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BERNSTEIN PROJECTOR OF COXETER DEPTH

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We dedicate this paper to our friend, colleague and teacher Marko Tadić on the occasion of his 70th birthday. In particular, both authors have enjoyed friendship with Marko for over four decades, and the second author thanks Marko for introducing him to the beautiful subject of representation theory.

Abstract. We decompose the Bernstein projector of Coxeter depth as a sum of the Bernstein projector of depth 0 and the projector on simple supercuspidal representations.

1. Introduction

Suppose k is a non-archimedean local field and $\mathbf{G}$ is a simple, simply connected group defined and split over k. Let $\mathcal{G} = \mathbf{G}(k)$ denote the group of k-rational points. Let $\mathcal{B}$ be the corresponding Bruhat-Tits building [4]. As a consequence of the work [5], [6] of Moy-Prasad on unrefined minimal K-types, for any irreducible representation (π, V) of $\mathcal{G}$, there is a nonnegative rational number $\rho(\pi)$ (called the depth of π). It can be characterized as the smallest rational number r so that there exists $x \in \mathcal{B}$ with $V^{\mathcal{G}_{x,r^+}} \neq \{0\}$, where $\mathcal{G}_{x,r^+}$ is the Moy-Prasad subgroup of $\mathcal{G}$ attached to x. Moreover, if we fix $r \geq 0$, then any smooth representation (π, V) of $\mathcal{G}$ can be decomposed, as a G-module,

$$V = V_r \oplus V_r'$$

where $V_r \subset V$ is a $\mathcal{G}$-submodule generated by $V^{\mathcal{G}_{x,r^+}}$ as x runs over $\mathcal{B}$ [2, Proposition 5.2]. As a consequence of the work [1] of Bernstein, there exist a $\mathcal{G}$-invariant distribution $P_{\leq r}$, an element of the Bernstein center, such that $\pi(P_{\leq r})$ is the projector on V_r, for every smooth representation (π, V).

In [3], Bezrukavnikov–Kazhdan–Varshavsky give a presentation of the distribution $P_{\leq r}$ using a refined simplicial structure on the building, as follows. If N is a natural number such that $r \in \mathbb{Z}/N$, divide each chamber C into

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$N^{\dim \mathcal{B}}$ congruent simplices (each simplex is thus obtained from C by rescaling by the factor $1/N$) and denote the building with this simplicial structure by $\mathcal{B}_N$. The key property of $\mathcal{B}_N$ are that the groups $\mathcal{G}_{x,r^+}$ are constant on its facets. Fix a Haar measure μ on $\mathcal{G}$, and for any open compact subgroup K of $\mathcal{G}$ define the $C_c(\mathcal{G})$ idempotent:

$$e_K \ := \ \frac{1}{\mu(K)} \ 1_K \ .$$

Then $P_{\le r}$ is an Euler–Poincaré sum

$$P_{\le r} \ = \ \sum_{F \subset \mathcal{B}_N} (-1)^{\dim(F)} \, e_{\mathcal{G}_{x,r^+}}$$

where x is (any) point in the facet F.

Observe that, for a fixed r, there are infinitely many N such that $r \in \mathbb{Z}/N$. Accordingly, there are many expressions for the depth $\le r$ projector $P_{\le r}$. However, the sign $(-1)^{\dim F}$ in the Euler-Poincare formula for $P_{\le r}$ is the Euler characteristic of the facet F, and additivity of the Euler characteristic implies that the different expressions are the same. For example, if r is an integer then we can use either $\mathcal{B}$ or $\mathcal{B}_N$ to write down the projector $P_{\le r}$. The equality of the expressions boils down to the fact that for every facet F of $\mathcal{B}$

$$(-1)^{\dim F} = \sum_{E \in \mathcal{B}_N, E \subset F} (-1)^{\dim E}.$$

Thus, more generally, if we break up $\mathcal{B}$ into a disjoint union of cells R such that the groups $\mathcal{G}_{x,r^+}$ are constant on each cell R, then we have an expression

$$P_{\le r} = \sum_{R \subset \mathcal{B}} (-1)^{\dim R} \cdot e_{\mathcal{G}_{x,r^+}}$$

where x is (any) point in R. In particular, one can use the canonical partition of $\mathcal{B}$ defined on each apartment $\mathcal{A}$ by hyperplanes $\psi = r$ cut out by all affine roots ψ. We shall use this observation to prove that $P_{\le r}$ is constant in the range $0 < r < 1/h$ where h is the Coxeter number attached to the root system Φ. It follows at once that $\mathcal{G}$ has no irreducible representations of positive depth less than $1/h$.

Next, as r approaches $1/h$, we compare $P_{\le r}$ and $P_{\le \frac{1}{h}}$. We obtain

$$P_{\le \frac{1}{h}} = P_{\le \frac{1}{h} - \epsilon} + P^{\mathrm{sc}}_{\frac{1}{h}}$$

where $P^{\mathrm{sc}}_{\frac{1}{h}}$ is the Bernstein projector associated to simple supercuspidal unrefined data [7]. This implies that all irreducible representations of $\mathcal{G}$ of the depth $1/h$ are super cuspidal, in fact induced from an open compact subgroup. (These results were previously obtained by Reeder and Yu, essentially a consequence of [8, Proposition 8.1].) Let $h' = 1 - \frac{1}{h}$. A similar analysis implies that $P_{\leq r}$ is constant in the range $h' < r < 1$ as well as

$$P_{\leq h'} = P_{\leq h'-\epsilon} + P^{\mathrm{sc}}_{h'}$$

where $P^{\mathrm{sc}}_{h'}$ is another super-cuspidal projector. Thus $\mathcal{G}$ has no irreducible representations of depth $h' < r < 1$, and those of the depth h' are supercuspidal, induced from an open compact subgroup.

We finish the introduction with the following remark. Quotient groups $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$ depend only on r modulo $\mathbb{Z}$. Hence our analysis is applicable to depths in the range $n < r < n + 1/h$ and $n + h' < r < n + 1$, where n is any non-negative integer. As a consequence, $\mathcal{G}$ has no irreducible representations of these depths and irreducible representations of the depths $n + 1/h$ and $n + h'$ are all super cuspidal.

2. Notation

Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$. Let $\mathcal{S} = \mathbf{S}(k)$. Let $\Phi(\mathcal{S})$ be the set roots of $\mathbf{G}$ with respect to $\mathbf{S}$, and let

$$\Psi(\mathcal{S}) \ := \ \{ \ \alpha + k \mid \alpha \ \in \ \Phi \ , \ k \in \mathbb{Z} \ \} \cup \{0, 1, 2 \ldots\}$$

be the set of affine roots (with respect to $\mathbf{S}$) where constants are also called imaginary roots. When clear, we abbreviate these two sets to Φ and Ψ respectively. For an affine root $\psi \in \Psi$, let $\mathcal{X}_\psi$ denote the attached affine root group in $\mathcal{G}$. If ψ is an imaginary root, that is $\psi = n$ a non-negative integer, then $\mathcal{X}_\psi = \mathcal{S}_n$ where

$$\mathcal{S} \supset \mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \ldots$$

is the filtration of $\mathcal{S}$ by principal congruence subgroups.

Let $x \in \mathcal{B}$. Pick an apartment $\mathcal{A}$ of $\mathcal{B}$ containing x. Let $\mathcal{S}$ be a maximal split torus stabilizing $\mathcal{A}$. Let $r \geq 0$. We have the Moy-Prasad groups

$$\mathcal{G}_{x,r} = \langle \mathcal{X}_\psi : \psi(x) \geq r \rangle \text{ and } \ \mathcal{G}_{x,r^+} = \langle \mathcal{X}_\psi : \psi(x) > r \rangle,$$

that is, the groups generated by affine root groups (including imaginary roots) $\mathcal{X}_\psi$ such that $\psi(x) \geq r$ and $\psi(x) > r$, respectively.

If χ is a character of $\mathcal{G}_{x,r}$ define the $C_c(\mathcal{G})$ idempotent:

$$e_{\mathcal{G}_{x,r},\chi} = \begin{cases} \frac{1}{\mu(\mathcal{G}_{x,r})} \ \chi & \text{on } \mathcal{G}_{x,r} \\[2ex] 0 & \text{off } \mathcal{G}_{x,r} \end{cases}$$

3. Two supercuspidal projectors

We define a projector $P^{\mathrm{sc}}_{\frac{1}{h}}$ to simple supercuspidal representationa. Let C be a chamber in an apartment $\mathcal{A}$ and let $\Delta = \{\psi_0, \ldots, \psi_\ell\}$ the corresponding set of simple affine roots. Then

$$n_0\psi_0 + \ldots + n_\ell\psi_\ell = 1$$

where $n_0 + \ldots + n_\ell = h$, is the Coxeter number. Let x_C be the barycenter of C, that is, the unique point in C such that $\psi_i(x) = \frac{1}{h}$ for all simple roots. Then

$$\mathcal{G}_{x_C,\frac{1}{h}}/\mathcal{G}_{x_C,\frac{1}{h}^+} = \mathcal{X}_{\psi_0}/\mathcal{X}_{\psi_0^+} \oplus \ldots \oplus \mathcal{X}_{\psi_\ell}/\mathcal{X}_{\psi_\ell^+}.$$

A character χ of $\mathcal{G}_{x_C,\frac{1}{h}}/\mathcal{G}_{x_C,\frac{1}{h}^+}$ is cuspidal if it is non-trivial on each of the $\ell+1$ summands. Then x_C is a vertex in the refined building $\mathcal{B}_h$ and (x_C, χ) is a cuspidal pair of depth $1/h$ in the sense of [7]. The group $\mathcal{G}$ acts with finitely many orbits on the collection of these cuspidal pairs and if follows from the main result [7] that the sum (over all chambers)

$$P^{\mathrm{sc}} = \sum_{C\subset\mathcal{B}} \sum_{\chi-\mathrm{cuspidal}} e_{\mathcal{G}_{x_C,\frac{1}{h}},\chi}$$

is an idempotent in the Bernstein center. Moreover, P^{sc} acts nontrivially only on a class of supercuspidal representations of $\mathcal{G}$, also known as simple supercuspidal representations.

We now define a second projector. The definition is similar. Observe that $\psi \mapsto \phi = 1 - \psi$ is an involution on the set of non-imaginary affine roots, and $\psi = r$ is equivalent to $\phi = 1 - r$. Let $\phi_i = 1 - \psi_i$, for the simple roots ψ_i, and let $h' = 1 - 1/h$. Then

$$\mathcal{G}_{x_C,h'}/\mathcal{G}_{x_C,h'^+} = \mathcal{X}_{\phi_0}/\mathcal{X}_{\phi_0^+} \oplus \ldots \oplus \mathcal{X}_{\phi_\ell}/\mathcal{X}_{\phi_\ell^+}.$$

A character χ of $\mathcal{G}_{x_C,h'}/\mathcal{G}_{x_C,h'^+}$ is cuspidal if it is non-trivial on each of $\ell+1$ summands. Then

$$P^{\mathrm{sc}}_{h'} = \sum_{C\subset\mathcal{B}} \sum_{\chi-\mathrm{cuspidal}} e_{\mathcal{G}_{x_C,h'},\chi}$$

is an idempotent in the Bernstein center, by the main result [7], that acts non-trivially on a class of supercuspidal representations.

4. Main work

In this section, as the first result, we shall prove that $P_{\leq r}$ is constant in the range $0 < r < \frac{1}{h}$. Recall that the simplicial decomposition of $\mathcal{B}$ on each apartment is cut out by hyperplanes $\psi = 0$ for all non-imaginary affine roots. Let us refine this by adding hyperplanes $\psi = r$ for all non-imaginary affine roots. Let $\mathcal{B}_r$ denote the resulting cell decomposition. It is clear that that the groups $\mathcal{G}_{x,r^+}$ are constant on cells.

Proposition 4.1. *Let h be the Coxeter number. Then $P_{\leq s} = P_{\leq r}$ for all $0 < s < r \leq 1/h$.*

Proof. Let C be a chamber in an apartment $\mathcal{A}$ and let $\Delta = \{\psi_0, \ldots, \psi_\ell\}$ the corresponding set of simple affine roots. For every facet F of C let Φ_F be the set of affine roots vanishing on F. It is a root system. Let Φ_F^+ be the roots positive on the interior of C, and let $\Delta_F \subset \Phi_F^+$ be the corresponding set of simple roots. Then Δ_F is a subset of Δ. In this way the facets of C correspond one-one to proper ($\subsetneq$) subsets of Δ, where C corresponds to empty set.

Assume $r < 1/h$. Let $\bar{C}$ denote the closure of C. In order to understand $\mathcal{B}_r$, we organize hyperplanes $\psi = r$ that intersect and partition $\bar{C}$ as follows. We have two steps:

First step:

In this step, we partition C according to its facets. To a facet F of C, define

$$X_F^r = \{x \in \bar{C} \mid \psi(x) \leq r \ \text{ for } \psi \in \Delta_F \text{ and } \psi(x) > r \ \text{ for } \psi \in \Delta \setminus \Delta_F\}.$$

For every $\psi \in \Delta$ and $x \in \bar{C}$, either $\psi(x) > r$ or $\psi(x) \leq r$. Thus we evidently have the disjoint union

$$\bar{C} = \cup_{F \subseteq \bar{C}} X_F^r. \tag{4.1}$$

Observe that X_C^r is an open set (hence a single cell of dimension ℓ). It is non-empty precisely since $r < 1/h$.

Let V_F be a closed positive Weyl chamber for Δ_F (so $\dim(V_F)+\dim(F) = \ell$). It may be convenient to realize V_F as a subset of the affine subspace $H_F \subset \mathcal{A}$ containing the barycenter x_F of F as the origin, perpendicular and complementary to F. Let $V_F^r \subset V_F$ be cut out by inequalities $\psi \leq r$ for all $\psi \in \Delta_F$. Observe that we have a homeomorphism

$$X_F^r \cong F \times V_F^r.$$

Indeed, the cross sections of X_F^r parallel to F are parameterized by points in V_F^r. Each cross section can be projected orthogonally on F where it can be identified by F by radial homothety centered at x_F.

Second step:

We decompose each X_F^r into cells cut out by all hyperplanes $\psi = r$ for all $\psi \in \Phi_F^+$. Under the above homeomorphism this amounts to decomposing V_F^r. In this way we obtained a cell decomposition of $\bar{C}$ such that the groups $\mathcal{G}_{x,r^+}$ are constant on each cell.

Next, we need the following lemma.

LEMMA 4.2. *Let $0 < s < r < 1/h$. For any face F of C, there exists a cell-preserving homeomorphism $\varphi : X_F^r \to X_F^s$ such that $\mathcal{G}_{x,r^+} = \mathcal{G}_{\varphi(x),s^+}$ for every $x \in X_F^r$.*

PROOF. Using the identifications $X_F^r = F \times V_F^r$ and $X_F^s = F \times V_F^s$, we define φ as identity on F and rescaling by the factor s/r on V_F^r. The resulting map sends half planes $\psi \leq r$ to halfplanes $\psi \leq s$, for $\psi \in \Phi_F^+$. The lemma follows. □

The maps constructed in the lemma patch together to give a homeomorphism $f : \mathcal{B}_r \to \mathcal{B}_s$ respecting the two cell decompositions, and $\mathcal{G}_{x,r^+} = \mathcal{G}_{f(x),s^+}$ for any $x \in \mathcal{B}$. Hence $P_{\leq r} = P_{\leq s}$, as desired.

□

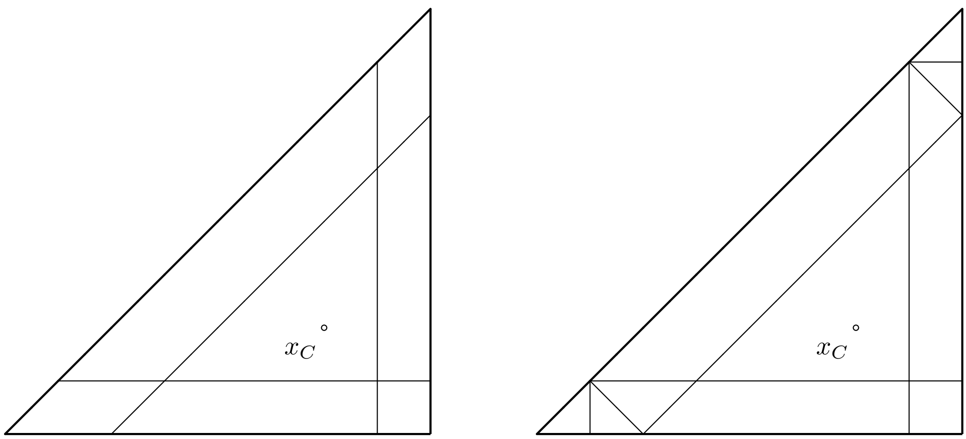

FIGURE 1. Illustration of the two steps in Proposition 4.1 for the root system C2 and $r = 1/8$

Next we analyze the projector $P_{\frac{1}{h}}$. Recall the cuspidal projector

$$P_{\frac{1}{h}}^{\mathrm{sc}} = \sum_{C \subset \mathcal{B}} \sum_{\chi-\text{cuspidal}} e_{\mathcal{G}_{x_C, \frac{1}{h}}, \chi}.$$

PROPOSITION 4.3. *Let h be the Coxeter number. Let $0 < r < 1/h$. Then*

$$P_{\leq \frac{1}{h}} = P_{\leq r} + P_{\frac{1}{h}}^{\mathrm{sc}}.$$

Proof. We have to understand how the decomposition (4.1) changes at $r = 1/h$. As we observed earlier, X_C^r is an open set that becomes the empty set at $r = 1/h$. Instead, at this depth, we have a new cell x_C, the barycenter of C. Thus we have

$$\bar{C} = \cup_{F \subsetneq C} \;\; X_F^{1/h} \; \cup \; \{x_C\}.$$

If F is a facet of C then X_F^r contains a cell

$$Y_F^r = \{x \in \bar{C} \mid \psi(x) = r \;\text{ for } \psi \in \Delta_F \text{ and } \psi(x) > r \;\text{ for } \psi \in \Delta \setminus \Delta_F\}.$$

Observe that $Y_C^r = X_C^r$, otherwise Y_F^r is boundary facet of the closure of X_C^r, homeomorphic to F by the homothety centered at x_C. In other words, if $\bar{X}_C^r$ denotes the closure of X_C^r, then

$$\bar{X}_C^r = \cup_{F \subseteq C} Y_F^r.$$

Now an analogue of Lemma 4.2 holds, except this time we have a homeomorphism from $X_F^r \setminus Y_F^r$ to $X_F^{1/h}$. These homomorphisms patch to a homomorphism

$$f : \mathcal{B}_r \setminus (\cup_{C \subset \mathcal{B}} \bar{X}_C^r) \to \mathcal{B}_{\frac{1}{h}} \setminus (\cup_{C \subset \mathcal{B}} \{x_C\})$$

Thus it follows that

$$P_{\leq 1/h} - P_{\leq r} = \sum_{C \subset \mathcal{B}} (e_{\mathcal{G}_{x_C, \frac{1}{h}+}} - \sum_{F \subset C} (-1)^{\dim F} \cdot e_{\mathcal{G}_{Y_F^r, r+}}).$$

The proposition follows from the following lemma:

Lemma 4.4. *For any chamber C in $\mathcal{B}$,*

$$e_{\mathcal{G}_{x_C, \frac{1}{h}+}} - \sum_{F \subset C} (-1)^{\dim F} \cdot e_{\mathcal{G}_{Y_F^r, r+}} = \sum_{\chi - cuspidal} e_{\mathcal{G}_{x_C, \frac{1}{h}}, \chi}$$

Proof. Observe that

$$\mathcal{G}_{x_C, \frac{1}{h}} / \mathcal{G}_{Y_F^r, r+} = \oplus_{\psi \in \Delta_F} \mathcal{X}_\psi / \mathcal{X}_{\psi+}.$$

Hence $e_{\mathcal{G}_{Y_F^r, r+}}$ is the sum of $e_{x_C, \chi}$ over all characters χ trivial on $\mathcal{X}_\psi$ for $\psi \in \Delta_F$. The lemma follows from a simple inclusion-exclusion argument. □

□

Corollary 4.5. *The group G has no representations of depth r, $0 < r < 1/h$. Irreducible representations of depth $1/h$ are all simple supercuspidal.*

Finally we consider $h' < r < 1$ where $h' = 1 - \frac{1}{h}$. Since $\psi \mapsto 1 - \psi$ is an involution on the set of non-imaginary affine roots it follows that $\mathcal{B}_r = \mathcal{B}_{1-r}$. Thus arguing in the same way as in the range $0 < r < \frac{1}{h}$, one sees that $P_{\leq r}$ is constant in the range $h' < r < 1$. As r crosses over h' (but slightly so) then

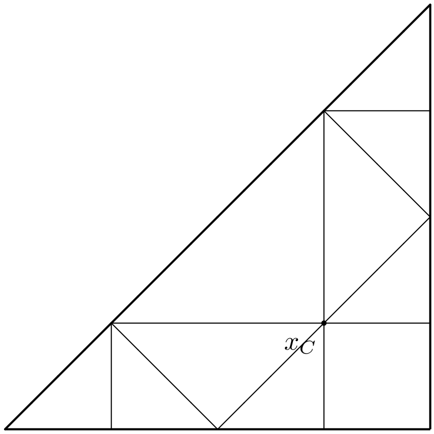

FIGURE 2. The canonical cell decomposition for the root system C2 and $r = 1/4$

the cell decomposition of $\mathcal{B}$ changes only at the barycenters of chambers. We leave details out for the readers to check that

$$P_{\leq h'} = P_{\leq h'-\epsilon} + P_{h'}^{\mathrm{sc}}.$$

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Bernsteinov projektor pridružen Coxeterovom broju

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Sažetak. U ovom radu dokazujemo da se Bernsteinov projektor na reprezentacije Coxeterove dubine dekomponira na sumu Bernsteinovog projektora na reprezentacije dubine 0 i projektora na jednostavne superkuspidalne reprezentacije.

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