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THE QUASIREGULAR PROJECTIVE PLANES OF ORDER 16

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Abstract. The projective planes of order 16 admitting a large ($\geq$ 137) quasiregular group of collineations are classified. The classification is done using the theorem of Dembowski and Piper [DP67] and a complete search by computer. No new planes are found.

1. Introduction

Let $\mathfrak{P}$ be a projective plane of order n and G a group of collineations of $\mathfrak{P}$. The group G is called *quasiregular* if it acts regularly on its point and line orbits, i.e. if the point-(line-)stabiliser is a normal subgroup in G. The quasiregular group is called *large* if $|G| > \frac{1}{2}(n^2+n+1)$. By [Dem68, 4.2.8, p.181], a group of collineations of a projective plane acts faithfully on at least one point or line orbit. Up to duality, we may assume that this is an orbit of points. So a large quasiregular group of collineations has exactly one orbit of size $|G|$ on points. One has:

Theorem 1.1 (Dembowski-Piper, [DP67], [Dem68, 4.2.10, p.182]). *Let G be a quasiregular group of collineations of the projective plane $\mathfrak{P}$ of order n. Denote by $m = m(G)$ the number of point (or line) orbits of G, and by $\mathbf{F} = \mathbf{F}(G)$ the substructure of the elements fixed by G. If $|G| > \frac{1}{2}(n^2+n+1)$, then there are only the following possibilities:*

DP$_a$ $|G| = n^2+n+1$, *$m = 1$ and $\mathbf{F} = \emptyset$. Here G is transitive.*
DP$_b$ $|G| = n^2$, *$m = 3$ and $\mathbf{F}$ is a flag.*
DP$_c$ $|G| = n^2$, *$m = n+2$ and $\mathbf{F}$ is either a unique line A and all $x \in A$ or dually a unique point c and all lines $X \in [c]$.*

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DP$_d$ $|G| = n^2 - 1$, $m = 3$ *and* **F** *is a unique non-incident point-line pair.*

DP$_e$ $|G| = n^2 - \sqrt{n}$, $m = 2$ *and* $\mathbf{F} = \emptyset$. *In this case, one point and one line orbit together form a Baer subplane.*

DP$_f$ $|G| = n(n-1)$, $m = 5$ *and* **F** *consists of two points* u, v, *the line containing* u *and* v *and one other line through one of* u, v.

DP$_g$ $|G| = (n-1)^2$, $m = 7$ *and* **F** *consists of the vertices and sides of a triangle.*

In the following we will describe a computer search which leads to the classification result of Theorem 1.2.

THEOREM 1.2. *Let* $\mathfrak{P}$ *be a projective plane of order* 16 *with a large quasiregular collineation group* G *and a regular point orbit. Then* $\mathfrak{P}$ *is a translation plane or the Mathon plane. Moreover the desarguesian plane occurs as types* **DP**$_a$, **DP**$_b$, **DP**$_c$, **DP**$_d$, **DP**$_f$ *or* **DP**$_g$. *The two semifield planes occur as types* **DP**$_b$ *or* **DP**$_c$. *The remaining translation planes occur only as type* **DP**$_c$ *and the Mathon plane has type* **DP**$_b$.

The translation planes of order 16 have been classified in [DR83], i.e. only types different from **DP**$_c$ have to be considered. In each of these cases, regular orbits on points and lines exist. This leads to the difference set construction which we describe in the next section.

A complete list of all groups and the types of planes they act on is given in the appendix.

2. Notation, relative difference sets

Let G be a finite group and $D, N \subseteq G$. We say that D is a *relative difference set with forbidden set* N, if there exists $0 < \lambda \in \mathbb{Z}$ such that every element of $G - N$ can be expressed in exactly λ ways as a quotient $d_1 d_2^{-1}$ with $d_1, d_2 \in D$ and no element of $N - 1$ can be written like this. In group ring notation: $DD^{(-1)} = k + \lambda(G - N)$.

We call $(|G|/|N|, |N|, k, \lambda)$ with $k = |D|$ the parameter tuple of the difference set D.

A relative difference set is called ordinary difference set or simply *difference set* if $N = 1$ holds. The incidence structure $\operatorname{dev} D := (G, \{Dg \mid g \in G\}, \in)$ is called *development of* D and G acts (via right multiplication) as a regular group of collineations on $\operatorname{dev} D$.

A set $D \subset G$ is called *partial relative difference set* with forbidden set $N \subseteq G$, if every element outside N can be written in at most λ ways as a quotient in D and no element of $N - 1$ is a quotient of two elements of D.

Two partial relative difference sets $D, D' \subseteq G$ are called *equivalent* if there is a $g \in G$ and $\phi \in \operatorname{Aut}(G)$ such that $D = (Dg)^{\phi}$. Two partial relative difference sets $D, D' \subseteq G$ are called *strongly equivalent* if they are equivalent and have the same forbidden set.

TABLE 1. The difference set parameters of Theorem 1.1

type	$(\lvert G\rvert/\lvert N\rvert, \lvert N\rvert, k, \lambda)$	
$\mathbf{DP}_a$	$(n^2+n+1, 1, n+1, 1)$	$N = 1$
$\mathbf{DP}_b$	$(n, n, n, 1)$	$N \trianglelefteq G$
$\mathbf{DP}_d$	$(n+1, n-1, n, 1)$	$N \trianglelefteq G$
$\mathbf{DP}_e$	$(n+\sqrt{n}+1, n-\sqrt{n}, n-1, 1)$	$N \trianglelefteq G$
$\mathbf{DP}_f$	$(n/2, 2n-2, n-1, 1)$	$N = A \cup B,\ A, B \trianglelefteq G$
		$\lvert B\rvert + 1 = \lvert A\rvert = n,\ A \times B = G$
$\mathbf{DP}_g$	$((n-1)^2/3n-5, 3n-5, n-1, 1)$	$N = A \cup B \cup C,\ A \times B = G$
		$\lvert A\rvert = \lvert B\rvert = \lvert C\rvert = n-1$

Obviously, equivalent relative difference sets induce isomorphic developments. Note that, in general $\operatorname{dev} D^{-1} \not\sim \operatorname{dev} D$. Instead, it is easy to see that $\operatorname{dev} D^{-1} \sim (\operatorname{dev} D)^d$, where $\cdot^d$ denotes the dual incidence structure.

For more about relative difference sets see [Pot95, Sch02, GJ03].

2.1. *Difference sets from projective planes.* Assume now that $\mathfrak{P}$ is a projective plane of order n and G a large quasiregular group of collineations which has a regular point and line orbit. We identify the elements in that point orbit with the elements of G. Let D be the intersection of the regular point orbit with a line from the regular line orbit. Then D is a relative difference set. In table 1, the parameters for the difference sets of the Dembowski-Piper theorem are given. The last column contains a description of the forbidden set. Observe that in all cases, N is either a normal subgroup of G or a union of normal subgroups. As $\mathbf{DP}_c$ defines translation planes, it does not have a description in terms of relative difference sets.

It is not difficult to see that in all cases $\operatorname{dev} D$ can be extended to a projective plane of order n and that this extension is uniquely determined. So in order to obtain Theorem 1.2 we have to classify all relative difference sets of the respective types for $n = 16$.

3. The search

Before we give an algorithm to calculate all relative difference sets for each of the cases of Theorem 1.1, we introduce an invariant for partial relative difference sets which will be used during the search.

3.1. *An invariant for partial relative difference sets.* The following lemma is a generalisation of a lemma of Bruck [Bru55] and the proof is very similar to the original one.

LEMMA 3.1. *Let G be a finite group and $U \trianglelefteq G$. Furthermore, let $D \subseteq G$ be a relative difference set with forbidden set N. We define $v_i := \lvert D \cap g_i\rvert$, where $\{g_1, \ldots, g_{\lvert G:U\rvert}\} = G/U = G^\rho$ and ρ is the natural homomorphism. Let $g_1 = U$ and $v_{ij} = \lvert D \cap g_i g_j\rvert$. Then*

1. $\sum v_i = k$.
2. $\sum v_i^2 = \lambda(|U| - |U \cap N|) + k$.
3. $\sum_j v_j v_{ij} = \lambda(|U| - |g_i \cap N|)$ *for* $i \neq 1$.
4. If $N \leq G$, *then*

$$\sum_j v_j v_{ij} = \begin{cases} \lambda(|U| - |U \cap N|), & \text{if } g_1 \neq g_i \in N^\rho, \\ \lambda|U|, & \text{if } g_i \notin N^\rho. \end{cases}$$

Note that $v_{ij} \in \{v_1, \ldots, v_{|G:U|}\}$ is just the coefficient of $g_i g_j \in G/U$ in $D^\rho \in \mathbb{Z}[G/U]$. As the right sides in equations 1.–4. are known, we may calculate solutions $(v_1, \ldots, v_{|G:U|})$ of these equations *before* we search for a difference set D. This results in a significant reduction of the search space.

DEFINITION 3.2. *Let* N, U, G *be as in Lemma 3.1. Let* $G/U = \{g_1, \ldots, g_{|G:U|}\}$ *with* $g_1 = U$ *and* $v = (v_1, \ldots, v_{|G:U|})$ *a solution of 1.–4. (with* $v_{ij} = v_{g_i g_j}$*). Then* v *is called* ordered signature *for* U *(relative to* N*). The corresponding multiset* $\|v\|$ *is called* admissible signature *for* U *(relative to* N*). Moreover we define a mapping* $s_U : \mathcal{P}(G) \to \mathbb{N}^{|G:U|}$ *by* $s_U(S)(i) = |S \cap g_i|$ *and denote by* $\sigma_U(S)$ *the multiset* $\|s_U(S)\|$.

For every relative difference set D and every $U \trianglelefteq G$, the tuple $s_U(D)$ is an ordered signature and the multiset $\sigma_U(D)$ is an admissible signature.

It is obvious that if $\mathcal{U}$ is a system of representatives of the $\mathrm{Aut}(G)$-orbits on $\{U \trianglelefteq G \mid |U| \in \mathcal{I}\}$ for some $\mathcal{I} \subseteq \mathbb{N}$, and S a partial relative difference set, we have

$$\|\big(\sigma_U(S)\big)_{U \in \mathcal{U}}\| = \|\big(\sigma_U((Sg)^\phi)\big)_{U \in \mathcal{U}}\| \text{ for all } g \in G \text{ and all } \phi \in \mathrm{Aut}(G).$$

So the mappings σ_U can be used as an invariant for equivalence classes of partial relative difference sets. And we may in some cases even be able to decide that a partial relative difference set can not be extended to a full relative difference set by comparing its image under σ_U to admissible signatures. Note that using $\mathrm{Aut}(G)_N$ instead of $\mathrm{Aut}(G)$ gives an invariant for strong equivalence.

3.2. *The algorithm.* We will now outline the algorithm used in the search for quasiregular projective planes of order 16. For each case of Theorem 1.1 other than $\mathbf{DP}_c$, all possible groups and the respective forbidden sets are calculated using the small groups library provided by GAP. Restrictions on the forbidden sets are given in table 1. For each group G and all possible forbidden sets N, relative difference sets are then constructed using the following method:

1. choose a set $\mathcal{V}$ of normal subgroups of G (representatives of $\mathrm{Aut}(G)_N$ orbits, for instance) and calculate admissible signatures for them
2. choose normal subgroup $U \notin \mathcal{V}$ of small index with "nice" admissible signatures

3. generate partial relative difference sets (of maximal length) in U. Use the mappings σ_V with $V \in \mathcal{V}$ to partition the found partial relative difference sets and a test for strong equivalence on the partition elements
4. continue generating partial difference sets by adding elements from non-trivial cosets modulo U as above (coset by coset) or by brute force search (omitting partitioning and equivalence tests)
5. generate projective planes from the relative difference sets found
6. run isomorphism test on planes

The algorithm was implemented for each of the relevant cases of Theorem 1.1. Depending on the special case, slightly different implementations were used for the above steps. For example, equivalence tests can be performed at step 3. before the maximal length of partial relative difference sets in U is reached. Some further aspects of the implementation will be discussed in the next section. A more detailed description is given in [Röd06a].

3.3. *Remarks concerning implementation.* The calculation was done using the computer algebra system GAP [GAP] and a package especially written for this purpose [Röd06b]. GAP's small groups library is used to look up all possible groups. Implementations are available at [Röd].

3.3.1. *Signatures and "nice" normal subgroups.* For normal subgroups of low index, it is possible to calculate not only admissible, but even ordered signatures. Subgroups for which ordered signatures are known are considered "nice" for step 2. of the algorithm. Of particular interest are those subgroups which do only have one ordered signature (up to permutations induced by right multiplication with group elements).

In case $\mathbf{DP}_a$, we also calculated ordered signatures for normal subgroups of larger index using basic representation arguments for some groups G (see [Röd06a] for details). Knowing ordered signatures enables us to use the mappings s_V for some V in step 3. This is much more restrictive than using just admissible signatures and σ_V.

3.4. *Special methods for case* $\mathbf{DP}_b$. In case $\mathbf{DP}_b$, all groups of order 2^8 have to be considered. A preliminary argument is used to reduce the number of groups in which a search for relative difference sets is actually conducted.

LEMMA 3.3. *Let $D \subseteq G$ be a relative difference set of order $n \equiv 0 \mod 2$ and type $\mathbf{DP}_b$ with forbidden subgroup $N \trianglelefteq G$. Let $\iota \in G$ be an involution. Then $\iota \in N$.*

PROOF. Let $\mathfrak{P}$ be a projective plane of type $\mathbf{DP}_b$ and order n. Let L_∞ the line fixed by G. Any involutorial collineation of $\mathfrak{P}$ is either planar or an elation [Dem68, 4.1.9].

The affine points, i.e. the points not on L_∞ form a regular orbit and are identified with the elemets of G. Let $\iota \in G$ be an involution. Then ι is not planar as ι has no affine fixed-point. Thus ι is an elation. Then ι fixes n affine lines which must be the cosets of N as ι acts fixed-point-freely on the affine lines $\{Dg \mid g \in G\}$. Hence $N\iota = N$ and thus $\iota \in N$. ∎

In particular, a relative difference set of type $\mathbf{DP}_b$ cannot exist in a group containing "too many" involutions. This also rules out some subgroups as forbidden subgroups (namely those not containing all involutions of the full group).

3.4.1. *Invariants for full projective planes.* Isomorphism tests are done using the number of Fano-subplanes as an invariant of projective planes. For type $\mathbf{DP}_b$, the group of translations is calculated to determine if a plane is a translation plane. By [Roy], the translation planes of order 16 are uniquely determined by the number of Fano-subplanes (all translation planes of order 16 are known by [DR83]). For the remaining case, explicit isomorphisms are constructed. The data for the known planes is available from [Roy].

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4. Appendix: detailed results

This appendix lists all groups that admit an action as a large quasiregular group of collineations on a projective plane of order 16. Because of its size the case $\mathbf{DP}_b$ will be treated in a separate section below.

4.1. *Cases* $\mathbf{DP}_a$–$\mathbf{DP}_g$ *without* $\mathbf{DP}_b$. Let $\mathfrak{P}$ be a projective plane of order 16 and G a group acting on $\mathfrak{P}$ as a large quasiregular group of collineations. If $\mathfrak{P}$ is of type

- $\mathbf{DP}_a$ then $G \simeq C_{273}$ or $G \simeq C_3 \ltimes C_{91} \simeq \langle a, b \mid |a| = 3, |b| = 91, b^a = b^{16}\rangle$. Note that there are two isomorphism types of groups $C_3 \ltimes C_{91}$, but only one admits an action of type $\mathbf{DP}_a$.
- $\mathbf{DP}_d$ then $G \simeq C_{255}$.
- $\mathbf{DP}_f$ then $G \simeq C_{15} \times C_2^4$.
- $\mathbf{DP}_g$ then $G \simeq C_{15}^2$.

In each of these cases, $\mathfrak{P}$ is the Desarguesian plane.

There is no projective plane of type $\mathbf{DP}_e$ and order 16. The planes of type $\mathbf{DP}_c$ are translation planes. For their classification we refer the reader to [DR83].

4.2. *Case* $\mathbf{DP}_b$. The following table contains the 590 groups of order 256 which admit relative difference sets of type $\mathbf{DP}_b$.

The column labels "Des", "Semi 2", "Semi 4" and "Mathon" stand for the Desarguesian plane, the semifield planes with kernel 2 and 4, respectively, and the Mathon plane. Note that a group acts on the Mathon plane if and only if it acts on the dual Mathon plane. So there is no extra column for the dual Mathon plane. And, of course, a mark means that there is an action on the corresponding plane. Groups which are not in the table do not admit a relative difference set of type $\mathbf{DP}_b$.

The groups are numbered as in the small groups library of GAP:

```
There are 56092 groups of order 256.
They are sorted by their ranks.
   1 is cyclic.
   2 - 541 have rank 2.
   542 - 6731 have rank 3.
   6732 - 26972 have rank 4.
   26973 - 55625 have rank 5.
   55626 - 56081 have rank 6.
   56082 - 56091 have rank 7.
   56092 is elementary abelian.
```

Group number 6732 is C_4^4.

Nr.	Des	Semi 2	Semi 4	Mathon
296	•	•	•	
331	•			•
420				•
843		•	•	•
855	•	•		
874				•
876				•
909				•
938		•	•	
947	•	•		
956				•
961				•
963		•	•	
978				•
980				•
985				•
1001				•
1038				•
1052				•
1053				•
1060		•	•	
1066				•
1081				•
1086		•	•	
1101				•
1104				•
1108				•
3322				•

Nr.	Des	Semi 2	Semi 4	Mathon
4509	•			•
5287				•
5688		•	•	
5848	•	•	•	
6732	•			
6738			•	
6753		•		
6756		•		
6760		•		
6769		•		
6774			•	
6775		•		
6781		•	•	
6785		•		
6792		•		
6794			•	
6800		•		
6807		•		
6814		•	•	
6817	•			
6821		•	•	
6822		•		
6838			•	
6842		•	•	
6843		•		
6844		•		
6848		•	•	
6851		•		

Nr.	Des	Semi 2	Semi 4	Mathon
6852		•		
6873		•		
6897		•		
6916		•		
6917		•		
6919		•	•	
6920		•		
6922		•		
6938		•	•	
6942		•		
6943		•		
6949		•	•	
6952		•	•	
6964		•		
6966		•		
6973		•		
6988		•		
6991			•	
6994		•		
6997		•		
7012		•	•	
7030		•		
7036		•		
7039		•		
7043		•		
7045		•		
7046			•	
7048		•		

Nr.	Des	Semi 2	Semi 4	Mathon
7049		●		
7050		●		
7053		●		
7057		●		
7071		●		
7079		●		
7080		●		
7082		●	●	
7093		●		
7101		●		
7103		●	●	
7109		●		
7111		●		
7114		●		
7121		●		
7130		●		
7139		●		
7143		●	●	
7148		●		
7149			●	
7150		●		
7151		●		
7152		●	●	
7156			●	
7162		●	●	
7167		●		
7174		●		
7179		●		
7180		●		
7191		●		
7202		●		
7205		●		
7211		●		
7214		●		
7224		●		
7226		●		
7227		●		
7233			●	
7235		●		
7238		●		
7240		●		
7258		●		

Nr.	Des	Semi 2	Semi 4	Mathon
7268		●		
7272			●	
7274		●		
7284		●		
7296		●		
7306		●		
7308		●		
7316		●		
7318		●		
7334		●		
7344		●		
7366			●	
7382		●		
7402		●		
7423		●		
7429	●			
7438		●	●	
7446		●		
7447		●		
7453		●		
7454		●	●	
7458		●		
7459	●	●	●	
7460		●		
7465		●		
7471		●	●	
7473		●		
7477		●	●	
7478		●		
7489		●		
7490		●		
7498	●			
7499		●		
7519		●		
7526			●	
7538		●		
7540		●		
7542		●		
7549		●		
7562	●			
7581		●		
7583		●		

Nr.	Des	Semi 2	Semi 4	Mathon
7585		●		
7586		●		
7587		●	●	
7589		●		
7593		●		
7596			●	
7602			●	
7622		●	●	
7626		●	●	
7636		●		
7638		●		
7651			●	
7652	●			
7656		●	●	
7691	●	●		
7697	●			
7698		●		
7767		●		
7769		●		
7775		●	●	
7779		●		
7788		●		
7792		●		
7838		●		
7839		●	●	
7851		●		
7855		●		
7860		●		
7866		●		
7869		●	●	
7872		●		
7926		●		
7930		●		
7938		●		
7950		●		
7963		●		
7980		●		
7982		●		
7988		●		
7996		●		
7999		●	●	
8001			●	

Nr.	Des	Semi 2	Semi 4	Mathon
8016			•	
8017		•		
8024		•		
8030		•		
8032		•		
8036			•	
8039		•		
8040		•		
8041			•	
8044			•	
8048		•		
8063		•		
8073		•		
8074		•		
8077		•	•	
8082		•	•	
8084		•		
8085		•	•	
8086		•	•	
8087		•		
8092		•		
8096		•	•	
8104			•	
8107		•	•	
8109		•		
8116		•	•	
8121		•		
8131		•		
8134		•		
8152		•		
8154		•		
8172		•		
8179			•	
8181		•		
8198		•		
8227		•		
8239		•	•	
8241		•	•	
8244		•		
8306		•		
8335		•		
8337		•		

Nr.	Des	Semi 2	Semi 4	Mathon
8348		•		
8355		•		
8362		•		
8370		•		
8402		•		
8423		•		
8425	•	•		
8488		•		
8491		•		
8498		•		
8509		•		
8518		•		
8521		•		
8524		•		
8530		•		
8546		•		
8561		•		
8562		•		
8569		•		
8584		•		
8589		•		
8651		•		
8671		•		
8673		•		
8679		•	•	
8680		•		
8686		•		
8687		•	•	
8691		•		
8695		•		
8708		•		
8712		•		
8717		•		
8728		•		
8735		•		
8740		•		
8748		•		
8750		•	•	
8754		•		
8755		•		
8766		•		
8767		•		

Nr.	Des	Semi 2	Semi 4	Mathon
8778		•		
8781		•		
8782			•	
8787		•		
8801		•		
8805		•		
8835		•		
8837	•	•		
8842		•	•	
8845		•	•	
8846		•	•	
8848		•		
8849		•		
8850		•		
8855		•		
8860		•		
8875		•		
8876		•		
8878		•		
8879		•		
8884			•	
8891		•		
8906		•		
8921		•		
8923			•	
8925		•		
8930		•		
8952		•		
8987		•		
8989		•		
8991	•			
8995		•		
8997		•		
9010	•			
9019		•		
9021		•		
9025		•		
9028		•		
9029		•		
9046		•		
9051		•	•	
9053		•		

Nr.	Des	Semi 2	Semi 4	Mathon
9054		•		
9056		•		
9063		•		
9069		•		
9070		•		
9074		•		
9081		•	•	
9083		•		
9084		•		
9096		•		
9097		•	•	
9098		•	•	
9100	•	•		
9101		•		
9104			•	
9110	•		•	
9116		•		
9118		•		
9125		•		
9126		•		
9127		•		
9128		•	•	
9131		•		
9138		•		
9143		•		
9150		•		
9151	•			
9153		•		
9154			•	
9155		•		
9156			•	
9158			•	
9160		•		
9162		•		
9164		•		
9166		•		
9167		•		
9168		•	•	
9172		•		
9173			•	
9174	•	•	•	
9176		•		

Nr.	Des	Semi 2	Semi 4	Mathon
9182		•		
9189		•		
9191		•	•	
9192		•		
9204		•		
9208		•	•	
9209		•		
9213		•		
9218		•		
9219		•		
9220			•	
9223		•		
9224		•		
9225		•		
9227		•		
9229		•		
9231		•		
9236		•		
9269		•	•	
9281		•		
9364			•	
9375		•		
9376		•		
9377		•		
9394		•		
9395		•		
9397		•		
9398		•		
9400		•		
9401		•		
9424		•		
9432		•		
9436		•		
9441		•		
9448		•		
9470		•		
9471		•		
9473		•		
9481		•		
9512		•		
9541		•		
9542		•		

Nr.	Des	Semi 2	Semi 4	Mathon
9553		•		
9558		•		
9564		•		
9582		•		
9586		•		
9598			•	
9599		•		
9613		•		
9617		•	•	
9618		•	•	
9674		•	•	
9675		•		
9676		•		
9683		•		
9697		•		
9698		•		
9699		•		
9701		•		
9704		•		
9709		•		
9714		•		
9720		•		
9722		•		
9727		•		
9732		•		
9733			•	
9738		•		
9740			•	
9745		•		
9748		•		
9754		•		
9757		•		
9758		•		
9764		•		
9771		•		
9772		•		
9778			•	
9780		•		
9784		•		
9794		•		
9801		•		
9814		•		

Nr.	Des	Semi 2	Semi 4	Mathon
9815		•		
9828		•		
9831		•		
9837		•		
9862		•	•	
9868		•		
9877			•	
9878		•		
9885		•		
9893		•	•	
9896		•		
9900		•		
9903		•		
9906		•		
9912	•			
9919		•		
9926			•	
9930		•	•	
9932		•		
9934		•		
9935		•		
9936		•		
9938		•		
9946		•		
9947		•		
9959		•		
9960		•		
9963		•		
9967		•		
9971		•		
9976		•		
9978		•		
9981		•		
9982		•		
9983		•		
9984			•	
9986		•		
9988		•		
9990		•		
9991		•		
10009		•		
10020		•		

Nr.	Des	Semi 2	Semi 4	Mathon
10022		•		
10024		•		
10030		•		
10039			•	
10041		•	•	
10042		•	•	
10043		•		
10060		•	•	
10066	•			
10069		•		
10073		•		
10100			•	
10116		•		
10120		•		
10142		•		
10150		•		
10166		•		
10173		•	•	
10179		•		
10190		•		
10197		•		
10198			•	
10200		•		
10206		•		
10207		•		
10234			•	
10244			•	
10246		•		
10254		•		
10263		•		
10266		•		
10268			•	
10269		•	•	
10272		•	•	
10277		•	•	
10283		•		
10285		•		
10287		•		
10294		•		
10295		•	•	
10296		•		
10297	•	•	•	
10313				•

Nr.	Des	Semi 2	Semi 4	Mathon
10437		•	•	
10528				•
10572				•
10636				•
10655	•	•		
10730		•		
10734			•	•
10739				•
10785				•
10796				•
10808				•
13317				•
13780				•
14204				•
14819				•
14829				•
27067		•		
27101		•		
27106		•		
27131		•		
27333		•	•	
27534		•	•	
27588		•		
27677		•		
27848			•	
27880		•		
27887		•		
27916		•		
27928		•		
27932			•	
29622	•			
29676		•		
29677		•		
45194		•		
45224		•		
45244		•		
45253		•		
45257		•		
45259		•	•	
45274	•	•	•	
53237				•
53830				•
53959				•

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