Determination of the steady state response of viscoelastically supported rectangular specially orthotropic plates with varying supported area

Gokhan Altintas and M. Bagci
Celal Bayar University, Department of Civil Engineering, 45140 Muradiye, Manisa, TURKEY
e-mail: gokhan.altintas@bayar.edu.tr

SUMMARY
The influence of the amount of the supported area on the free and forced vibration properties of anisotropic plate is presented. Using the energy based finite difference method, the problem is modelled by a kind of finite difference element. Due to the significance of the fundamental frequency of the plate, its variation was investigated with respect to the amount of the supported area on the plate, mechanical properties of plate material and translational spring coefficient of supports. The steady state response of viscoelastically supported plates was also investigated numerically for various damping coefficients and amounts of supported areas. Numerical results are obtained to investigate the effect of the ratio of the plate system. In the numerical examples, the natural frequency parameters and steady state responses to a sinusoidally varying force are assessed for the fundamental mode. Results showing effect of supported area ratios of plate indicate that variation of ratio of supported area of plate system is very significant. Convergence studies are made. Many new results have been presented. Considered problems are solved within the frame work of Kirchhoff-Love hypothesis.

Key words: Orthotropic plate, viscoelastic support, supported area, damping coefficient, vibration analysis.

1. INTRODUCTION
It is generally accepted that classical support conditions employed in the analysis of rectangular plate behaviour represent only limiting mathematical conditions. The actual boundary conditions of a real system are mostly not classical, for example in ship plating, machine tables, circuit boards, solar panels, bridge decks, aircraft and marine structures supports generally accepted are elastic. In many branches of modern industry, these panels and plates are fabricated from composite materials. Therefore, the present investigation may be considered to be a problem of the mechanics of elements fabricated from composite materials.

Lot of work has been undertaken for the analysis of a rectangular plate in the case of free and forced vibrations in literature. Extensive investigation has been carried out on the analysis of the free vibration of rectangular plates having classical boundary conditions [1-6] and elastically restrained edges [7-31] have been widely analyzed. Viscoelastically supported plates were studied by several researchers for point supported plate systems. Yamada and co-workers [32] studied free vibrations of elastically point-supported plates and forced vibrations of viscoelastically point-supported isotropic plates. Kocatürk and Altintas [33, 34] extended Yamada’s [32] problem in case of anisotropic plates by using finite difference technique.

In this paper, plate problems are studied particularly for the case of boundary conditions elastically and viscoelastically restrained against translation. To represent many practical applications on industrial structures supports are located on areas as wide bands...
is specially orthotropic. Given
coincide with the
displacement of the mid-surface of the plate
2. ANALYSIS
Consider a viscoelastically supported loaded plate
with side lengths \( L_X \), \( L_Y \) and thickness \( h \) subjected to
a concentrated force as shown in Figure 1. Translational stiffness and damping coefficients
were assigned equally per supported area. Supports are
obtained by using Kelvin-Voight type point support in
every discrete area on finite difference mesh. By
obtained by using Kelvin-Voight type point support in
a concentrated force as shown in Figure 1.

Fig. 1 Viscoelastically supported plate subjected to
concentrated force

The elastic symmetry axis of the plate material
coincide with the \( 0X \) and \( 0Y \) axes. Therefore the plate
is specially orthotropic. Given \( W \) is the lateral
displacement of the mid-surface of the plate

\[
U = \frac{D_{XX}}{2} \int \int \left[ \frac{\partial^2 W}{\partial X^2} \right]^2 + 2\nu_{XY} \frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial Y^2} + e \left( \frac{\partial^2 W}{\partial Y^2} \right)^2 + 4D_{66} \frac{D_{XX}}{\partial Y^2} \left( \frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial Y^2} \right)^2 \right] dX dY
\]

and maximum kinetic energy of the plate is:

\[
T = \frac{1}{2} \int \int \left[ \rho h \omega^2 \right] W dX dY
\]

where \( D_{XX}, D_{YY} \) and \( D_{66} \) are expressed as follows:

\[
D_{XX} = \frac{(E_X h^3)}{12}, \quad D_{YY} = \frac{(E_Y h^3)}{12}, \quad D_{66} = \frac{(G_{XY} h^3)}{12}
\]

where \( G_{XY} \) is shear modulus. \( E_X', E_Y' \) are derived using:

\[
v_{XX}E_Y' = v_{XY}E_Y, \quad e = E_Y/E_X, \quad E_X' = E_X(1-v^2_{XX}e), \quad E_Y' = E_Ye(1-v^2_{XX}e)
\]

The additional strain energy and dissipation function
per viscoelastic support is:

\[
F_s = 1/2 k' W^2, \quad D=1/2 c' (W_S)^2
\]

where \( k' \) and \( c' \) is spring coefficient and damping
coefficient per viscoelastic support, \( E_X, E_Y \) are Young’s
moduli in the \( 0X \) and \( 0Y \) directions, respectively, and
\( v_{XX} \) is the Poisson’s ratio for the strain response in the \( X \)
direction due to an applied stress in the \( Y \) direction. The
total energy of the whole plate can be found by summing
the entire area of plate with supports and external force.
The potential energy from external force is:

\[
U_i = -F_i W_i
\]

where \( F_i \) and \( W_i \) are external forces and corresponding
displacements.

Introducing the following non-dimensional parameters:

\[
\tilde{X} = \frac{X}{a}, \quad \tilde{Y} = \frac{Y}{b}, \quad \alpha = \frac{a}{b}, \quad \tilde{w}(x,y) = w(x,y)e^{\alpha iY} = W/a, \quad i = \sqrt{-1}
\]

the above energy expressions can be written as:

\[
U_{m,n} = \frac{D_{m}}{2} \left[ \frac{1}{\alpha X^2} \left( \tilde{w}_{m-n} - 2\tilde{w}_{m,n} + \tilde{w}_{m+n} \right) \right]^2 + \frac{2\alpha \nu_{XX}}{\Delta X^2 \Delta Y^2} \left( \tilde{w}_{m-n} - 2\tilde{w}_{m,n} + \tilde{w}_{m+n} \right) \left( \tilde{w}_{m+n,1} - 2\tilde{w}_{m,n} + \tilde{w}_{m+n,1} \right) + \frac{4\alpha D_{66}}{D_{XX}} \left( \tilde{w}_{m-n,1} - 2\tilde{w}_{m,n} + \tilde{w}_{m+n,1} \right) \left( \tilde{w}_{m,n-1} - 2\tilde{w}_{m,n} + \tilde{w}_{m,n+1} \right) \right] \Delta X \Delta Y
\]

\[
T_{m,n} = \frac{\rho \alpha h}{2} \left( \tilde{w}_{m,n} \right) \Delta X \Delta Y, \quad F_i = \frac{1}{2} a' k' \tilde{w}_{m,n}, \quad D = \frac{1}{2} a' c' \left( \tilde{w}_{m,n} \right)^2, \quad F_e = -aQ\tilde{w}_{m,n}
\]
The derivative terms were approximated in terms of discrete displacements at gridpoints (see Figure 2) by using the following finite difference operators:

\[
\begin{align*}
\frac{\partial^2 \Theta}{\partial x^2}_{m,n} &= \frac{1}{\Delta x^2} \left( \Theta_{m-1,n} - 2\Theta_{m,n} + \Theta_{m+1,n} \right) \\
\frac{\partial^2 \Theta}{\partial y^2}_{m,n} &= \frac{1}{\Delta y^2} \left( \Theta_{m,n-1} - 2\Theta_{m,n} + \Theta_{m,n+1} \right) \\
\frac{\partial^2 \Theta}{\partial x \partial y}_{m,n} &= \frac{1}{4 \Delta x \Delta y} \left( \Theta_{m-1,n-1} - \Theta_{m+1,n-1} - \Theta_{m-1,n+1} + \Theta_{m+1,n+1} \right)
\end{align*}
\]

The energy for the whole plate can be found by summing over the entire area of the plate. Thus:

\[
\begin{align*}
U &= \sum_{m=1}^{N_x} \sum_{n=1}^{N_y} U_{m,n}, \quad T = \sum_{m=1}^{N_x} \sum_{n=1}^{N_y} T_{m,n}, \\
F_s &= \sum F_{si}, \quad D = \sum D_{ei}, \quad F_e = \sum F_{ei}
\end{align*}
\]

where \( N \times N \) is the total number of area elements on the plate.

The governing differential equation obtained from the Lagrange’s equation is given as:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial T}{\partial \bar{w}_{mn}} \right) - \frac{\partial (T - U)}{\partial \bar{w}_{mn}} - \frac{\partial D}{\partial \bar{w}_{mn}} + \frac{\partial F_s}{\partial \bar{w}_{mn}} + \frac{\partial F_e}{\partial \bar{w}_{mn}} &= 0
\end{align*}
\]

where \( \bar{w}_{mn} \) is the \( m, n \) discrete displacement and the overdot stands for the partial derivative with respect to time. Introducing the following non-dimensional parameters,

\[
k_j = \frac{k_j a^2}{b D_{11}}, \quad \gamma_j = \frac{c_j}{\sqrt{\rho \delta D_{11}}}, \quad \lambda^2 = \rho \omega a^4 D_{11}, \quad q = \frac{F_a}{D_{11}}
\]

and remembering that \( \Psi(x_j, x_2, t) = w(x_j, x_2) e^{i \omega t} \), which was given in Eq. (7), by using Eq. (11) for the mesh point \( m, n \) with Eq. (8a-e) results in the following expression:

\[
\begin{align*}
\left[ -\frac{4}{\alpha \Delta x^2} \left( w_{m-1,n} - 2w_{m,n} + w_{m+1,n} \right) - \frac{4V_{xx} \alpha}{\Delta x^2 \Delta y^2} \left( w_{m,n-1} - 2w_{m,n} + w_{m,n+1} \right) - \frac{4V_{yy} \alpha}{\Delta x^2 \Delta y^2} \left( w_{m-1,n} - 2w_{m,n} + w_{m+1,n} \right) - \frac{2 \lambda^2}{\alpha} w_{mn} \right] \Delta x \Delta y + \\
\left[ \frac{2}{\alpha \Delta x} \left( w_{m-1,n} - 2w_{m,n} + w_{m+1,n} \right) + \frac{2V_{xx} \alpha}{\Delta x^2 \Delta y^2} \left( w_{m-1,n-1} - 2w_{m-1,n} + w_{m+1,n} \right) \right] \Delta x \Delta y + \\
\left[ \frac{2}{\alpha \Delta x} \left( w_{m,n} - 2w_{m-1,n} + w_{m-2,n} \right) + \frac{2V_{yy} \alpha}{\Delta x^2 \Delta y^2} \left( w_{m-1,n-1} - 2w_{m-1,n} + w_{m+1,n} \right) \right] \Delta x \Delta y + \\
\left[ \frac{2V_{xx} \alpha}{\Delta x^2 \Delta y^2} \left( w_{m-1,n-1} - 2w_{m-1,n} + w_{m+1,n} \right) + \frac{2 \lambda^2}{\alpha} D_{yy} \right] \Delta x \Delta y + \\
\left[ \frac{\alpha D_{xx}}{\alpha \Delta x^2 \Delta y^2} \left( w_{m-2,n} - 2w_{m-1,n} + w_{m,n} \right) \right] \Delta x \Delta y + \\
\left[ -\frac{\alpha D_{xx}}{\alpha \Delta x^2 \Delta y^2} \left( w_{m-2,n} - 2w_{m-1,n} + w_{m,n} \right) \right] \Delta x \Delta y + \\
\left[ \frac{\alpha D_{xx}}{\alpha \Delta x^2 \Delta y^2} \left( w_{m,n} - 2w_{m-1,n} + w_{m-2,n} \right) \right] \Delta x \Delta y + \\
\left[ \frac{\alpha D_{xx}}{\alpha \Delta x^2 \Delta y^2} \left( w_{m,n} - 2w_{m-1,n} + w_{m-2,n} \right) \right] \Delta x \Delta y + \\
\left[ \frac{2(\kappa + i \lambda y)}{\alpha} \right] \Delta x \Delta y - \frac{\alpha q}{\Delta x \Delta y} \quad i = \sqrt{-1}
\end{align*}
\]
where $\alpha_s$ and $\alpha_Q$ are taken values 0, 1 depending on existence of support, load respectively on pivotal point $m$, $n$. One may prefer element and system matrix forms similar to finite element method. In the form of Eq. (13), there is no need for any manipulation to express final form of solution of algebraic equations.

For the whole mesh points, by using Eq. (13), the following set of linear algebraic equations is obtained which can be expressed in the following matrix form:

$$Aw + i\lambda yBw - \lambda^2 Cw = q$$  (14)

where $A$, $B$, and $C$, are coefficient matrices obtained by using Eq. (13) for all mesh points. For free vibration analysis, when the external force and damping of the supports are zero in Eq. (14), this situation results in a set of linear homogeneous equations that can be expressed in the following matrix form:

$$Aw - \lambda^2 Cw = 0$$  (15)

Numbering the mesh points is shown in Figure 2. By decreasing the dimensionless mesh widths, the accuracy can be increased.

The total magnitude of the reaction forces of the supports is given by:

$$N = \sum_{j=1}^{N} P_j = \sum_{j=1}^{N} (k_j + i\epsilon_j w)aw$$  (16)

and therefore the force transmissibility at the supports is determined by:

$$T_R = \sum P_j / F = \sum (k_j + i\epsilon j \lambda) w / q$$  (17)

The number of unknown displacements is $(N+2)^2$, where $N^2$ is the mesh size in the plate region.

3. NUMERICAL RESULTS

The steady state response to a concentrated force acting on an orthotropic square plate, which is viscoelastically supported on all edges is calculated numerically. Viscoelastic supports are given equally per supported area. A brief investigation of the free vibration of an elastically supported plate is necessary for a better understanding of the responses presented in this study. The natural frequencies of the elastically supported plate are determined by calculating the eigenvalues, assuming that the damping parameter of the supports and external force are zero. Poisson ratio $\nu$, is taken 0.3 in all numerical calculations.

In Table 1 the convergence of the fundamental mode is presented for the following $E_2/E_1$ ratios 0.6, 0.8, 1 respectively. Tabulated results in Table 1 were obtained for simple supported plate ($\kappa=\infty$). Frequency parameter is monotonic from below for all $E_2/E_1$ ratios. Convergence properties are not affected by $A_{sup}/A_{plt}$ ratio and they are not shown here. Due to the lack of the comparable results for different $A_{sup}/A_{plt}$ and $E_2/E_1$ ratios, only fundamental frequency of simple supported isotropic plate was compared in Table 1.

Gorman also studied elastically supported plate but results are given graphically in Ref. [11].

<table>
<thead>
<tr>
<th>$E_2/E_1$</th>
<th>$A_{sup}/A_{so}$=0.6</th>
<th>$A_{sup}/A_{so}$=0.8</th>
<th>$A_{sup}/A_{so}$=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa=1$</td>
<td>0.999 0.999 0.999</td>
<td>0.999</td>
<td>0.999</td>
</tr>
<tr>
<td>$\kappa=10$</td>
<td>3.158 3.148 3.123</td>
<td>3.095</td>
<td>3.065</td>
</tr>
<tr>
<td>$\kappa=100$</td>
<td>9.886 9.57 8.993</td>
<td>8.335</td>
<td></td>
</tr>
<tr>
<td>$\kappa=1000$</td>
<td>27.862 22.389 17.555</td>
<td>14.461</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$E_2/E_1$</th>
<th>$A_{sup}/A_{so}$=0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa=1$</td>
<td>0.999 0.999 0.999</td>
</tr>
<tr>
<td>$\kappa=10$</td>
<td>3.16 3.15 3.13</td>
</tr>
<tr>
<td>$\kappa=100$</td>
<td>9.903 9.63 9.112</td>
</tr>
<tr>
<td>$\kappa=1000$</td>
<td>28.256 22.976 18.167</td>
</tr>
</tbody>
</table>

In Figure 4 it is seen that transmissibilities are obtained as a function of frequency parameter for sets of $A_{sup}/A_{plt}$ and $\gamma$ for $\kappa=1000$. One can note that peak values of transmissibility curves for $\gamma$ increases while $A_{sup}/A_{plt}$ ratio decreases. This means an effect of damper decreases for $A_{sup}/A_{plt}$ ratio also decreases.
Figure 5 shows that the peak values of force transmissibilities occur for variation of damping parameter. When damping parameter increases transmissibilities decrease rapidly where the minimum peak values occur. In Figure 5 plotted for $\kappa=100$ minimum peak values occur, smaller magnitudes of $\gamma$ than in Figure 5 are plotted for $\kappa=1000$. When $\kappa$ increases some lines for example line “$p$” and line “$d$” in Figure 5 do not have minimum peak values in the range of fundamental mode.
Table 3  The frequencies at which the peak values of the force transmissibilities occur

<table>
<thead>
<tr>
<th>$\kappa$=100</th>
<th>$E_2/E_1$=0.6</th>
<th>$\gamma$=5</th>
<th>$T_0$</th>
<th>$\lambda$</th>
<th>$T_0$</th>
<th>$\lambda$</th>
<th>$T_0$</th>
<th>$\lambda$</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>$A_{sup}/A_{plt}$=0.8</td>
<td></td>
<td>$A_{sup}/A_{plt}$=0.7</td>
<td></td>
<td>$A_{sup}/A_{plt}$=0.6</td>
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</tr>
<tr>
<td>$\kappa$=100</td>
<td>$E_2/E_1$=0.6</td>
<td>$\gamma$=5</td>
<td>4.229</td>
<td>8.838</td>
<td>3.699</td>
<td>9.175</td>
<td>3.264</td>
<td>9.417</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma$=10</td>
<td>2.606</td>
<td>9.461</td>
<td>2.291</td>
<td>9.803</td>
<td>2.025</td>
<td>9.867</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma$=50</td>
<td>3.029</td>
<td>17.482</td>
<td>2.480</td>
<td>19.719</td>
<td>1.946</td>
<td>22.130</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma$=100</td>
<td>4.464</td>
<td>19.431</td>
<td>3.114</td>
<td>22.636</td>
<td>2.214</td>
<td>26.929</td>
</tr>
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<td>$\gamma$=5</td>
<td>1.288</td>
<td>2.676</td>
<td>1.271</td>
<td>2.606</td>
<td>1.258</td>
<td>2.530</td>
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<tr>
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<td>$\gamma$=10</td>
<td>1.344</td>
<td>15.508</td>
<td>1.314</td>
<td>18.206</td>
<td>1.091</td>
<td>2.662</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma$=50</td>
<td>2.333</td>
<td>17.966</td>
<td>2.326</td>
<td>20.599</td>
<td>1.830</td>
<td>22.652</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\gamma$=100</td>
<td>4.387</td>
<td>19.553</td>
<td>3.056</td>
<td>22.725</td>
<td>2.169</td>
<td>26.962</td>
</tr>
<tr>
<td>$\kappa$=10</td>
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<td>$\gamma$=5</td>
<td>3.996</td>
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<td>3.517</td>
<td>9.231</td>
<td>3.132</td>
<td>9.430</td>
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<tr>
<td></td>
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<td>$\gamma$=10</td>
<td>2.476</td>
<td>9.599</td>
<td>2.186</td>
<td>9.840</td>
<td>1.949</td>
<td>9.802</td>
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<tr>
<td></td>
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<td>$\gamma$=50</td>
<td>2.833</td>
<td>17.875</td>
<td>2.359</td>
<td>19.941</td>
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<td>21.916</td>
</tr>
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<td>4.092</td>
<td>20.569</td>
<td>2.944</td>
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<td>28.601</td>
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<tr>
<td>$\kappa$=10</td>
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<td>1.279</td>
<td>2.636</td>
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<td>2.577</td>
<td>1.253</td>
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<tr>
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<td>$\gamma$=10</td>
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<td>$\gamma$=50</td>
<td>2.655</td>
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<td>1.749</td>
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<td></td>
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<td>$\gamma$=100</td>
<td>4.016</td>
<td>20.717</td>
<td>2.889</td>
<td>24.131</td>
<td>2.102</td>
<td>28.687</td>
</tr>
</tbody>
</table>

Lines plotted for $\gamma$ and changes of frequency parameter are not shown in Figure 5. Table 3 depicts frequency parameters and transmissibility values where the peak values occur for the set of $\kappa, \gamma, E_2/E_1$. Frequency parameter tends to be close to the natural frequency of the plate for lower values of $\gamma$.

4. CONCLUSIONS

This paper presents, to the authors’ knowledge, the first known vibration analysis of viscoelastically supported anisotropic plate with varying supported area. The model can be used to simulate the actual boundary conditions of the plates. A simple numerical method has been presented to determine natural frequencies of
plates. The convergence studies are made. Frequency parameter is from below for different anisotropy ratios. Fundamental frequency was determined depending on the amount of supported area, spring coefficient and anisotropy. The effect of supported area ratio on fundamental frequency of plate is significant for higher stiffness coefficients of springs for all anisotropy ratios. The effect of the amount of supported area and damping coefficient on response curves is investigated. The effect of dampers increases when the supported area increases. The peak values of transmissibilities were plotted to determine where the minimum peak values were obtained. Numerical and graphical results have revealed the effects of supported area variations on the free and forced vibrations of anisotropic plate. It is believed that these novel results will be useful to designers in the various types of practical applications.

5. REFERENCES

ODREĐIVANJE REAKCIJE MIRNOG STANJA VISKOELASTIČNIH PODUPRTIH PRAVOKUTNIH, NAROČITO ORTOTROPNIH PLOČA KOJE SU PODUPRTE NA RAZLIČITIM PODRUČJIMA

SAŽETAK


Ključne riječi: ortotropna ploča, viskoelastični oslonac, poduprta površina, koeficijent prigušenja, analiza vibracija.