Economic Research-Ekonomska Istraživanja

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/rero20

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To cite this article: Paulo Rotella Junior, Luiz Célio Souza Rocha, Rogério Santana Peruchi, Giancarlo Aquila, Edson de Oliveira Pamplona, Karel Janda & Arthur Leandro Guerra Pires (2023) Robust portfolio optimization: a stochastic evaluation of worst-case scenarios, Economic Research-Ekonomska Istraživanja, 36:3, 2165525, DOI: 10.1080/1331677X.2023.2165525

To link to this article: https://doi.org/10.1080/1331677X.2023.2165525

Published online: 17 May 2023.

ECONOMIC RESEARCH-EKONOMSKA ISTRAŽIVANJA
2023, VOL. 36, NO. 3, 2165525
https://doi.org/10.1080/1331677X.2023.2165525

Robust portfolio optimization: a stochastic evaluation of worst-case scenarios

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ABSTRACT
This article presents a new approach for building robust portfolios based on stochastic efficiency analysis, by using the Chance Constrained Data Envelopment Analysis (CCDEA) model and periods of market downturn, i.e. worst-state market. The model is able to accommodate investors who exhibit different risk behaviors and the empirical analysis is done on assets traded on the Brazil Stock Exchange, B3 (Brasil, Bolsa, Balcão). The results confirm that the proposed model achieved portfolios that at the same time reduced systematic risk and maximized portfolio returns when working with worse market state data and higher levels of risk aversion. A higher level of risk aversion also led to better risk-return ratios, which can be seen in higher Sharpe ratio values.

ARTICLE HISTORY
Received 13 May 2022
Accepted 2 January 2023

KEYWORDS
Robust optimization; stochastic evaluation; chance constrained DEA; worst-case markets; portfolios

JEL CODES
G11; G14; C38; C61

1. Introduction

Diversification is a critical factor in reducing non-systematic risk in portfolio selection theory. There has been a growing amount of published research seeking to reconcile the benefits of diversification with investment practices (Brito, 2023; Kim et al., 2015). Portfolio selection is a broad theory that also involves the issue of capital allocation among a given number of assets so that the investment provides a higher return while minimizing risks i.e. a risk-adjusted return that is satisfactory for investors, similar to Markowitz's proposal (Jalota et al., 2023; Leung et al., 2012).

Seventy years after the development of the Markowitz model (1952), this classic approach of mean-variance is still one of the most used academic models in asset allocation and management, and has given rise to many new approaches

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(Gonçalves et al., 2022; Leung et al., 2012), that have been developed by various academics (Bouaddi & Moutanabbir, 2023; Chen et al., 2020; Jalota et al., 2023; Leung et al., 2012; López Prol & Kim, 2022). Whether for researchers or for investors, the investment selection process remains a major challenge for financial management (Ali et al., 2019; Markowitz, 2014).

Among several tools for efficiency measurement e.g. conventional statistical methods, non-parametric methods and artificial intelligence methods, Data Envelopment Analysis (DEA) can effectively measure the relative efficiency of Decision Making Units (DMUs), which employ multiple inputs to produce multiple outputs (Emrouznejad & Tavana, 2014; Shi & Wang, 2020).

DEA is a non-parametric method that has been broadly used in different types of companies and organizations, helping managers from diverse areas, including the financial area (Azadi et al., 2015; Emrouznejad & Tavana, 2014; Smętek et al., 2022). More recently, DEA continues to be used in efficiency evaluation and building portfolios (Amin & Hajjami, 2020; Atta Mills & Anyomi, 2022; Choi & Min, 2017; Edirisinghe & Zhang, 2010; Lim et al., 2014; Rotela Junior et al., 2015).

Since then, variations of the classic DEA models have been presented. Among these models, some seek to include approximate information or uncertainty, using a DEA model with Fuzzy coefficients (Azadi et al., 2015; Gong et al., 2021), or models like those proposed by Sengupta (1987), which combine Chance-Constrained Programming (CCP), from Charnes and Cooper (1963), with the DEA model (Jin et al., 2014).

The most widespread and traditional models of portfolio optimization theory, such as the reference models presented by Markowitz (1952), and Sharpe (1963), are recognized for being sensitive to small variations in data, and not considered robust (Kim et al., 2014, 2015). Consequently, researchers started to develop mathematical techniques on robust optimization. Several approaches have been used to increase the robustness of traditional models using mean-variance, and these approaches usually deal with solving max-min problems (Won & Kim, 2020; Xidonas et al., 2017). These techniques allow investors to incorporate risk into their portfolio optimization process considering estimate errors (Baltas & Yannacopoulos, 2019; Fabozzi et al., 2007, 2010; Sehgal & Mehra, 2020). Robust portfolio optimization has quickly become a widely applied approach among investors to incorporate uncertainty into their financial models (Kim et al., 2018).

Other relevant information for portfolio optimization theory was presented by Kim et al. (Kim et al., 2014, 2015, 2018). The authors state that the so-called robust models are achieved based on information coming from bear market periods. In other words, this information is more relevant than information coming from peak market periods, when seeking robustness. More recently, other researchers have analyzed market downturn conditions when proposing portfolio optimization models (Ashrafi & Thiele, 2021; Yu et al., 2019).

This study seeks to present a method for robust portfolio optimization based on the stochastic analysis of asset efficiency, using asset information taken from worst-case market scenarios. We used data from the Brazilian Stock Exchange (B3 - *Brasil, Bolsa, Balcão*).

In general terms, the presented model is the result of a combination of different mathematical techniques: Hierarchical Clustering, Chance-Constrained DEA (CCDEA), and the Sharpe approach. Hierarchical Clustering will be used to form different clusters, which contribute to diversifying the portfolio. The CCDEA model will allow us to stochastically identify the efficient assets in each group. Sharpe's approach will allow us to optimally allocate efficient assets in the portfolios. It is worth mentioning that information from worst-case market scenarios will be used.

The novelty of the study focuses on the fact that it allows the insertion of investor risk aversion in the portfolio optimization model. Furthermore, it contributes to the theory of robust portfolio optimization by showing the importance of worst-state market information.

2. Chance constrained DEA

DEA has been gaining more popularity as a non-parametric efficiency technique for measuring the performance of financial assets, as can be seen in recent studies such as Ghahtarani et al. (2022), Gong et al. (2021), Adam and Branda (2020) and Choi and Min (2017). The most used classic models in literature are deterministic and do not consider errors in random variables. According to Azadi and Saen (2012), the generalized randomness in the evaluation processes comes from data collection errors.

One of the first attempts to fill this gap involved developing Chance Constrained Programming in mathematical models for DEA (Charnes & Cooper, 1959), to incorporate stochastic variations in the data.

Saen and Azadi (2011), define Chance Constrained Programming (CCP) as a type of approach for stochastic optimization, appropriate for solving optimization problems with random variables included in the constraints, and sometimes in the objective function, as was done by Charnes and Cooper (1959) . The major contribution can be found in the research carried out by Sengupta (1987). The CCP can effectively reflect the reliability of satisfying a system with constraints under risky conditions. The CCP does not require that all restrictions are completely satisfied, they can be satisfied according to established probabilities (Azadi et al., 2012; Saen & Azadi, 2011).

The stochastic DEA model formulation is presented according to equations (1) through (4), where the i-th DMU, $\hat{x}_i = (\hat{x}_{i1}, \hat{x}_{i2}, \ldots, \hat{x}_{ia})^T$ and $\hat{y}_i = (\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ib})^T$ respectively denote the stochastic variables for the input and output vectors, where $i = 1, \ldots, n$. The objective function of the stochastic model is formulated by equation (1), where 'E' represents an expected value from the sum of weighted $\hat{y}_{iq}(q = 1, \ldots, b)$.

$$\max E\left(\sum_{q=1}^{b} u_q \hat{y}_{0q}\right) \tag{1}$$

s.t.:

$$E\left(\sum_{p=1}^{a} v_p \hat{x}_{0p}\right) = 1 \tag{2}$$

$$P\left(\frac{\sum_{q=1}^{b} u_q \hat{y}_{iq}}{\sum_{p=1}^{a} v_p \hat{x}_{ip}} \leq \beta_i\right) \geq 1 - \alpha_i \qquad i = 1, 2, \ldots, n \tag{3}$$

$$u_q, v_p \geq 0 \tag{4}$$

u_q and v_p respectively indicate the weight of the multipliers associated with the q-th output and the p-th input. $u_1, \ldots, u_b$, $v_1, \ldots, v_a$ are the weights that will be calculated by the optimization model. P is the probability, and superscript '^' means that $\hat{x}_{ip}$ and $\hat{y}_{iq}$ are random variables. Regarding the constraints, the model states that the proportion equal or inferior to β_i represents an efficiency level expected for the i-th DMU, the variation of which is in [0,1] (Cooper et al., 1996; Jin et al., 2014; Rotela Junior et al., 2015). α_i is the risk criterion adopted by the decision maker. 1-α_i is the probability of meeting the constraint requirements, which is a confidence level whose variance is [0,1] (Jin et al., 2014; Rotela Junior et al., 2015).

The formulation needs to be rewritten as proposed by Charnes and Cooper (1963), to provide a viable model from a computational point of view. Randomness is considered in this proposal, and the stochastic variable $\hat{x}_{ip}$ for each input can be represented as $\hat{x}_{ip} = \bar{x}_{ip} + \alpha_{ip}\xi$, where p presents a variation in the interval [1,b], and i is in [1,n]; $\bar{x}_{ip}$ is the expected value of $\hat{x}_{ip}$ and α_{ip} is the standard deviation (Rotela Junior et al., 2015).

Similarly, the stochastic variable $\hat{y}_{iq}$ for each output can be represented as $\hat{y}_{iq} = \bar{y}_{iq} + b_{iq}\xi$, where q has a variation in [1,a], and i in [1,n]; $\bar{y}_{iq}$ is the expected value of $\hat{y}_{iq}$ and b_{iq} is the standard deviation. Thus, it is assumed that the random variable ξ follows a normal distribution, since part of the stochastic disorders suggest that the errors are the result of data collection.

In order to make the model solution simpler, it is convenient to present its equivalent deterministic formulation. The objective function (as presented in equation (1)), can be remodeled according to equation (5):

$$E\left(\sum_{q=1}^{b} u_q \hat{y}_{0q}\right) = \sum_{q=1}^{b} u_q \bar{y}_{0q} \tag{5}$$

The model constraints, according to equations (2) and (3), when including the stochastic process, will be rewritten in equations (6) and (7):

$$E\left(\sum_{p=1}^{a} v_p \hat{x}_{0p}\right) = \sum_{p=1}^{a} v_p \bar{x}_{0p} = 1 \tag{6}$$

$$P\left(\frac{\sum_{q=1}^{b} u_q \hat{y}_{iq}}{\sum_{p=1}^{a} v_p \hat{x}_{ip}} \leq \beta_i\right) = P\left(\sum_{q=1}^{b} u_q \hat{y}_{iq} - \beta_i \sum_{p=1}^{a} v_p \hat{x}_{ip} \leq 0\right) \geq 1 - \alpha_i \quad i = 1, 2, \ldots, n \tag{7}$$

M_i and V_i are the average and variance of each random variable. These can be expressed as equations (8) and (9):

$$M_i = \sum_{q=1}^{b} u_q \bar{y}_{iq} - \beta_i \sum_{p=1}^{a} v_p \bar{x}_{ip} \tag{8}$$

$$V_i = \left(\sum_{q=1}^{b} u_q \bar{y}_{iq} - \beta_i \sum_{p=1}^{a} v_p \bar{x}_{ip} \right)^2 \sigma^2 \tag{9}$$

In this way, the random variable $\left(\frac{\left(\sum_{q=1}^{b} u_q \hat{y}_{iq} - \beta_i \sum_{p=1}^{a} v_p \hat{x}_{ip} \leq 0 \right) - M_i}{\sqrt{V_i}} \right)$ follows a normal distribution of mean zero and variance one. Thus, equation (7) can be expressed according to equation (10), or even in its equivalent form as expressed by equation (13).

$$\frac{-M_i}{\sqrt{V_i}} \geq \phi^{-1}(1 - \alpha_i) \qquad i = 1, 2, \ldots, n \tag{10}$$

Equation (10) can be written as equation (11):

$$\sum_{p=1}^{a} v_p \beta_i \left(\bar{x}_{ip} + \phi^{-1}(1 - \alpha_i) \alpha_{ip} \sigma \right) - \sum_{q=1}^{b} u_q \left(\bar{y}_{iq} + \phi^{-1}(1 - \alpha_i) b_{iq} \sigma \right) \geq 0 \quad i = 1, 2, \ldots, n \tag{11}$$

In this model, ϕ represents a function of standard normal distribution, and ϕ^{-1} is the inverse of the function. Finally, the CCDEA optimization model can be discussed. In this manner, the original proposal is presented as a linear model, according to equations (12)-(15):

$$\max E \left(\sum_{q=1}^{b} u_q \bar{y}_{0q} \right) \tag{12}$$

s.t.:

$$\sum_{p=1}^{a} v_p \bar{x}_{0p} = 1 \tag{13}$$

$$\sum_{p=1}^{a} v_p \beta_i \left(\bar{x}_{ip} + \phi^{-1}(1 - \alpha_i) \alpha_{ip} \sigma \right) - \sum_{q=1}^{b} u_q \left(\bar{y}_{iq} + \phi^{-1}(1 - \alpha_i) b_{iq} \sigma \right) \geq 0 \quad i = 1, 2, \ldots, n \tag{14}$$

$$u_q, v_p \geq 0 \tag{15}$$

The model of multipliers extends DEA applications to the financial area and helps in decision making. Besides the deterministic situation, efficiency can be measured considering random variables. The desired confidence levels of the model can be defined according to different situations of practical application, according to the particularities of each case.

3. Portfolio selection

Solutions for portfolio optimization are often influenced by poorly specified models or by errors in approximation, estimation, or even due to incomplete information. As demonstrated by Black and Litterman (1992), a small variation in the expected return of assets can result in a large alteration in the allocation of investments in an optimized portfolio. In other words, classic models for portfolio optimization are not robust because they are susceptible to small variations in data (Kim et al., 2015). As a matter of fact, Kim et al. (2014), affirmed that the main reason for questioning the Markowitz (1952) model is because of its high sensitivity towards small variations in data values.

Researchers have begun to incorporate risk by estimating errors directly in the portfolio optimization process using mathematical techniques for robust optimization. Different from traditional approaches where the data for the structure of portfolio allocation are deterministic, robust portfolio optimization incorporates the notion that these data have been estimated with errors (Fabozzi et al., 2007, 2010). In this case, data like the expected return and asset covariance are not traditional predictions but rather sets of probabilities e.g. confidence intervals. The two best-known methodologies for dealing with risk are robust and stochastic optimization (Xidonas et al., 2020).

It was identified that the correlation between assets tends to increase during periods of low market (bear market) performance, so investors cannot benefit from diversification when it is most needed. Worse still, correlation within capital markets has been increasing in recent periods (Kim et al., 2015). Some solutions have been proposed to overcome this problem, like employing variables that are less sensitive to historic data, or inserting risk sets on the parameters of traditional models (Fabozzi et al., 2007).

Of the several approaches for increasing robustness in the mean-variance model, robust portfolio optimization applies robust optimization techniques for asset allocation by solving max-min problems (Maciel, 2021; Won & Kim, 2020; Xidonas et al., 2017). According to Kim et al. (2015), even though worst-case optimization seems to be a natural extension of the mean-variance model for achieving robustness, more in-depth analysis on the importance of concentration in worst-case market scenarios has not been conducted.

The main contribution of the work conducted by Kim et al. (2015) was to demonstrate the importance of information on asset returns on the worst-performing days for achieving a robust portfolio. In other words, instead of selecting assets that always perform well in both bear markets and bull markets, assets that perform well in bear markets are usually considered. Therefore, these researchers believe that robustness can be achieved when worst-case market information is considered.

Furthermore, it is known that low beta assets perform better than high beta assets in crisis periods (market declines), since low beta assets reduce the overall risk and offer better returns (Kim et al., 2015).

4. Materials and methods

In order to establish a strategy for robust portfolio optimization using stochastic efficiency analysis, we used the following observations as starting points:

i. Preliminary results indicate that DEA is well suited for determining portfolio composition. DEA allows assets to be evaluated using criteria that represent investor interests, supporting the classic models of mean and variance;
ii. DEA models that consider data randomness allow investors to reduce the search space for assets that perform well, by presenting good data discrimination;
iii. Risk variation in stochastic efficiency analysis can meet the needs of investors with different attitudes towards risk;
iv. One difficulty in applying CCDEA to portfolio optimization (depending on the data) is the contradictory behavior of the constraints, making it difficult to identify efficient assets;
v. Hierarchical Clustering allows individuals or assets to be grouped based on the similarity or dissimilarity of these initial groups;
vi. Robust portfolio models achieve robustness by concentrating on information gathered from crisis periods or market recessions i.e. poor performance days are fundamental for building portfolios that perform well under any market conditions.

4.1. Selecting the variables

After defining the object of study, we set out to select a set of indicators that will be used in the efficiency analysis. Then, following the premises presented in the extensive DEA literature, the variables in this study will be defined as outputs or inputs.

Researchers have evaluated the impact of correlation on efficiency analyses, especially with a view to reducing model complexity and redundancy. Siriopoulos and Tziogkidis (2010) state that correlated inputs and outputs do not significantly affect the efficiency results of classic DEA models. The statement agrees with Charnes et al. (1994), who in their book use correlated variables in their studies. Therefore, we did not see a need to perform correlation tests for the possible input and output variables.

We chose to use input and output variables present in literature for DEA applications in the stock market (Powers & Mcmullen, 2002; Rotela Junior et al., 2014, 2015). It should be remembered that low beta assets have better returns while reducing the overall risk of the portfolio in worst-case market scenarios (Kim et al., 2015).

For this study, we chose to use the asset return, asset liquidity, and earnings per share (EPS) as output model variables. For input variables we chose to use the beta, the price-earnings (PE), and volatility (Kim et al., 2015; Powers & Mcmullen, 2002; Rotela Junior et al., 2014, 2015). For more information on calculating indicators, see Economatica (Economatica, 2022).

4.2. Sample selection and data collection

The sample comprised assets traded on the Brazilian Stock Exchange B3 (Brasil, Bolsa, Balcão). The B3 was founded after a merger between BM&FBOVESPA and Cetip, with participation in the Bovespa Index (Ibovespa).

Kim et al. (2015) used daily data for the return of the market index to identify worst-case market periods. The authors classified all the returns of the index in increasing order within a time interval. They then divided this period into n other periods. Within the longer period, Kim et al. (2015) defined n as ten, and for defining the worst-case market period, the tenth part was selected, corresponding to the smallest values presented for the index (i.e. 10% referring to the worst values). We used the same approach adopted by Kim et al. (2015), to identify crisis market periods.

We then proceeded to selecting the sample, however, we observed that only 61 companies in the Ibovespa index had all the necessary information for conducting the efficiency analysis. It is worth noting that Ibovespa index is a benchmark index of stocks traded on the Brazilian stock exchange. All information on the selected variables were collected using the software Economática®.

The information in this study corresponds to daily data for a period of 5 years. To validate the results, daily information from a 6-month period was used. It is important to highlight that the period used in this study was prior to the global covid-19 pandemic. It is worth noting that there is still no concrete evidence of the effects of the pandemic on stock markets.

The cumulative return was calculated for each proposed portfolio for the validation phase based on the results of the models adopted for optimization (identifying the ideal share).

5. Results and discussion

5.1. Data preparation and cluster analysis

We started by collecting data, and then proceeded using a proposal from Kim et al. (2015) by classifying the returns from Ibovespa in increasing order for the period adopted in this study. The remaining spreadsheet information followed such classification, and n was set equal to four for defining the worst-case market period, giving the model more than three hundred daily information pieces.

We calculated the average and variance for each of the variables adopted for the efficiency analysis using the data collected for each of the proposed scenarios i.e. worst-case ($n = 4$), and complete market information ($n = 1$).

We observed that the number of efficient assets was very reduced, even when varying the risk criterion. This led us to believe that the CCDEA model was composed of highly divergent constraints, making it more difficult to properly discriminate the analysis. Next, different forms of Hierarchical Clustering were tested, and the most viable option for each one of the considered scenarios was to group the DMUs by degree of similarity, considering the average and variance of the six variables adopted in this study.

This made it possible to group the DMUs by increasing the degree of similarity between the groups in which the efficiencies can be analyzed. Figure 1 shows the DMU grouping using Hierarchical Clustering for complete asset information ($n = 1$).

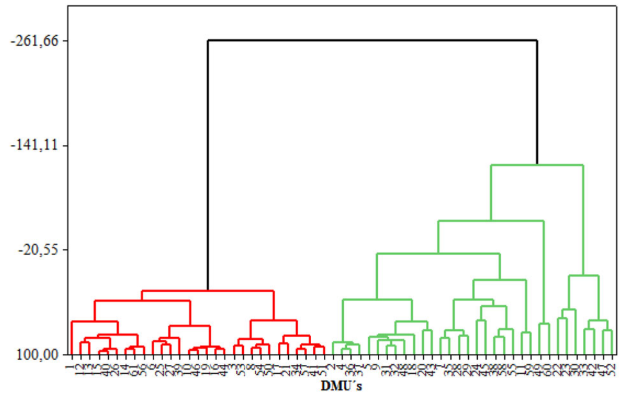

Figure 1. Dendrogram of the grouping considering complete market data information.
Source: Authors.

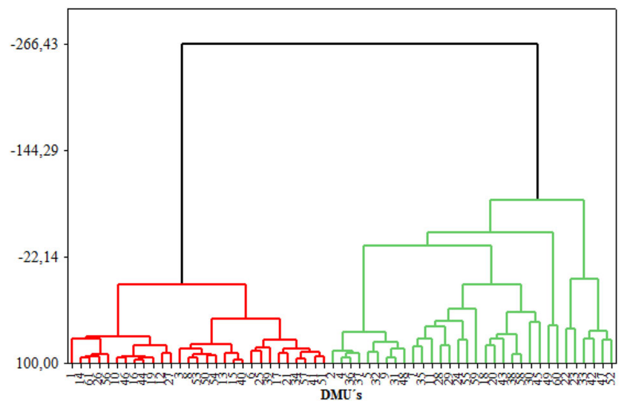

Figure 2. Dendrogram of the grouping considering information from worst-case market periods.
Source: Authors.

Figure 2 shows the DMU grouping for worst-case market periods, defined as n = 4. These Figures were obtained using the Minitab® software program.

Two groupings were performed for each of the two proposed scenarios for the assets that make up the more similar groups.

Grouping can be done in a larger number of groups, however, given the number of variables, the model required a minimum number of DMUs for good data discrimination (Cooper et al., 2007), and so only two groups were formed. This solution can be adopted to facilitate meeting the constraints in the CCDEA model, since they respect the recommendations in the model application.

Table 1 shows the descriptive statistics of the DMU input and output variables that comprise group 1 and 2 considering information on the total market state. Table 2 presents the same information for worst-case market periods.

Similar to other studies, negative data were transformed by adding a value that makes the most negative value in the series positive for the same variable, without changing the efficiency analysis (Cook & Zhu, 2014). It is worth mentioning that the input and output variables were independent in this study.

Table 1. Descriptive statistics of the variables considering total market state information.

Group 1	Return		Liquidity		EPS		Beta		PE		Volatility	
	μ_1	σ_1^2	μ_2	σ_2^2	μ_3	σ_3^2	μ_4	σ_4^2	μ_5	σ_5^2	μ_6	σ_6^2
Mean	0.07	3.52	0.65	0.09	7.03	9.19	0.63	0.01	19.11	132.21	1.87	0.09
Median	0.07	3.39	0.59	0.03	6.31	1.89	0.61	0.01	19.49	18.88	1.77	0.08
Standard Deviation	0.03	0.78	0.26	0.22	3.13	18.73	0.18	0.01	9.55	327.64	0.25	0.06
Minimum	0.01	2.17	0.29	0.00	2.55	0.18	0.30	0.00	1.00	1.98	1.45	0.01
Maximum	0.13	4.79	1.30	1.19	14.42	92.35	0.99	0.02	42.94	1706.57	2.31	0.27
Group 2												
Mean	−0.02	6.02	1.62	0.27	3.36	286.57	1.05	0.01	12.27	3365.30	2.39	0.19
Median	−0.02	5.41	1.04	0.10	6.32	31.91	1.06	0.01	10.24	1393.75	2.33	0.14
Standard Deviation	0.06	2.29	1.63	0.59	10.38	650.30	0.22	0.01	15.98	4451.30	0.45	0.15
Minimum	−0.17	3.10	0.35	0.01	−26.73	0.54	0.59	0.00	−30.46	2.07	1.70	0.03
Maximum	0.08	10.70	7.05	2.59	17.39	3173.20	1.50	0.05	47.87	14978.71	3.29	0.60

Source: Authors.

Table 2. Descriptive statistics of the variables considering information from the worst-case market periods.

Group 1	Return		Liquidity		EPS		Beta		PE		Volatility	
	μ_1	σ_1^2	μ_2	σ_2^2	μ_3	σ_3^2	μ_4	σ_4^2	μ_5	σ_5^2	μ_6	σ_6^2
Mean	−0.98	3.15	0.65	0.09	7.12	9.46	0.63	0.01	19.39	123.06	1.86	0.09
Median	−0.97	3.12	0.60	0.03	6.32	2.12	0.60	0.01	19.27	18.35	1.77	0.07
Standard Deviation	0.29	0.70	0.26	0.22	3.13	20.24	0.18	0.01	8.48	318.48	0.24	0.06
Minimum	−1.52	1.95	0.29	0.01	2.62	0.20	0.29	0.00	7.77	2.14	1.45	0.01
Maximum	−0.46	4.53	1.31	1.19	14.61	102.62	0.98	0.02	40.20	1686.56	2.29	0.23
Group 2												
Mean	−1.86	4.42	1.62	0.31	3.25	308.54	1.05	0.01	12.70	3152.21	2.37	0.20
Median	−1.78	4.12	1.04	0.10	6.44	35.82	1.06	0.01	10.61	1127.46	2.31	0.14
Standard Deviation	0.39	1.93	1.63	0.69	10.92	692.02	0.22	0.01	18.73	4112.77	0.44	0.15
Minimum	−2.79	2.01	0.36	0.01	−29.27	0.58	0.60	0.00	−30.61	1.89	1.70	0.03
Maximum	−1.14	9.85	7.10	3.17	18.03	3312.95	1.51	0.05	65.25	13506.95	3.25	0.64

Source: Authors.

5.2. Efficiency analysis

Efficiency assessments were carried out for the proposed groups. The efficiency level (β_i) was equal to 1. We observed in the studied data that a good range of discrimination was obtained in the analysis units when the risk criterion (α_i) was varied between 0.5 and 0.6. This range will change according to the data submitted to the CCDEA model.

The variation within the range stipulated in the previous step may be related to investor's aversion to risk. In this study, a variation of 1% was chosen within the relevant range defined for probability variation to fulfill constraints (1-α_i), generating eleven portfolios for each market state.

Table 3 shows the efficiency results with different compliance probability levels (defined by 1- α_i) of the constraints for groups 1 and 2, respectively, when supplied with worst-case market scenario information.

Table 3. Descriptive statistics for efficiencies considering full market information.

Group 1											
$(1\text{-}\alpha_i)$	40%	41%	42%	43%	44%	45%	46%	47%	48%	49%	50%
Mean	1.25	1.19	1.13	1.08	1.04	1.00	0.96	0.93	0.90	0.88	0.86
Median	1.24	1.17	1.11	1.06	1.02	0.99	0.96	0.95	0.92	0.89	0.87
Standard Deviation	0.17	0.16	0.15	0.15	0.14	0.14	0.13	0.13	0.12	0.12	0.11
Minimum	0.97	0.92	0.88	0.83	0.80	0.76	0.73	0.70	0.68	0.67	0.66
Maximum	1.56	1.49	1.40	1.34	1.27	1.22	1.17	1.12	1.08	1.04	1.00
Group 2											
Mean	2.25	2.07	1.93	1.81	1.69	1.50	1.34	1.22	1.10	1.00	0.82
Median	1.80	1.68	1.58	1.47	1.37	1.25	1.11	1.02	0.96	0.90	0.82
Standard Deviation	2.12	2.02	1.84	1.77	1.71	1.37	1.16	1.04	0.81	0.63	0.15
Minimum	1.10	1.02	0.96	0.90	0.83	0.74	0.66	0.60	0.55	0.51	0.47
Maximum	13.53	12.77	11.70	11.24	10.81	8.77	7.52	6.77	5.39	4.34	1.00

Source: Authors.

Table 4. Descriptive statistics for the efficiencies considering worst-case scenario market information.

Group 1											
$(1\text{-}\alpha_i)$	40%	41%	42%	43%	44%	45%	46%	47%	48%	49%	50%
Mean	0.96	0.95	0.94	0.92	0.91	0.90	0.89	0.88	0.87	0.86	0.85
Median	0.97	0.95	0.93	0.92	0.90	0.89	0.87	0.87	0.87	0.86	0.85
Standard Deviation	0.15	0.15	0.14	0.14	0.13	0.13	0.13	0.12	0.12	0.12	0.12
Minimum	0.70	0.68	0.69	0.68	0.68	0.67	0.66	0.66	0.66	0.65	0.65
Maximum	1.22	1.20	1.17	1.15	1.13	1.10	1.08	1.06	1.04	1.02	1.00
Group 2											
Mean	1.25	1.14	1.09	1.03	0.99	0.95	0.89	0.85	0.82	0.79	0.75
Median	1.08	1.03	0.97	0.93	0.91	0.88	0.85	0.83	0.81	0.79	0.76
Standard Deviation	1.09	0.79	0.70	0.62	0.57	0.54	0.38	0.31	0.27	0.25	0.21
Minimum	0.39	0.39	0.38	0.38	0.37	0.36	0.32	0.33	0.35	0.31	0.31
Maximum	6.79	5.05	4.47	4.01	3.68	3.48	2.37	1.83	1.49	1.34	1.00

Source: Authors.

We evaluated the efficiency of the proposed groups considering the risk criteria adopted. Table 3 shows the efficiency results for groups 1 and 2, respectively, when submitted to the CCDEA model under different probability levels for fulfilling the model constraints, supplied with complete market information taken from the stipulated period.

Likewise, Table 4 presents the efficiency results for groups 1 and 2, respectively. However, the CCDEA model was supplied with information taken from market downturns. It is important to highlight that the likelihood of complying with the constraints of the optimization model increases by reducing the risk criterion (α_i), making the model more critical. Therefore, fewer assets will be efficient.

The assets pre-selected by the CCDEA model will be submitted to the Sharpe approach (Sharpe, 1963) for optimal allocation within the portfolio. A similar strategy of pre-specification for assets was adopted by Chakrabarti (2021).

5.3. Asset *al*location and comparative analysis

Since traditional DEA models ignore diversification between investment opportunities (Adam & Branda, 2020), Sharpe's model could be necessary to build the portfolio.

Table 5. Results by risk criteria based on complete market information.

	TS-1	TS-2	TS-3	TS-4	TS-5	TS-6	TS-7	TS-8	TS-9	TS-10	TS-11
α_i	60%	59%	58%	57%	56%	55%	54%	53%	52%	51%	50%
β	0.702	0.700	0.691	0.691	0.763	0.713	0.680	0.689	0.668	0.674	0.616
R_E	1.07%	1.07%	1.06%	1.07%	1.09%	1.07%	1.06%	1.06%	1.05%	1.06%	1.03%
SD	8.80%	8.51%	8.71%	8.90%	8.69%	8.87%	8.97%	8.95%	8.70%	8.61%	9.65%
R	−3.05%	−2.68%	−3.33%	−2.10%	−1.00%	−0.22%	0.35%	0.39%	2.10%	1.54%	2.96%
SR	−0.467	−0.440	−0.504	−0.356	−0.240	−0.145	−0.079	−0.074	0.120	0.055	0.200
N	57	54	52	48	45	41	34	26	19	17	12
AAR	−1.59%	−1.12%	−1.84%	0.05%	1.48%	2.57%	3.90%	4.70%	7.44%	6.53%	9.31%

Source: Authors.

Table 6. Results by risk criterion based on information from periods of market downturns.

	WS-1	WS-2	WS-3	WS-4	WS-5	WS-6	WS-7	WS-8	WS-9	WS-10	WS-11
α_i	60%	59%	58%	57%	56%	55%	54%	53%	52%	51%	50%
β	0.458	0.458	0.446	0.444	0.444	0.438	0.432	0.429	0.429	0.429	0.429
R_E	0.96%	0.96%	0.96%	0.96%	0.96%	0.96%	0.95%	0.95%	0.95%	0.95%	0.95%
SD	7.67%	7.67%	8.16%	8.05%	8.05%	8.38%	8.36%	8.32%	8.32%	8.32%	8.32%
R	2.52%	2.52%	2.65%	3.85%	4.03%	5.34%	5.73%	5.73%	5.73%	5.73%	5.73%
SR	0.202	0.203	0.206	0.358	0.358	0.366	0.525	0.575	0.575	0.575	0.575
N	11	11	11	10	10	10	9	8	8	8	8
AAR	5.58%	5.58%	5.42%	6.62%	6.62%	7.24%	8.91%	9.16%	9.16%	9.16%	9.16%

Source: Authors.

Efficient assets from Table 3 were submitted to Sharpe's proposal taking information from the total market state into account for each risk criterion (α_i) adopted. It is interesting to observe that not all efficient assets considered will be used in the allocation when submitted to the Sharpe model.

Eleven portfolios were proposed from varying the risk criterion for the total market state (TS) information. These were identified as TS-1 to TS-11, to simplify discussion. Another eleven portfolios were proposed according to the risk criterion (α_i) for the worst-case (WS) market information. It is interesting to notice that there were fewer efficient assets when the model was supplied with WS market information. These portfolios were identified as WS-1 to WS-11, to simplify discussion.

We use the Capital Asset Pricing Model (CAPM), presented by Sharpe (1964), to analyze the results and to identify any abnormal returns (Rotela Junior et al., 2014, 2015). The accumulated abnormal return of the portfolios was obtained from information collected during the validation period and daily information was used. Furthermore, the Sharpe ratio was used, being the most commonly used metric for measuring and comparing portfolio performance (Homm & Pigorsch, 2012; Kourtis, 2016).

Table 5 presents the adopted risk criterion (α_i), portfolio beta (β), return results (R_E and R), standard deviation (SD), Sharpe ratio (S_R), number of assets (N), and the accumulated abnormal return (AAR) for each TS optimized portfolio. Table 6 presents the same content, but for WS optimized portfolios.

The main discussion shows the importance of information taken from periods of market crisis and recession and how this information contributes to robust portfolio optimization. Tables 5 and 6 analyze and compare the TS optimized portfolios (portfolios TS-1 to TS-11) and WS optimized portfolios (portfolios WS- 1 to WS-11). The portfolios are compared in pairs according to the α_i value adopted.

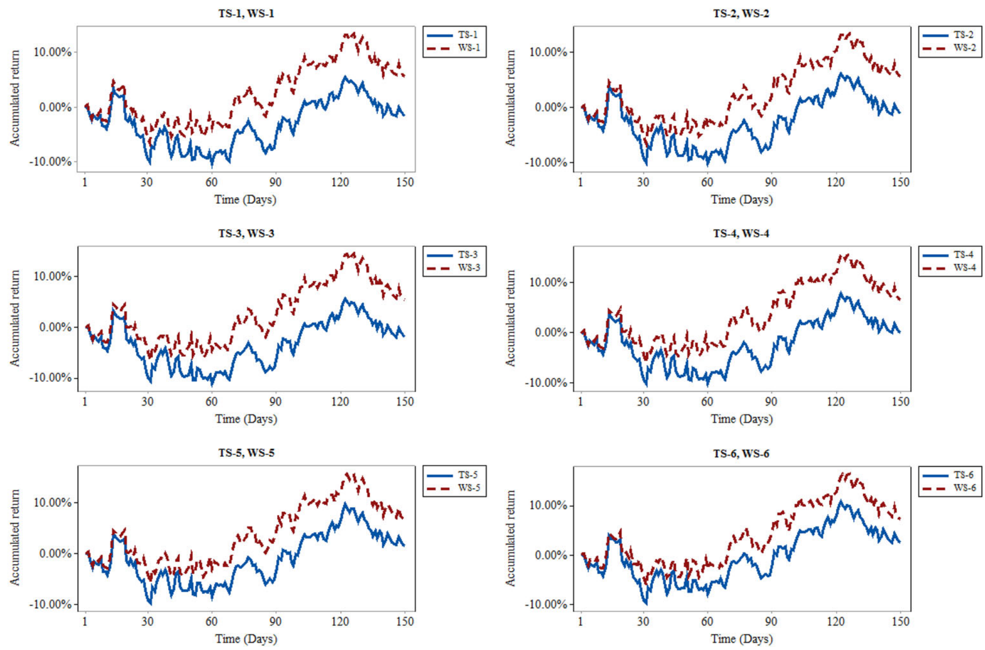

Figure 3. Accumulated return of the pairs of portfolios considering risk criterion variation (60% - 55%).
Source: Authors.

Tables 5 and 6 present the expected return values for the portfolios that were calculated as presented before. We needed to calculate the beta values (β) for each of the portfolios (also shown in the Tables). For the TS optimized portfolios (TS-1 to TS-11), the expected returns (R_E) vary from 1.03% to 1.09% a.m. For the WS optimized portfolios (WS-1 to WS-11), the expected returns (R_E) were concentrated between 0.95% and 0.96% a.m. And, the obtained mean profitability (R) were −3.05%, −2.68%, −3.33%, −2.10%, −1.00%, −0.22%, 0.35%, 0.39%, 2.10%, 1.54%, and 2.96% for TS-1 to TS-11, respectively. The obtained mean profitability (R) were 2.52%, 2.52%, 2.65%, 3.85%, 4.03%, 5.34%, 5.73%, 5.73%, 5.73%, 5.73%, and 5.73% for WS-1 to WS-11, respectively.

Figures 3 and 4 show the accumulated return in pairs established according to the probability level (1-α_i) of fulfilling constraints from the CCDEA model. Figure 3 shows the AAR of the portfolio pairs when a risk range (α_i) criterion of 60% to 55% is adopted. Figure 4 shows the AAR of the portfolio pairs when a risk range (α_i) criterion of 54% to 50% is adopted.

It is important to highlight that the WS optimized portfolios (WS-1 to WS-11) had better results when reading the Sharpe ratio (S_R) considering different α_i values. Regardless of the risk criterion adopted, the beta values of the proposed portfolios were lower than portfolios TS-1 to TS-11.

After developing the accumulated return graphs, we conducted a statistical test to compare the obtained series of abnormal accumulated returns for each pair of portfolios associated by the risk criterion. We decided to use the Mann-Whitney non-parametric test. The P-value results obtained in these tests (when portfolios are

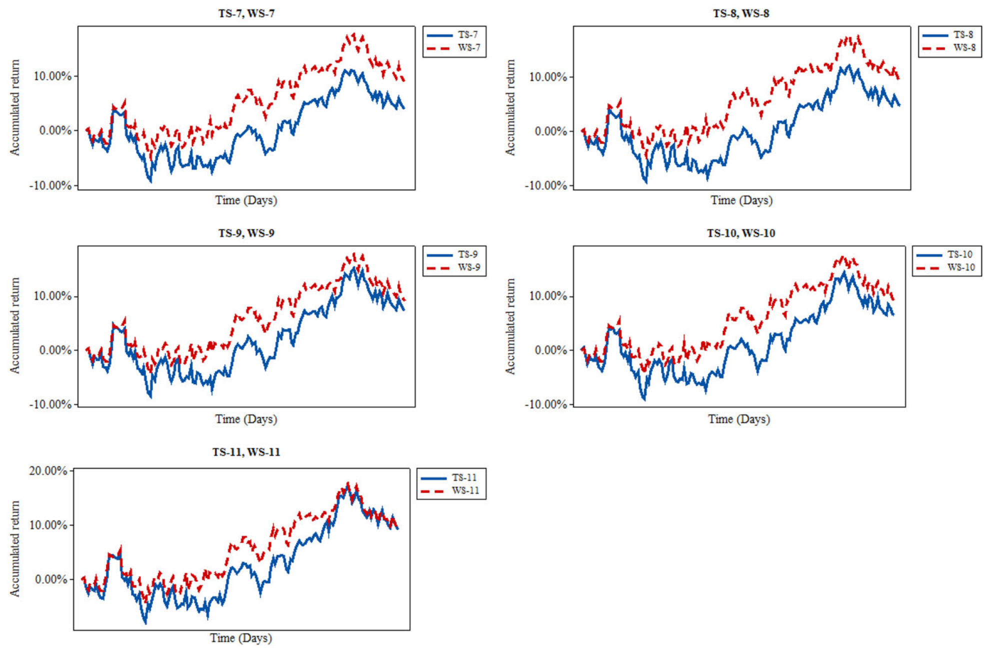

Figure 4. Accumulated return of the pairs of portfolios considering risk criterion variation (54% - 50%).
Source: Authors.

analyzed in pairs) were less than 0.05. This result allows us to affirm that the abnormal accumulated return of WS optimized portfolios is statistically higher than the accumulated returns of TS optimized portfolios for the entire period. Figure 5 presents a boxplot diagram for the pairs of portfolios.

The results have shown that the proposed method supplied with WS information has better results than TS information for the total period.

However, to confirm the applicability of this method, we needed to compare these portfolios with other portfolios build using classic models for portfolio optimization. Models presented by Markowitz (1952), and Sharpe (1963), were used for this comparison.

TS ($n = 1$) information assets were ideally allocated using the Markowitz (1952) model and by maximizing the Sharpe ratio of the assets. This portfolio was named the Comparative Markowitz (MC) portfolio. Using the same set of information, assets were ideally allocated using the Sharpe model (1963). This portfolio was named the Comparative Sharpe (SC) portfolio. Table 7 shows the share of assets in the comparative portfolios. Additionally, the shares are again presented for four of the eleven proposed portfolios, with two of them (TS-1 and TS-11) supplied with the same set of information as the Comparative portfolios, and two other portfolios (WS-1 and WS-11) from WS ($n = 4$) market periods. For this comparison, we chose to consider only portfolios with risk criteria 60% and 50%, which are limit values for the adopted range.

The same validation period employed in the previous comparison was used to validate the results obtained in the portfolios.

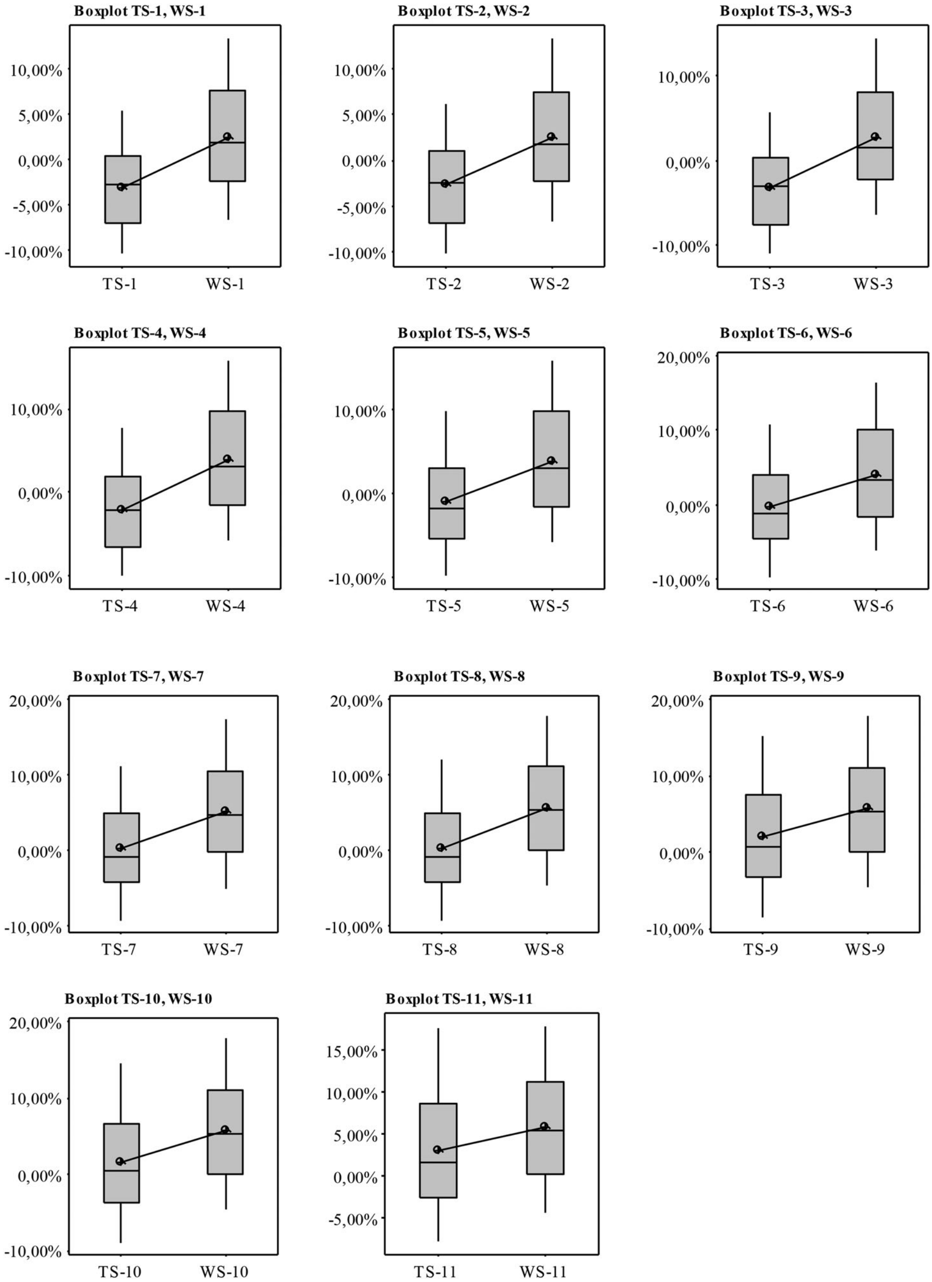

Figure 5. Boxplot of the accumulated returns from pairs of portfolios, by risk criterion (60% - 50%).
Source: Authors.

Table 8 shows some parameters for the proposed portfolios (TS-1, TS-11, WS-1, and WS-11) and the comparative portfolios (MC and SC), like the adopted risk (α_i), the portfolio beta (β), the return results (R_E and R), the standard deviation (SD), the Sharpe ratio (S_R), and the number of assets (N) that comprise the portfolio.

Table 7. Assets allocation in the comparative and proposed portfolios.

(α_i) Portfolios	– MC	– SC	60% TS-1	60% WS-1	50% TS-11	50% WS-11
DMU1	0.171	0.046	0.046	0.176	0.139	0.235
DMU2	0.000	0.014	0.015	0.000	0.061	0.000
DMU3	0.000	0.025	0.025	–	–	–
DMU4	0.000	0.024	0.024	0.000	0.091	0.000
DMU5	0.000	0.007	0.008	0.000	–	–
DMU6	0.090	0.038	0.038	0.093	–	–
DMU7	0.000	0.017	0.017	0.000	–	–
DMU8	0.000	0.017	0.017	–	–	–
DMU9	0.000	0.012	0.013	–	–	–
DMU10	0.000	0.027	0.028	–	–	–
DMU11	0.000	0.017	0.017	0.027	0.063	0.067
DMU12	0.120	0.033	0.033	0.095	0.096	0.127
DMU13	0.000	0.020	0.021	0.056	0.074	0.101
DMU14	0.000	0.028	0.029	–	–	–
DMU15	0.000	0.019	0.019	0.041	–	–
DMU16	0.014	0.029	0.030	–	–	–
DMU17	0.000	0.022	0.023	–	–	–
DMU18	0.000	0.003	0.003	–	–	–
DMU19	0.103	0.032	0.032	–	–	–
DMU20	0.000	0.006	0.007	–	–	–
DMU21	0.000	0.013	–	–	–	–
DMU22	0.000	0.004	0.004	0.000	–	–
DMU23	0.000	0.006	0.007	0.000	–	–
DMU24	0.000	0.004	0.004	0.000	0.046	0.019
DMU25	0.083	0.027	0.027	–	–	–
DMU26	0.000	0.020	0.020	0.097	–	–
DMU27	0.106	0.028	0.028	–	–	–
DMU28	0.000	0.011	0.011	–	–	–
DMU29	0.000	0.012	0.012	0.000	–	–
DMU30	0.000	0.001	0.001	–	–	–
DMU31	0.000	0.006	0.006	–	–	–
DMU32	0.000	0.006	0.006	–	–	–
DMU33	0.000	0.005	0.005	–	–	–
DMU34	0.032	0.023	0.023	–	–	–
DMU35	0.000	0.012	0.012	0.000	–	–
DMU36	0.000	0.021	0.021	0.000	–	–
DMU37	0.000	0.021	0.021	0.000	–	0.000
DMU38	0.000	0.010	0.010	0.000	–	–
DMU39	0.000	0.021	0.022	–	–	–
DMU40	0.000	0.021	0.021	0.045	–	–
DMU41	0.000	0.021	0.021	–	–	–
DMU42	0.000	0.003	0.003	–	–	–
DMU43	0.000	0.006	0.006	–	–	–
DMU44	0.000	0.020	0.021	–	–	–
DMU45	0.000	0.000	0.000	0.000	–	–
DMU46	0.017	0.029	0.029	–	–	–
DMU47	0.000	0.000	0.000	–	–	–
DMU48	0.000	0.003	0.003	–	–	–
DMU49	0.000	0.005	0.005	0.000	0.046	0.000
DMU50	0.042	0.023	0.024	–	–	–
DMU51	0.000	0.022	0.022	–	–	–
DMU52	0.000	0.000	0.000	–	–	–
DMU53	0.000	0.013	0.013	–	–	–
DMU54	0.000	0.023	0.023	0.046	–	–
DMU55	0.000	0.011	0.011	0.000	0.050	0.015
DMU56	0.043	0.038	0.038	0.166	0.134	0.214
DMU57	0.000	0.020	0.020	–	–	–
DMU58	0.169	0.001	0.001	0.000	–	–
DMU59	0.009	0.011	0.011	0.000	–	–
DMU60	0.000	0.011	0.012	0.000	0.077	0.000
DMU61	0.000	0.033	0.034	0.158	0.123	0.222

Source: Authors.

Table 8. Results for comparative portfolios and proposed portfolios.

	MC	SC	TS-1	WS-1	TS-11	WS-11
α_i	–	–	60%	60%	50%	50%
β	0.475	0.702	0.702	0.458	0.616	0.429
R_E	0.96%	0.96%	0.96%	0.96%	0.95%	0.95%
SD	7.20%	8.60 %	8.80%	7.67%	9.65%	8.32%
R	−1.00%	−3.00%	−3.00%	2.52%	2.96%	5.73%
SR	−0.272	−0.460	−0.467	0.202	0.200	0.575
N	13	58	57	11	12	8
AAR	1.35%	−1.57%	−1.59%	5.58%	9.31%	9.16%

Source: Authors.

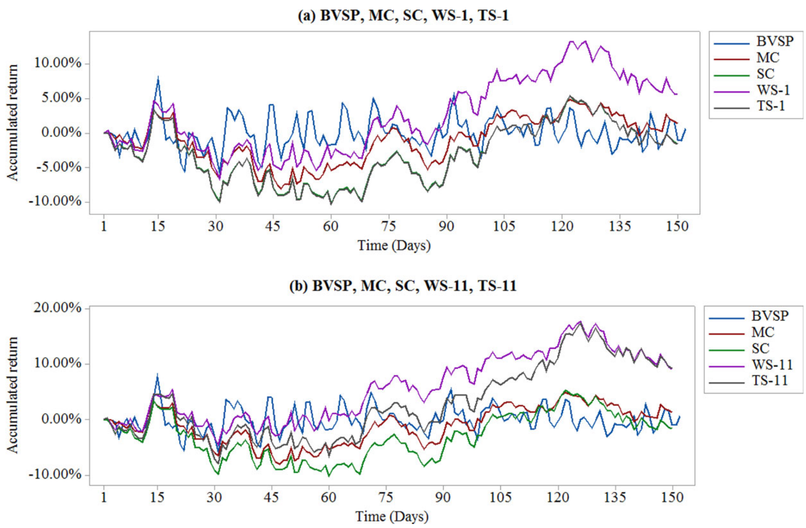

Figure 6. Accumulated return of proposed and comparative portfolios.
Source: Authors.

Next, the portfolios optimized according to this proposal are compared to those formulated from the classical and deterministic models of portfolio theory.

It is noteworthy that the Comparative Markowitz portfolios (MC) and Comparative Sharpe portfolios (SC) had only 13 and 58 assets after the optimization, and S_{Rs} equal to −0.272 and −0.460, respectively. Again, we see that WS optimized portfolios perform better.

Portfolios WS-1 and WS-11 had beta values equal to 0.458 and 0.429, while Comparative portfolios MC and SC had beta values equal to 0.475 and 0.702.

The cumulative return for the portfolios (AAR) was obtained based on the information from the period considered in the validation. Again, the results shown in Table 8 show abnormal returns. Figure 6 shows the accumulated returns for the analyzed portfolios.

Figure 6a shows the accumulated returns of the proposed portfolios (when a risk criterion (α_i)=60% is adopted) and the comparative portfolios: MC, SC, and the Ibovespa Brazil Sao Paulo Stock Exchange Index (BVSP). Figure 6b shows the accumulated return for proposed portfolios (when a risk criterion (α_i)=50% is adopted) and the same comparative portfolios.

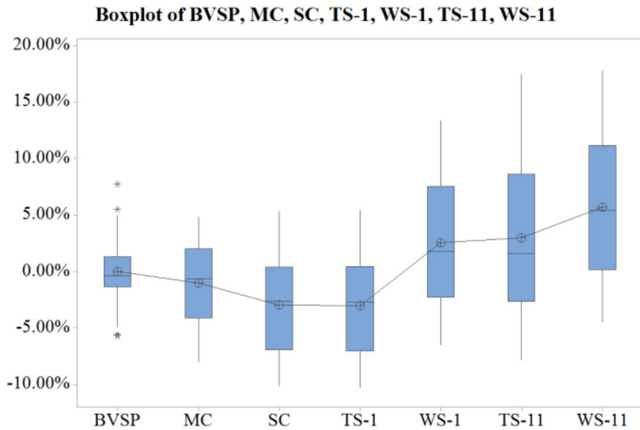

Figure 7. Boxplot diagram for proposed and comparative portfolios.
Source: Authors.

After presenting the accumulated return graphs, we performed a statistical test for the comparison between the series of accumulated returns obtained for the analyzed portfolios. We used the Kruskal-Wallis test (one-way analysis of variance), which allowed us to determine whether the medians of two or more groups differed. The *P-value* results obtained were less than 0.05. We see that the accumulated returns of WS optimized portfolios (WS-1 and WS-11) are statistically higher than the accumulated returns of TS optimized portfolios (TS-1 and TS-11), when using the same risk criterion. They are also statistically higher than the accumulated returns of the BVSP, MC and SC comparative portfolios.

Figure 7 shows the boxplot diagram of the analyzed portfolios.

6. Conclusion

We were able to reduce the search space when identifying efficient assets. This study used stochastic information for the different adopted variables. Subsequently, the efficient assets were submitted to approaches that promoted ideal asset allocation within the portfolios. It is interesting to note that both commonly used and fundamentalist variables were considered in the asset allocation.

We identified that it is possible to represent more flexible models by considering the different risk profiles of investors (conservative or risky) by varying the probability of meeting the constraints ($1\text{-}\alpha i$) in the CCDEA model.

In general, portfolios formed using our proposed method (WS-1 to WS-11) performed better as measured by the Sharpe ratio (S_R), and according to the accumulation of abnormal returns in the validation period. The averages of the series of abnormal returns were statistically higher than for the comparative portfolios.

The proposed model confirms that worse market state data generate portfolios with lower beta values, thus reducing systematic risk. Furthermore, higher levels of risk aversion also contribute to the formation of portfolios with lower beta values. Paradoxically, the proposed model achieved portfolios that at the same time reduced systematic risk and maximized portfolio returns when working with worse market state data and higher levels of risk aversion. A higher level of risk aversion also led to better risk-return ratios, which can be seen in higher Sharpe ratio values.

Finally, we suggested that this proposed method be applied to different stock markets, to more mature stock markets, to more assets, with data coming from longer historical series, and to validate data from different periods. We also suggest that future research compare this proposed method with other methods of robust portfolio optimization.

Acknowledgements

This paper is part of a project GEOCEP that has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No. 870245. The authors would like to thank the Brazilian National Council for Scientific and Technological Development - CNPq Brazil (Processes 308021/2019-3, and 302751/2020-3), and the Coordination for the Improvement of Higher Education Personnel - Capes Brazil. The authors also thank the Czech Science Foundation - Czech Republic (grant number 22-19617S) for the financial support and research incentive. The views expressed here are those of the authors and not necessarily those of our institutions. All remaining errors are solely our responsibility.

Disclosure statement

No potential conflict of interest was reported by the authors.

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