEMMA 2.1. *One can define the same convex hull by polynomial $W_{\mathfrak{g}}$–invariant inequalities.*

PROOF. Suppose that $P_1, \dots, P_n$ is a Weyl group orbit of polynomials (in our case the P_i are linear). Let $\sigma_1, \dots, \sigma_n$ be their symmetric combinations. The σ_i are obviously $W_{\mathfrak{g}}$–invariant, and we claim that the inequalities $P_i \geq 0$ define the same set as the inequalities $\sigma_i \geq 0$. Clearly, if all $P_i(x) \geq 0$, then also all $\sigma_i(x) \geq 0$. Conversely, suppose all $\sigma_i(x) \geq 0$. Each $P_i(x)$ satisfies the equation

$$t^n - \sigma_1(x)t^{n-1} + \sigma_2(x)t^{n-2} - \cdots + (-1)^n \sigma_n(x) = 0.$$

The coefficients of this equation alternate, so if $P_i(x) < 0$, then all terms are of the same sign, a contradiction. (This simplification of our original proof is due to Vogan.) $\square$

By Harish-Chandra isomorphism, we can associate $z_i \in Z(\mathfrak{g})$ to each σ_i. Since σ_i is also $W_{\mathfrak{k}}$–invariant, it moreover defines $\zeta(z_i) \in Z(\mathfrak{k}_\Delta)$. In the proof of Vogan's conjecture, the main point was to write each $z\otimes 1-\zeta(z)$ as $Da+aD$ for suitable $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. Here we want to write

$$z_i \otimes 1 - \zeta(z_i) = \sum_j b_j b_j^* \qquad \text{for some } \; b_j \in U(\mathfrak{g}) \otimes C(\mathfrak{p}). \tag{2.1}$$

The star-operation is defined on $\mathfrak{g}$ as the involutory antiauthomorphism whose restriction to $\mathfrak{g}_0$ is -1 and it is extended uniquely to involutive antiautomorphisms of the algebras $U(\mathfrak{g})$ and $C(\mathfrak{p})$. Furthermore, $(u \otimes c)^* = u^* \otimes c^*$ for $u \in U(\mathfrak{g})$ and $c \in C(\mathfrak{p})$. Then for any $b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ b^* is formally adjoint to b in any unitary module, so each bb^* acts automatically as positive operator, and therefore $z \otimes 1 - \zeta(z)$ is positive if (2.1) holds.

Example 2.2. Let $G = SU(2,1)$. The (real) Lie algebra of $SU(2,1)$ is $\mathfrak{g}_0 = \mathfrak{su}(2,1)$. The complexification of $\mathfrak{g}_0$ is $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$. The Weyl group is the group of permutations of coordinates, i.e. $W_{\mathfrak{g}} = S_3$.

As usual, $\mathfrak{h}_{\mathbb{R}}^*$ is identified with

$$\{(x,y,z) \in \mathbb{R}^3 \,|\, x+y+z=0\}.$$

The half sum of positive roots is $\rho = (1,0,-1)$. The convex hull of ρ is defined by

$$-1 \le x,y,z \le 1.$$

One orbit for $W_{\mathfrak{g}} = S_3$ is

$$x+1 \ge 0, \quad y+1 \ge 0, \quad z+1 \ge 0,$$

and the other is

$$-x+1 \ge 0, \quad -y+1 \ge 0, \quad -z+1 \ge 0.$$

Making symmetric combinations for each orbit and removing the redundant inequalities leads to the system

$$\begin{aligned} xyz + xy + xz + yz + 1 &\ge 0, \\ -xyz + xy + xz + yz + 1 &\ge 0. \end{aligned}$$

A basis for $\mathfrak{g}$ is given by:

$$\begin{aligned} &H_1 = e_{11} - e_{33}, \quad H_2 = e_{22} - e_{33}, \quad E = e_{12}, \quad F = e_{21}, \\ &E_1 = e_{13}, \quad E_2 = e_{23}, \quad F_1 = e_{31}, \quad F_2 = e_{32}, \end{aligned}$$

where e_{ij} denotes the usual matrix unit: it has the ij entry equal to 1 and all other entries equal to 0. Then the element $z \otimes 1 - \zeta(z)$ of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ corresponding to the first inequality can be written as

(2.2)
$$2H_1^2\otimes E_1F_1-4H_1H_2\otimes E_1F_1+6EF\otimes E_1F_1-6E_2F_2\otimes E_1F_1+$$
$$2H_1E\otimes E_2F_1+2H_2E\otimes E_2F_1+6E_1F_2\otimes E_2F_1+2H_1F\otimes E_1F_2+$$
$$2H_2F\otimes E_1F_2+6E_2F_1\otimes E_1F_2-4H_1H_2\otimes E_2F_2+2H_2^2\otimes E_2F_2+$$
$$6EF\otimes E_2F_2-6E_1F_1\otimes E_2F_2+4H_1E_1F_1\otimes 1-8H_2E_1F_1\otimes 1+12EE_2F_1\otimes 1+$$
$$12FE_1F_2\otimes 1-8H_1E_2F_2\otimes 1+4H_2E_2F_2\otimes 1-6H_1\otimes E_1F_1+$$
$$6H_2\otimes E_1F_1-6E\otimes E_2F_1-6F\otimes E_1F_2-12E_1F_1\otimes 1-12E_2F_2\otimes 1.$$

We would like to write this element as $\sum_j b_jb_j^*$.

3. A method for finding positive invariants for $SU(2,1)$

Now the idea is to find as many positive K-invariant elements as we can for the case $G=SU(2,1)$. The procedure described below would work equally well in the other rank two cases, $SO(4,1)$ and $Sp(4,\mathbb{R})$.

Let us denote
$$H=e_{11}-e_{22},\quad z=\frac{1}{3}\left(e_{11}+e_{22}-2e_{33}\right),$$
and let E,F,E_1,E_2,F_1,F_2 be as in Example 2.2.

The commutation relations are given by

$$\begin{aligned}
&[H,E_1]=E_1,\quad [z,E_1]=E_1,\quad [H,E_2]=-E_2,\quad [z,E_2]=E_2\\
&[H,F_1]=-F_1,\quad [z,F_1]=-F_1,\quad [H,F_2]=F_2,\quad [z,F_2]=-F_2\\
&[H,E]=2E,\quad [z,E]=0,\quad [H,F]=-2F,\quad [z,F]=0.
\end{aligned}\tag{3.1}$$

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to the usual Cartan involution $\theta(X)=-X^*$. Then
$$\mathfrak{k}=\mathrm{span}_{\mathbb{C}}\{H,E,F,z\}\cong\mathfrak{gl}(2,\mathbb{C}),\qquad\text{and}\qquad \mathfrak{p}=\mathrm{span}_{\mathbb{C}}\{E_1,E_2,F_1,F_2\}.$$
The semisimple part of $\mathfrak{k}$ is
$$\mathfrak{k}_s=\mathrm{span}_{\mathbb{C}}\{H,E,F\}\cong\mathfrak{sl}(2,\mathbb{C}),$$
with H, E and F corresponding to the standard basis of $\mathfrak{sl}(2,\mathbb{C})$, while the center of $\mathfrak{k}$ is equal to $\mathbb{C}z$.

For $n\in\mathbb{N}$, $m\in\frac{1}{3}\mathbb{Z}$, let $V_{(n,m)}$ denote the irreducible $\mathfrak{k}$–module with a highest weight vector x_n such that
$$H\cdot x_n=nx_n,\quad z\cdot x_n=mx_n.$$
Then we have
$$V_{(n,m)}=\mathrm{span}_{\mathbb{C}}\{x_n^{(m)},x_{n-2}^{(m)},\dots,x_{-n+2}^{(m)},x_{-n}^{(m)}\},$$
where $x_{n-2i}^{(m)}=F^i\cdot x_n^{(m)}$, $i\in\{1,\dots,n\}$, and then we have

$$
\begin{aligned}
&H \cdot x^{(m)}_{n-2i} = (n-2i)x^{(m)}_{n-2i}, \qquad i \in \{0,1,\dots,n\};\\
&z \cdot x^{(m)}_{n-2i} = m x^{(m)}_{n-2i}, \qquad i \in \{0,1,\dots,n\};\\
&F \cdot x^{(m)}_{n-2i} = x^{(m)}_{n-2(i+1)}, \qquad i \in \{0,1,\dots,n-1\}; \qquad F \cdot x^{(m)}_{-n} = 0;\\
&E \cdot x^{(m)}_{n-2i} = i(n-i+1)x^{(m)}_{n-2(i-1)}, \qquad i \in \{1,\dots,n\}; \qquad E \cdot x^{(m)}_{n} = 0.
\end{aligned}
$$

We have

$$H^* = H,\ z^* = z,\ E^* = F,\ F^* = E,\ E_i^* = -F_i\ F_i^* = -E_i,\ i \in \{1,2\}.$$

Since $(u \otimes c)^* = u^* \otimes c^*$ for $u \in U(\mathfrak{g}), c \in C(\mathfrak{p})$, it follows that for an element $x^{(m)}_{n-2i} \in V_{(n,m)} \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ we have

$$
\begin{aligned}
H \cdot (x^{(m)}_{n-2i})^* &= [H^*, (x^{(m)}_{n-2i})^*] = [x^{(m)}_{n-2i}, H]^* = -[H, x^{(m)}_{n-2i}]^*\\
&= -((n-2i)x^{(m)}_{n-2i})^* = -(n-2i)(x^{(m)}_{n-2i})^*.
\end{aligned}
\tag{3.2}
$$

Therefore

$$H \cdot (x^{(m)}_{n-2i}(x^{(m)}_{n-2i})^*) = (H \cdot x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^* + x^{(m)}_{n-2i}(H \cdot (x^{(m)}_{n-2i})^*) = 0. \tag{3.3}$$

Similar calculations show that

$$z \cdot (x^{(m)}_{n-2i})^* = -m(x^{(m)}_{n-2i})^*; \qquad z \cdot (x^{(m)}_{n-2i}(x^{(m)}_{n-2i})^*) = 0. \tag{3.4}$$

Furthermore, we have

$$
\begin{aligned}
E \cdot (x^{(m)}_{n-2i})^* &= [F^*, (x^{(m)}_{n-2i})^*] = [x^{(m)}_{n-2i}, F]^*\\
&= -[F, x^{(m)}_{n-2i}]^* = -(x^{(m)}_{n-2(i+1)})^*, \quad i \in \{0,1,\dots,n-1\}\\
E \cdot (x^{(m)}_{-n})^* &= 0
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
F \cdot (x^{(m)}_{n-2i})^* &= [E^*, (x^{(m)}_{n-2i})^*] = [x^{(m)}_{n-2i}, E]^* = -[E, x^{(m)}_{n-2i}]^*\\
&= -i(n-i+1)(x^{(m)}_{n-2(i-1)})^*, \quad i \in \{1,\dots,n\}\\
F \cdot (x^{(m)}_{n})^* &= 0.
\end{aligned}
\tag{3.6}
$$

For $V_{(n,m)} = \operatorname{span}_{\mathbb{C}}\{x^{(m)}_n, x^{(m)}_{n-2}, \dots, x^{(m)}_{-n+2}, x^{(m)}_{-n}\} \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$, let us denote

$$V^*_{(n,m)} = \operatorname{span}_{\mathbb{C}}\{(x^{(m)}_n)^*, (x^{(m)}_{n-2})^*, \dots, (x^{(m)}_{-n+2})^*, (x^{(m)}_{-n})^*\}] \subset U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

It follows from (3.2), (3.4), (3.5) and (3.6) that $V^*_{(n,m)}$ is an irreducible $\mathfrak{k}$-module with a highest weight vector $(x^{(m)}_{-n})^*$. It is clear that an element of the form

$$\sum_{i=0}^{n} \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^* \in V_{(n,m)} \cdot V^*_{(n,m)}$$

is positive, for all choices of nonnegative integers $\alpha_0, \dots, \alpha_n$ (which are not simultaneously 0). Our next goal is to find (enough) conditions on the constants α_i to ensure that $\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^*$ is $\mathfrak{k}$–invariant.

THEOREM 3.1. *With notation as above, suppose that* $\alpha_{i-1} = i(n-i+1)\alpha_i$ *for all* $i \in \{1, \dots, n\}$. *Then the element*

$$\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^* \in V_{(n,m)} \cdot V^*_{(n,m)}$$

is $\mathfrak{k}$*-invariant.*

PROOF. It follows from (3.3) and (3.4) that

$$H \cdot (\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^*) = 0 \qquad \text{and} \qquad z \cdot \sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^* = 0.$$

Furthermore, it follows from (3.6) that

$$\begin{aligned} F \cdot (\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^*) &= \alpha_0 x^{(m)}_{n-2} (x^{(m)}_n)^* - \alpha_n \cdot n \cdot x^{(m)}_{-n} (x^{(m)}_{-n+2})^* \\ &+ \sum_{i=1}^{n-1} \alpha_i ((x^{(m)}_{n-2(i+1)})(x^{(m)}_{n-2i})^* - i(n-i+1)(x^{(m)}_{n-2i})(x^{(m)}_{n-2(i-1)})^*) \\ &= \sum_{i=0}^{n-1} (\alpha_i - (i+1)(n-i)\alpha_{i+1})(x^{(m)}_{n-2(i+1)})(x^{(m)}_{n-2i})^*. \end{aligned}$$

Therefore, if $\alpha_{i-1} = i(n-i+1)\alpha_i$ for all $i \in \{1, \dots, n\}$, or equivalently

$$\alpha_i - (i+1)(n-i)\alpha_{i+1} = 0$$

for all $i \in \{0, 1, \dots, n-1\}$, then $F \cdot (\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^*) = 0$.

We now use (3.5) to conclude

$$\begin{aligned} E \cdot (\sum_{i=0}^n \alpha_i (x^{(m)}_{n-2i})(x^{(m)}_{n-2i})^*) &= -\alpha_0 x^{(m)}_n (x^{(m)}_{n-2})^* + n\alpha_n x^{(m)}_{-n+2} (x^{(m)}_{-n})^* \\ &+ \sum_{i=1}^{n-1} \alpha_i (i(n-i+1)(x^{(m)}_{n-2(i-1)})(x^{(m)}_{n-2i})^* - (x^{(m)}_{n-2i})(x^{(m)}_{n-2(i+1)})^*) \\ &= \sum_{i=0}^{n-1} (-\alpha_i + (i+1)(n-i)\alpha_{i+1})(x^{(m)}_{n-2i})(x^{(m)}_{n-2(i+1)})^* = 0. \end{aligned}$$

∎

Using the above theorem, we can find many positive invariants. In fact, for any highest weight $\mathfrak{sl}(2, \mathbb{C})$-module V contained in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ we get a new positive invariant. For example, we can get some familiar positive K-invariants such as minus the $\mathfrak{p}$–Laplacian $-(E_1F_1+E_2F_2)$, and $\mathrm{Cas}_{\mathfrak{k}}$.

EXAMPLE 3.2. Let $V=\mathrm{span}_{\mathbb{C}}\{E_1,E_2\}\subset U(\mathfrak{g})$ be the two-dimensional $\mathfrak{sl}(2,\mathbb{C})$–module with highest weight 1 and highest weight vector E_1. Then $V^*=\mathrm{span}_{\mathbb{C}}\{E_1^*,E_2^*\}\subset U(\mathfrak{g})$ is the two-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight 1 and highest weight vector $E_2^*=-F_2$. Therefore the positive invariant is $-(E_1F_1+E_2F_2)$.

Similarly, we get $\mathrm{Cas}_{\mathfrak{k}}$ from the three-dimensional $\mathfrak{sl}(2,\mathbb{C})$–module with highest weight 2 and highest weight vector E. A few more complicated examples can be found below.

EXAMPLE 3.3. Let

$$V=\mathrm{span}_{\mathbb{C}}\{EE_1,-HE_1+EE_2,-2FE_1-2HE_2,-6FE_2\}\subset U(\mathfrak{g})$$

be the highest weight $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight 3 and highest weight vector EE_1. Then

$$V^*=\mathrm{span}_{\mathbb{C}}\{-F_1F,F_1H-F_2F,2F_1E+2F_2H,6F_2E\}\subset U(\mathfrak{g})$$

is the highest weight $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight 3 and highest weight vector $6F_2E$. The corresponding positive invariant is therefore

$$\begin{aligned}
&\frac{1}{12}[36(EE_1)(-F_1F)+12(-HE_1+EE_2)(F_1H-F_2F)\\
&\quad+3(-2FE_1-2HE_2)(2F_1E+2F_2H)+(-6FE_2)(6F_2E)]\\
&=-H_1^2E_1F_1+2H_1H_2E_1F_1-H_2^2E_1F_1-4EFE_1F_1-H_1^2E_2F_2\\
&\quad+2H_1H_2E_2F_2-H_2^2E_2F_2-4EFE_2F_2\\
&\quad+3H_1E_1F_1-3H_2E_1F_1+2EE_2F_1+2FE_1F_2+H_1E_2F_2-H_2E_2F_2.
\end{aligned}$$

EXAMPLE 3.4. Let

$$\begin{aligned}
V=\mathrm{span}_{\mathbb{C}}\{&E_1F_2\otimes E_1,(E_2F_2-E_1F_1)\otimes E_1+E_1F_2\otimes E_2,\\
&-2E_2F_1\otimes E_1+2(E_2F_2-E_1F_1)\otimes E_2,-6E_2F_1\otimes E_2\}\subset U(\mathfrak{g})\otimes C(\mathfrak{p})
\end{aligned}$$

be the highest weight $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight 3 and highest weight vector $E_1F_2\otimes E_1$. Then

$$\begin{aligned}
V^*=\mathrm{span}_{\mathbb{C}}\{&-E_2F_1\otimes F_1,(E_1F_1-E_2F_2)\otimes F_1-E_2F_1\otimes F_2,\\
&2E_1F_2\otimes F_1+2(E_1F_1-E_2F_2)\otimes F_2,6E_1F_2\otimes F_2\}\subset U(\mathfrak{g})\otimes C(\mathfrak{p})
\end{aligned}$$

is the highest weight $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight 3 and highest weight vector $6E_1F_2\otimes F_2$. The corresponding positive invariant is therefore

$$\begin{aligned}
&\frac{1}{12}[36(E_1F_2\otimes E_1)(-E_2F_1\otimes F_1)\\
&+12((E_2F_2-E_1F_1)\otimes E_1+E_1F_2\otimes E_2)((E_1F_1-E_2F_2)\otimes F_1-E_2F_1\otimes F_2)\\
&+3(-2E_2F_1\otimes E_1+2(E_2F_2-E_1F_1)\otimes E_2)(2E_1F_2\otimes F_1\\
&+2(E_1F_1-E_2F_2)\otimes F_2)+(-6E_2F_1\otimes E_2)(6E_1F_2\otimes F_2)]\\
=&-E_1^2F_1^2\otimes E_1F_1-2E_1E_2F_1F_2\otimes E_1F_1-E_2^2F_2^2\otimes E_1F_1-E_1^2F_1^2\otimes E_2F_2\\
&-2E_1E_2F_1F_2\otimes E_2F_2-E_2^2F_2^2\otimes E_2F_2+H_1E_1F_1\otimes E_1F_1\\
&+3H_2E_1F_1\otimes E_1F_1-EE_2F_1\otimes E_1F_1-FE_1F_2\otimes E_1F_1+H_1E_2F_2\otimes E_1F_1\\
&+H_2E_2F_2\otimes E_1F_1-EE_1F_1\otimes E_2F_1+H_1E_1F_2\otimes E_2F_1\\
&+H_2E_1F_2\otimes E_2F_1-EE_2F_2\otimes E_2F_1-FE_1F_1\otimes E_1F_2+H_1E_2F_1\otimes E_1F_2\\
&+H_2E_2F_1\otimes E_1F_2-FE_2F_2\otimes E_1F_2+H_1E_1F_1\otimes E_2F_2+H_2E_1F_1\otimes E_2F_2\\
&-EE_2F_1\otimes E_2F_2-FE_1F_2\otimes E_2F_2+3H_1E_2F_2\otimes E_2F_2\\
&+H_2E_2F_2\otimes E_2F_2-4E_1F_1\otimes E_1F_1-2E_2F_2\otimes E_1F_1-2E_1F_2\otimes E_2F_1\\
&-2E_2F_1\otimes E_1F_2-2E_1F_1\otimes E_2F_2-4E_2F_2\otimes E_2F_2.
\end{aligned}$$

So far, we have found 80 positive invariants and are still working on expressing the element (2.2) as a positive linear combination of these (and possibly other) positive invariants.

The list of invariants we have constructed so far can be found at the link `https://drive.google.com/file/d/1FKEj4FjJ_sI5waqG6bojcCnffOOZqLBR/view?usp=sharing`.

Acknowledgements.

P. Pandžić and A. Prlić are both supported by the QuantiXLie Center of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

References

[1] J. Adams, M. van Leeuwen, P. Trapa and D. Vogan, *Unitary representations of real reductive groups,* Astérisque **417** (2020), 174 pp.

[2] T. Enright, R. Howe and N. Wallach, *A classification of unitary highest weight modules,* in: Representation theory of reductive groups (Park City, Utah, 1982), Birkhäuser, Boston, 1983, 97–143.

[3] J.-S. Huang and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan,* J. Amer. Math. Soc. **15** (2002), 185–202.

[4] J.-S. Huang and P. Pandžić, Dirac Operators in Representation Theory, Math. Theory Appl., Birkhäuser, Boston, 2006.

[5] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`

[6] H. P. Jakobsen, *Hermitian symmetric spaces and their unitary highest weight modules,* J. Funct. Anal. **52** (1983), 385–412.

[7] P. Pandžić, *Dirac operators and unitarizability of Harish-Chandra modules*, Math. Commun. **15** (2010), 273–279.
[8] P. Pandžić, A. Prlić, G. Savin, V. Souček and V. Tuček, *On the classification of unitary highest weight modules in the exceptional cases*, in preparation.
[9] P. Pandžić, A. Prlić, V. Souček and V. Tuček, *Dirac inequality for highest weight Harish-Chandra modules I*, Math. Inequalities Appl. **26** (2023), 233–265.
[10] P. Pandžić, A. Prlić, V. Souček and V. Tuček, *Dirac inequality for highest weight Harish-Chandra modules II*, Math. Inequalities Appl. **26** (2023), 729–760.
[11] P. Pandžić, A. Prlić, V. Souček and V. Tuček, *On the classification of unitary highest weight modules*, `arXiv:2305.15892`.
[12] R. Parthasarathy, *Dirac operator and the discrete series,* Ann. of Math. (2) **96** (1972), 1–30.
[13] R. Parthasarathy, *Criteria for the unitarizability of some highest weight modules,* Proc. Indian Acad. Sci. Sect. A Math. Sci. **89** (1980), 1–24.
[14] S. A. Salamanca-Riba and D. A. Vogan, Jr., *On the classification of unitary representations of reductive Lie groups,* Ann. of Math. (2) **148** (1998), 1067–1133.
[15] D. A. Vogan, Jr., *Dirac operators and unitary representations,* 3 talks at MIT Lie groups seminar, Fall 1997.
[16] D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with nonzero cohomology,* Compositio Math. **53** (1984), 51–90.
[17] K. D. Wong, *On the unitary dual of $U(p,q)$: proof of a conjecture of Salamanca-Riba and Vogan,* `arxiv:2210.08684`.

Pojačanje Diracove nejednakosti

Pavle Pandžić i Ana Prlić

Sažetak. Objašnjavamo ideju koja vodi ka mogućem dokazu slutnje Salamanca-Ribe i Vogana. Ova slutnja, također nazvana slutnja konveksne ljuske, pojačava dobro poznatu Partahasarathyjevu Diracovu nejednakost, koja je bila korisna u nekoliko parcijalnih klasifikacija unitarnih reprezentacija realnih reduktivnih grupa. Ideja koju predstavljamo proizašla je iz suradnje s Davidom Renardom.

Pavle Pandžić
Department of Mathematics, Faculty of Science, University of Zagreb
Bijenička 30, 10000 Zagreb, Croatia
E-mail: `pandzic@math.hr`

Ana Prlić
Department of Mathematics, Faculty of Science, University of Zagreb
Bijenička 30, 10000 Zagreb, Croatia
E-mail: `anaprlic@math.hr`

Received: 14.6.2023.
Revised: 29.7.2023.
Accepted: 19.9.2023.