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ON THE EXTENSIBILITY OF DIOPHANTINE TRIPLES $\{k-1, k+1, 4k\}$ FOR GAUSSIAN INTEGERS

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Abstract. In this paper, we prove that if $\{k-1, k+1, 4k, d\}$, where $k \in \mathbb{Z}[i]\backslash\{0, \pm 1\}$, $d \in \mathbb{Z}[i]$, is a Diophantine quadruple in $\mathbb{Z}[i]$, i.e. if $(k-1)d+1$, $(k-1)d+1$, $4kd+1$ are perfect squares in $\mathbb{Z}[i]$, then $d = 16k^3 - 4k$.

1. Introduction

The set of non-zero elements $\{a_1, a_2, \ldots, a_m\}$ in a commutative ring R with 1 is called *Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square in R for all $1 \leq i < j \leq m$. Let us mention the most famous historical examples of such sets: the first rational quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ found by Diophantus of Alexandria in third century AD, the first integer quadruple $\{1, 3, 8, 120\}$ found by Fermat in the seventeenth century, the first rational sextuple $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$ found by Gibbs ([13]). There exist families of such sets, for instance quadruples $\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\}$ (where F_k is k-th Fibonacci number) and $\{k-1, k+1, 4k, 16k^3 - 4k\}$ (which both represent a generalization of the Fermat's quadruple).

We mention some further important results for $R = \mathbb{Z}$. In 1969, Baker and Davenport in [2] showed that the Diophantine triple $\{1, 3, 8\}$ extends uniquely to the quadruple $\{1, 3, 8, 120\}$. Obviously, this result implies that $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő in [10] proved that the Diophantine pair $\{1, 3\}$ can be extended to infinitely many quadruples, but it cannot be extended to a quintuple. Arkin, Hoggatt and Strauss showed that each Diophantine triple can be extended

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to quadruple (see [1]). Moreover, the following conjecture is very plausible: *If* $\{a, b, c\}$ *is a Diophantine triple, then there exists unique positive integer* d *such that* $d > \max\{a, b, c\}$ *and* $\{a, b, c, d\}$ *is a Diophantine quadruple.* This conjecture has already been proved for a large class of Diophantine triples (see [8] and [9]). Furthermore, as a consequence it was obtained that there is no Diophantine sextuple in $\mathbb{Z}$ and that there are only finitely many Diophantine quintuples ([9]). Concerning the family of triples $\{k-1, k+1, 4k\}$ in $\mathbb{Z}$, Dujella showed in [6] that for $k \neq 0, \pm 1$ this triple can be extended uniquely to a quadruple $\{k-1, k+1, 4k, 16k^3-4k\}$. Further, if $\{k-1, k+1, 16k^3-4k, d\}$, $k \in \mathbb{N}$, $k \neq 1$, is a Diophantine quadruple then $d \in \{4k, 64k^5-48k^3+8k\}$ (see [5]). Fujita in [12] showed that the Diophantine pair $\{k-1, k+1\}$, $k \in \mathbb{N}$, $k \neq 1$, cannot be extended to a Diophantine quintuple.

Problems concerning Diophantine m-tuples are also considered in other rings, like polynomial rings or rings of integers in quadratic number fields, and especially in the ring of Gaussian integers (for references refer to Dujella's *Diophantine m-tuple page*, `http://www.math.hr/~duje/dtuples.html`).

In this paper we deal with the extensibility of the family of triples $\{k-1, k+1, 4k\}$ in $\mathbb{Z}[i]$. We prove the following theorem.

THEOREM 1.1. *Let* $k \in \mathbb{Z}[i]\backslash\{0, \pm 1\}$ *and let* $\{k-1, k+1, 4k, d\}$ *be a Diophantine quadruple in* $\mathbb{Z}[i]$. *Then* $d = 16k^3 - 4k$.

In Section 2 we show that the original problem of extending the triple $\{k-1, k+1, 4k\}$ is equivalent to the problem of solving the following system of two Diophantine equations:

$$(k+1)x^2 - (k-1)y^2 = 2, \; 4kx^2 - (k-1)z^2 = 3k+1. \tag{1.1}$$

Solutions of each equation in (1.1) form linear recurrence sequences. If (1.1) is solvable then these sequences have the same initial term ($x_0 = 1$ which is related to a trivial solution of (1.1)), for all parameters $k \in \mathbb{Z}[i]$, $|k| > 5$. This is shown in Section 3 using some congruence conditions modulo $2k-1$ and $4k(k-1)$. In Section 4 we apply an analog of Bennett's theorem on simultaneous rational approximations of square roots which are close to one by rationals in the case of imaginary quadratic fields ([14]) and obtain that all solutions of (1.1), for $|k| \geq 350$, are $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ and $(x, y, z) = (\pm(4k^2 - 2k - 1), \pm(4k^2 + 2k - 1), \pm(8k^2 - 1))$. In Section 5, we solve our problem for $5 < |k| < 350$ by transforming the exponential equations into inequalities for linear forms in three logarithms of algebraic numbers, then applying Baker's theory on linear forms ([3]) and, finally, we reduce the upper bound for the solution of (1.1) by using a version of Baker-Davenport's reduction method ([2]) in Section 6.

All other cases ($1 \leq |k| \leq 5$) are solved separately in the last two sections. In the case $k = i$, instead of (1.1) we solve the following system of Pellian equations

$$y^2 + ix^2 = i + 1, \qquad z^2 - (2-2i)x^2 = -1 + 2i. \tag{1.2}$$

The set of solutions of (1.2) is described using [11] and then the same procedure as in Section 5 is applied. For some parameters $1 < |k| \leq 5$, we obtain, perhaps surprisingly, some extra solutions.

The proof of Theorem 1.1 will follow directly from Propositions 6.2, 7.1 and 8.1 below, together with the simple observation that $\{k-1, k+1, 4k, d\}$ is a Diophantine quadruple in $\mathbb{Z}[i]$ if and only if $\{-k-1, -k+1, -4k, -d\}$ is a Diophantine quadruple.

All computations are performed in *Mathematica 5.2*.

2. Solving a system of Diophantine equations

Let $k \in \mathbb{Z}[i] \backslash \{0, \pm 1\}$. Our aim is to determine all Diophantine quadruples of the form $\{k-1, k+1, 4k, d\}$ in $\mathbb{Z}[i]$. Thus, we have to solve the system

$$(k-1)d + 1 = x^2, \qquad (k+1)d + 1 = y^2, \qquad 4kd + 1 = z^2, \tag{2.1}$$

in $d, x, y, z \in \mathbb{Z}[i]$. By eliminating d in (2.1), we obtain the following system of Diophantine equations

$$(k+1)x^2 - (k-1)y^2 = 2, \tag{2.2}$$

$$4kx^2 - (k-1)z^2 = 3k + 1. \tag{2.3}$$

It can be seen that the system of equations (2.2) and (2.3) is equivalent to the system (2.1). Indeed, if $x, y, z \in \mathbb{Z}[i]$ are the solutions of (2.2) and (2.3), than it follows that

$$(k+1)(x^2 - 1) = (k-1)(y^2 - 1), \qquad 4k(x^2 - 1) = (k-1)(z^2 - 1).$$

So, d is well defined by

$$d = \frac{x^2 - 1}{k - 1} = \frac{y^2 - 1}{k + 1} = \frac{z^2 - 1}{4k}. \tag{2.4}$$

Lemma 2.1. *If d is given by (2.4), where $(x, y, z) \in \mathbb{Z}[i]^3$ is a solution of (2.1), then $d \in \mathbb{Z}[i]$.*

Proof. We have to show that $d \in \mathbb{Z}[i]$. According to (2.2), we obtain that $(k+1)x^2 \equiv 2 (\mathrm{mod}\ (k-1))$, i.e. $2x^2 \equiv 2 (\mathrm{mod}\ (k-1))$. Thus, we have that $2d \in \mathbb{Z}[i]$. Besides that, $2d$ can be represented as a difference of two squares of Gaussian integers. Hence, $2d$ must be of the form $2m + 2ni$ or of the form $2d = 2m + 1 + 2ni$, where $m, n \in \mathbb{Z}$ (see ([16, p. 449])). Suppose that $2d = 2m + 1 + 2ni$. We can obtain a contradiction by showing that at least one of the numbers $(k-1)d+1$, $(k+1)d+1$ and $4kd+1$ is not a perfect square in $\mathbb{Z}[i]$. Let us note that k is of the form $2l + 1$, $l \in \mathbb{Z}[i]$, because $(k-1)d$ is a Gaussian integer. If we assume that $2d \equiv 1 (\mathrm{mod}\ 4)$ and $l \equiv 0 (\mathrm{mod}\ 4)$, i.e. $k \equiv 1 (\mathrm{mod}\ 8)$, then

$$y^2 = (k+1)d + 1 \equiv 2 (\mathrm{mod}\ 4),$$

and this is a contradiction since $y^2 \mod 4 \in \{0, 1, 3, 2i\}$. Similarly, we verify all the other possibilities ($2d \mod 4 \in \{3, 1+2i, 3+2i\}$ and $l \mod 4 \in \{1, 2, 3\}$). Hence, we conclude that $2d$ must be of the form $2m + 2ni$, i.e that d is a Gaussian integer. ☐

According to Lemma 2.1, we have that the system of equations (2.2) and (2.3) is equivalent to (2.1).

Our further step is to solve the system of equations (2.2) and (2.3) in $\mathbb{Z}[i]$. The following lemma describes the set of all solutions of the equation (2.2) in $\mathbb{Z}[i]$. In order to get expressions of the form $\sqrt{c}$, $c \in \mathbb{Z}[i]$, uniquely determined, we assume that $\sqrt{c}$ always has positive imaginary part or is a non-negative real number.

Lemma 2.2. *Let $k \in \mathbb{Z}[i] \backslash \{0, \pm 1, \pm i\}$. Then there exist $i_0 \in \mathbb{N}$ and $x_0^{(i)}, y_0^{(i)} \in \mathbb{Z}[i]$, $i = 1, \dots, i_0$, such that*

(i) *$(x_0^{(i)}, y_0^{(i)})$ is a solution of (2.2) for all $i = 1, \dots, i_0$,*

(ii) *the estimates*

$$|x_0^{(i)}|^2 \leq \frac{2|k-1|}{|k|-1}, \tag{2.5}$$

$$|y_0^{(i)}|^2 \leq \frac{2}{|k-1|} + \frac{2|k+1|}{|k|-1}, \tag{2.6}$$

hold for all $i = 1, \dots, i_0$,

(iii) *for each solution (x, y) of (2.2) there exist $i \in \{1, \dots, i_0\}$ and $m \in \mathbb{Z}$ such that*

$$x\sqrt{k+1} + y\sqrt{k-1} = (x_0^{(i)}\sqrt{k+1} + y_0^{(i)}\sqrt{k-1})(k + \sqrt{k^2-1})^m.$$

Proof. If (x, y) is a solution of (2.2), than (x_m, y_m) obtained by

$$x_m\sqrt{k+1} + y_m\sqrt{k-1} = (x\sqrt{k+1} + y\sqrt{k-1})(k + \sqrt{k^2-1})^m \tag{2.7}$$

is also a solution of (2.2) for all $m \in \mathbb{Z}$.

Let (x^*, y^*) be an element of the sequence $(x_m, y_m)_{m \in \mathbb{Z}}$ (defined by (2.7)) such the absolute value $|x^*|$ is minimal. We put

$$\begin{aligned} x'\sqrt{k+1} + y'\sqrt{k-1} &= (x^*\sqrt{k+1} + y^*\sqrt{k-1})(k + \sqrt{k^2-1}), \\ x''\sqrt{k+1} + y''\sqrt{k-1} &= (x^*\sqrt{k+1} + y^*\sqrt{k-1})(k + \sqrt{k^2-1})^{-1} \\ &= (x^*\sqrt{k+1} + y^*\sqrt{k-1})(k - \sqrt{k^2-1}). \end{aligned}$$

Due to minimality of $|x^*|$, we have that

$$\begin{aligned} |x^*| &\leq |x'| = |x^*k + y^*(k-1)|, \\ |x^*| &\leq |x''| = |x^*k - y^*(k-1)|. \end{aligned}$$

At least one of the expressions $|x^*k+y^*(k-1)|$ and $|x^*k-y^*(k-1)|$ must be greater or equal to $|x^*||k|$, since $|x^*k+y^*(k-1)|+|x^*k-y^*(k-1)| \geq 2|x^*||k|$. Let us assume that $|x^*k+y^*(k-1)| \geq |x^*||k|$. Hence,

$$|(x^*k)^2 - (y^*(k-1))^2| \geq |x^*|^2|k|,$$

and

$$|(x^*)^2 + 2(k-1)| \geq |x^*|^2|k|.$$

Immediately, we obtain the estimate for $|x^*|$,

$$|x^*|^2 \leq \frac{2|k-1|}{|k|-1}.$$

This implies the estimate for $|y^*|$,

$$|(k-1)(y^*)^2| = |(k+1)(x^*)^2 - 2| \leq |k+1|\frac{2|k-1|}{|k|-1} + 2.$$

It is obvious that there exists only finitely many pairs (x^*, y^*) such that above estimates are fulfilled. Finally, according to the definition of (x^*, y^*), there exist $m \in \mathbb{Z}$ such that

$$x^*\sqrt{k+1} + y^*\sqrt{k-1} = (x\sqrt{k+1} + y\sqrt{k-1})(k + \sqrt{k^2-1})^m.$$

Therefrom, we obtain that

$$x\sqrt{k+1} + y\sqrt{k-1} = (x^*\sqrt{k+1} + y^*\sqrt{k-1})(k + \sqrt{k^2-1})^{-m}.$$

□

The solutions $(x_0^{(i)}, y_0^{(i)})$, $i = 1, \ldots, i_0$, defined in Lemma 2.2, will be called *fundamental solutions* of the equation (2.2).

Analogously, all solutions of (2.3) are given by the following lemma.

Lemma 2.3. *Let $k \in \mathbb{Z}[i]\backslash\{0,1\}$. Then there exist $j_0 \in \mathbb{N}$ and $x_1^{(j)}, z_1^{(j)} \in \mathbb{Z}[i]$, $j = 1, \ldots, j_0$, such that*

(i) *$(x_1^{(j)}, z_1^{(j)})$ is a solution of (2.3) for all $j = 1, \ldots, j_0$,*
(ii) *the estimates*

$$|x_1^{(j)}|^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}, \tag{2.8}$$

$$|z_1^{(j)}|^2 \leq \frac{4|k||3k+1|}{|2k-1|-1} + \frac{|3k+1|}{|k-1|}, \tag{2.9}$$

hold for all $j = 1, \ldots, j_0$,

(iii) *for each solution* (x, z) *of (2.3) there exist* $j \in \{1, \dots, j_0\}$ *and* $n \in \mathbb{Z}$ *such that*

$$x\sqrt{4k} + z\sqrt{k-1} = (x_1^{(j)}\sqrt{4k} + z_1^{(j)}\sqrt{k-1})(2k-1+\sqrt{4k(k-1)})^n.$$

Now, we create the sequences

$$v_0^{(i)} = x_0^{(i)}, \qquad v_1^{(i)} = kx_0^{(i)} + (k-1)y_0^{(i)}, \qquad v_{m+2}^{(i)} = 2kv_{m+1}^{(i)} - v_m^{(i)}, \tag{2.10}$$

$$v'^{(i)}_0 = x_0^{(i)}, \qquad v'^{(i)}_1 = kx_0^{(i)} - (k-1)y_0^{(i)}, \qquad v'^{(i)}_{m+2} = 2kv'^{(i)}_{m+1} - v'^{(i)}_m, \tag{2.11}$$

for all $m \in \mathbb{N}_0$ and $i = 1, \dots, i_0$. If (x, y) is a solution of (2.2), then there exist a nonnegative integer m and $i \in \{1, \dots, i_0\}$ such that $x = v_m^{(i)}$ or $x = v'^{(i)}_m$. Similarly, if (x, z) is a solution of (2.3), then there exist $n \geq 0$ and $j \in \{1, \dots, j_0\}$ such that $x = w_n^{(j)}$ or $x = w'^{(j)}_n$, where

$$w_0^{(j)} = x_1^{(j)}, \quad w_1^{(j)} = (2k-1)x_1^{(j)} + (k-1)z_1^{(j)}, \quad w_{n+2}^{(j)} = 2(2k-1)w_{n+1}^{(j)} - w_n^{(j)}, \tag{2.12}$$

$$w'^{(j)}_0 = x_1^{(j)}, \ w'^{(j)}_1 = (2k-1)x_1^{(j)} - (k-1)z_1^{(j)}, \ w'^{(j)}_{n+2} = 2(2k-1)w'^{(j)}_{n+1} - w'^{(j)}_n. \tag{2.13}$$

LEMMA 2.4. *Let* $k \in \mathbb{Z}[i]$ *and* $|k| > 3$. *Then all fundamental solutions of equation (2.2) are* $(x_0, y_0) = (\pm 1, \pm 1)$. *Furthermore, if* (x, y) *is a solution of this equation, then there exists non-negative integer* m *such that* $x = v_m$ *or* $x = -v_m$, *where the sequence* (v_m) *is given by*

$$v_0 = 1, \qquad v_1 = 2k-1, \qquad v_{m+2} = 2kv_{m+1} - v_m, \quad m \in \mathbb{N}_0. \tag{2.14}$$

PROOF. Suppose that (x_0, y_0) is a fundamental solution of (2.2). Then the estimate (2.5) implies that

$$|x_0|^2 \leq 2(1 + \frac{2}{|k|-1}) < 4.$$

Hence, $|x_0|^2 \in \{0, 1, 2\}$. Obviously, $(x_0, y_0) = (\pm 1, \pm 1)$ are the solutions of (2.2) for every k. Also, the following cases may appear:

- $(x_0, y_0) = (0, \pm(1+i))$, $k = 1+i$,
- $(x_0, y_0) = (0, \pm(1-i))$, $k = 1-i$,
- $(x_0, y_0) = (\pm(1+i), 0)$, $k = -1-i$,
- $(x_0, y_0) = (\pm(1-i), 0)$, $k = -1+i$,
- $(x_0, y_0) = (0, \pm i)$, $k = 3$,
- $(x_0, y_0) = (\pm i, 0)$, $k = -3$.

Evidently, these cases do not satisfy the condition $|k| > 3$. The rest of the assertion follows immediately from (2.10) and (2.11). ∎

Before proceeding further, let us recapitulate our results: For $|k| > 3$, the problem of solving (2.2) and (2.3) is reduced to solve the equations

$$v_m = \pm w_n, \ v_m = \pm w'_n, \quad m, n \geq 0, \tag{2.15}$$

where we omitted the upper index (j).

3. Congruence method

In this section, we will determine all fundamental solutions of the equation (2.3) under the assumption that one of the equations in (2.15) is solvable. We will apply the congruence method which was first introduced by Dujella and Pethő in [6].

Lemma 3.1. *If (x_1,z_1) is a fundamental solutions of (2.3), then*

$$x_1 \bmod (2k-1) \in \{0,1,-1\} \text{ or } z_1(k-1) \bmod (2k-1) \in \{0,1,-1\}.$$

Proof. We have

$$(v_m \bmod (2k-1))_{m\geq 0} = (1,0,-1,-1,0,1,1,0,-1,-1,\ldots),$$
$$(w_n \bmod (2k-1))_{n\geq 0} = (x_1, z_1(k-1), -x_1, -z_1(k-1), x_1, z_1(k-1), \ldots),$$
$$(w'_n \bmod (2k-1))_{n\geq 0} = (x_1, -z_1(k-1), -x_1, z_1(k-1), x_1, -z_1(k-1), \ldots).$$

These congruence relations are obtained by induction from (2.14), (2.12) and (2.13), respectively. The rest follows immediately from (2.15). ∎

Lemma 3.2. *Let $k\in\mathbb{Z}[i]$ and $|k|>5$. If at least one of the equations in (2.15) is solvable, then all fundamental solutions of the equation (2.3) are $(x_1,z_1)=(\pm1,\pm1)$ and the related sequences (w_n) and (w'_n) are given by*

$$w_0=1,\ w_1=3k-2,\ w_{n+2}=2(2k-1)w_{n+1}-w_n, \tag{3.1}$$
$$w'_0=1,\ w'_1=k,\ w'_{n+2}=2(2k-1)w'_{n+1}-w'_n, \tag{3.2}$$

for $n\in\mathbb{N}_0$.

Proof. The proof consists of the analysis of the cases given in Lemma 3.1.

• Case: $x_1\equiv 0 (\text{mod } (2k-1))$

In this case, we have that $x_1=u(2k-1)$ for some $u\in\mathbb{Z}[i]$. Hence, $|x_1|\geq|2k-1|$ or $x_1=0$. If $x_1\neq0$, then (2.8) implies that

$$|2k-1|^2 \leq |x_1|^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}. \tag{3.3}$$

Therefrom, we obtain that

$$2(2|k|-1)^2(|k|-1) \leq |2k-1|^2(|2k-1|-1) \leq |k-1||3k+1| \leq (|k|+1)(3|k|+1).$$

Obviously, $2(2|k|-1)^2(|k|-1)-(|k|+1)(3|k|+1)>0$, for $|k|>3$ and this is in contrary with (3.3). So, for $|k|>3$ there is no non-zero fundamental solution (x_1,z_1) such that $x_1\equiv0(\text{mod } (2k-1))$ and $x_1\neq0$.

The equation (2.3) has the solution of the form $(0,z_1)$ if and only if $k\in\{0,-1,1\pm i,5\}$.

• Case: $x_1\equiv\pm1(\text{mod } (2k-1))$

Let us assume that $x_1 = u(2k-1) \pm 1$ for some $u \in \mathbb{Z}[i]$. If $x_1 \neq \pm 1$, then $|x_1| \geq |2k-1| - 1$. According to (2.8), we obtain

$$(|2k-1|-1)^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}. \tag{3.4}$$

Further, if $|k| > 3$, then $(|2k-1|-1)^3 - |k-1||3k+1| \geq 8(|k|-1)^3 - (|k|+1)(3|k|+1) > 0$, but this contradicts (3.4). Hence, under the assumptions $|k| > 3$ and $x_1 \equiv \pm 1 (\text{mod } (2k-1))$, all fundamental solutions of (2.3) are $(x_1, z_1) = (\pm 1, \pm 1)$.

• Case: $z_1(k-1) \equiv 0 (\text{mod } (2k-1))$

We have that a solution with $z_1 = 0$ is a solution of (2.3) if and only if $k = 1$. If we assume that $z_1 \neq 0$, then $z_1 = u(2k-1)$ for some $u \in \mathbb{Z}[i] \backslash \{0\}$ (because $k-1$ and $2k-1$ are relatively prime). So, $|z_1| \geq |2k-1|$. As in the previous cases, according to (2.9), we obtain that there is no non-zero fundamental solution of (2.3) such that $z_1(k-1) \equiv 0 (\text{mod } (2k-1))$ and $|k| > 4$.

• Case: $z_1(k-1) \equiv \pm 1 (\text{mod } (2k-1))$

In this case we obtain $z_1 \equiv \mp 2 (\text{mod } (2k-1))$. The solution of (2.3) is of the form $(x_1, \pm 2)$ if and only if $k = 1$. If $z_1 \neq \pm 2$, then $z_1 = u(2k-1) \pm 2$ for some $u \in \mathbb{Z}[i] \backslash \{0\}$. According to (2.9), we get that there is no fundamental solution of (2.3) such that $z_1(k-1) \equiv \pm 1 (\text{mod } (2k-1))$ and $|k| > 5$. □

Lemma 3.3. *The sequences (v_m), (w_n) and (w'_n) defined by (2.14), (3.1) and (3.2), respectively, satisfy the following congruences*

$$\begin{aligned}
(v_m \bmod 4k(k-1))_{m \geq 0} &= (1, 2k-1, 2k-1, 1, 1, 2k-1, 2k-1 \ldots), \\
(w_n \bmod 4k(k-1))_{n \geq 0} &= (1, 3k-2, -2k+3, 5k-4, -4k+5, 7k-6, \\
&\qquad 6k+7, \ldots), \\
(w'_n \bmod 4k(k-1))_{n \geq 0} &= (1, k, 2k-1, 2-k, 4k-3, -3k+4, 6k-5, \\
&\qquad -5k+6, \ldots).
\end{aligned}$$

Proof. This can be verified by induction. □

Lemma 3.4. *Let $k \in \mathbb{Z}[i]$, $|k| > 5$ and let $x \in \mathbb{Z}[i] \backslash \{\pm 1\}$ be a solution of the system of equations (2.2) and (2.3). Then there exist $m, n \in \mathbb{N}$, $n \equiv 0$ or $\pm 2 \mod 4k$, such that $x = \pm v_m$ with*

$$v_m = w_n \text{ or } v_m = w'_n,$$

where (v_m), (w_n) and (w'_n) are given by (2.14), (3.1) and (3.2), respectively.

Proof. If $v_m = \pm w_{2n+1}$ or $v_m = \pm w'_{2n+1}$, then Lemma 3.1 implies that $z_1(k-1) (\text{mod } (2k-1)) \in \{0, 1, -1\}$. But, there is no solution z_1 of (2.3) which satisfies these conditions.

Let $v_m = w_{2n}$. Then, according to Lemma 3.3, two cases may arise: $-2nk+2n+1 \equiv 1(\mathrm{mod}\ 4k(k-1))$ or $-2nk+2n+1 \equiv 2k-1(\mathrm{mod}\ 4k(k-1))$. Let us analyze each of them.

- If $-2nk+2n+1 \equiv 1(\mathrm{mod}\ 4k(k-1))$, then $-2n(k-1) \equiv 0(\mathrm{mod}\ 4k(k-1))$, i.e. $2n \equiv 0(\mathrm{mod}\ 4k)$.
- If $-2nk+2n+1 \equiv 2k-1(\mathrm{mod}\ 4k(k-1))$, then $-2n(k-1)-2(k-1) \equiv 0(\mathrm{mod}\ 4k(k-1))$. Hence, $2n \equiv -2(\mathrm{mod}\ 4k)$.

If we assume that $v_m = -w_{2n}$, then the following possibilities occur:

- If $2nk-2n-1 \equiv 1(\mathrm{mod}\ 4k(k-1))$, i.e. if $2nk-2n-2 = 4k(k-1)\zeta$ for some $\zeta \in \mathbb{Z}[i]$, then $(k-1)(n-2k\zeta) = 1$. But, this equation is not solvable in $\mathbb{Z}[i]$ for $|k| > 5$.
- If $2nk-2n-1 \equiv 2k-1(\mathrm{mod}\ 4k(k-1))$, then $2nk-2n = 2k+4k(k-1)\zeta$ for some $\zeta \in \mathbb{Z}[i]$. Therefrom, we obtain that $(k-1)(n-2k\zeta-1) = 1$ and this equation has no solution in $\mathbb{Z}[i]$ for $|k| > 5$.

Similarly, we show that the assumption $v_m = w'_{2n}$ implies that $2n \equiv 0(\mathrm{mod}\ 4k)$ or $2n \equiv 2(\mathrm{mod}\ 4k)$. Also, the assumption $v_m = -w'_{2n}$ leads to a contradiction. ∎

Now, observe that $v_0 = w_0 = w'_0 = 1$ and $v_2 = w'_2 = -1-2k+4k^2$. So, $(x,y,z) = (\pm1,\pm1,\pm1)$ and $(x,y,z) = (\pm(4k^2-2k-1), \pm(4k^2+2k-1), \pm(8k^2-1))$ are solutions of the system of equations (2.2) and (2.3). Solutions with $x = \pm1$ are not interesting for us, because they correspond to $d = 0$ which presents a trivial extension of the triple $\{k-1,k+1,4k\}$. On the other hand, solutions with $x = \pm(4k^2-2k-1)$ correspond to $d = 16k^3-4k$. Since we intend to prove that this is the unique nontrivial extension of the triple $\{k-1,k+1,4k\}$, we have to show that the system of equations (2.2) and (2.3) has no other solutions, but those given above. Our next step is to determine an upper bound for all solutions of (2.2) and (2.3) that are different from the previous ones.

Lemma 3.5. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If (x,y,z) is a solution of the system of equations (2.2) and (2.3) and if $x \in \mathbb{Z}[i]\backslash\{\pm1, \pm(4k^2-2k-1)\}$, then*

$$|x| \geq (4|k|-3)^{4|k|-3}.$$

Proof. According to Lemma 3.4, there exists $n > 2$, $n \equiv 0(\mathrm{mod}\ 4k)$ or $n \equiv \pm2(\mathrm{mod}\ 4k)$, such that $x = \pm w_n$ or $x = \pm w'_n$. The sequence $(|w_n|)$ is increasing. Let us show this by induction. Obviously, $|w_0| \leq |w_1|$. Now, assume that $|w_n| \leq |w_{n+1}|$. From (3.1), we have that

$$|w_{n+2}| \geq |2(2k-1)w_{n+1}| - |w_n| \geq (|2(2k-1)|-1)|w_{n+1}| \geq |w_{n+1}|.$$

Analogously, we obtain that $(|w'_n|)$ is an increasing sequence.

Now, let us show that $|w_n| \geq (4|k|-3)^{n-1}$ for all $n \in \mathbb{N}$. It can be easily verified that it is true for $n = 1$. Let us assume that the above inequality is true for some $n \in \mathbb{N}$. According to (3.1), we obtain that

$$|w_{n+1}| \geq (4|k|-2)|w_n| - |w_{n-1}| = (4|k|-3)|w_n| + |w_n| - |w_{n-1}|.$$

So, using the fact that $(|w_n|)$ is an increasing sequence, we get

$$|w_{n+1}| \geq (4|k|-3)(4|k|-3)^{n-1} \geq (4|k|-3)^n.$$

Further, we have that $|n| \geq 4|k|-2$, because $n \equiv 0 (\mathrm{mod}\ 4k)$ or $n \equiv -2 (\mathrm{mod}\ 4k)$ and $n \neq 0, 2$. Hence, $|w_n| \geq (4|k|-3)^{4|k|-3}$.

The same can be proved for the sequence (w'_n). □

4. An application of a theorem on simultaneous approximations

In this section, we prove that if the parameter $|k|$ is large enough, then $x = \pm 1$ and $x = \pm(4k^2 - 2k - 1)$ give all solutions of the system of equations (2.2) and (2.3). In order to get this, we apply the following generalization of Bennett's theorem [4] on simultaneous rational approximations of square roots which are close to one.

Theorem 4.1 ([14]). *Let $\theta_i = \sqrt{1 + \frac{a_i}{T}}$, $i = 1, 2$, with a_1 and a_2 pairwise distinct quadratic integers in the imaginary quadratic field K and let T be an algebraic integer of K. Further, let $M = \max\{|a_1|, |a_2|\}$, $|T| > M$ and*

$$\begin{aligned} l &= \frac{27}{64}\frac{|T|}{|T|-M}, \\ L &= \frac{27}{16|a_1|^2|a_2|^2|a_1-a_2|^2}(|T|-M)^2 > 1, \\ p &= \sqrt{\frac{2|T|+3M}{2|T|-2M}}, \\ P &= 16\frac{|a_1|^2|a_2|^2|a_1-a_2|^2}{\min\{|a_1|^3, |a_2|^3, |a_1-a_2|^3\}}(2|T|+3M). \end{aligned}$$

Then

$$\max\left(\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right) > c|q|^{-\lambda},$$

for all algebraic integers $p_1, p_2, q \in K$, where

$$\begin{aligned} \lambda &= 1 + \frac{\log P}{\log L}, \\ c^{-1} &= 4pP(\max\{1, 2l\})^{\lambda-1}. \end{aligned}$$

First, let us show the following technical lemma.

LEMMA 4.2. *Let $k \in \mathbb{Z}[i]$, $|k| > 5$ and let $(x,y,z) \in \mathbb{Z}[i]^3$ be a solution of the system of equations (2.2), (2.3). Furthermore, let*

$$\theta_1^{(1)} = \pm\sqrt{\frac{k+1}{k-1}}, \qquad \theta_1^{(2)} = -\theta_1^{(1)},$$
$$\theta_2^{(1)} = \pm\sqrt{\frac{k}{k-1}}, \qquad \theta_2^{(2)} = -\theta_2^{(1)},$$

where the signs are chosen such that

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \le \left|\theta_1^{(2)} - \frac{y}{x}\right|, \qquad \left|\theta_2^{(1)} - \frac{z}{2x}\right| \le \left|\theta_2^{(2)} - \frac{z}{2x}\right|.$$

Then, we obtain

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \le \frac{2}{\sqrt{|k^2-1|}} \cdot \frac{1}{|x|^2},$$
$$\left|\theta_2^{(1)} - \frac{z}{2x}\right| \le \frac{1}{4}\frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|x|^2}.$$

PROOF. We have

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| = \left|(\theta_1^{(1)})^2 - \frac{y^2}{x^2}\right| \cdot \left|\theta_1^{(1)} + \frac{y}{x}\right|^{-1} = \frac{2}{|k-1||x|^2}\left|\theta_1^{(2)} - \frac{y}{x}\right|^{-1}.$$

Because of the assumptions on $\theta_1^{(1)}$ and $\theta_1^{(2)}$, we get

$$\left|\theta_1^{(2)} - \frac{y}{x}\right| \ge \frac{1}{2}\left(\left|\theta_1^{(1)} - \frac{y}{x}\right| + \left|\theta_1^{(2)} - \frac{y}{x}\right|\right) \ge \frac{1}{2}\left|\theta_1^{(1)} - \theta_1^{(2)}\right| = \left|\sqrt{\frac{k+1}{k-1}}\right|.$$

Hence,

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \le \frac{2}{|k-1||x|^2}\left|\sqrt{\frac{k-1}{k+1}}\right|.$$

Similarly, the second inequality is obtained. □

Now, we will apply Theorem 4.1 on $\theta_1^{(1)}$ and $\theta_2^{(1)}$. In our case, we have $a_1 = 2$, $a_2 = 1$, $T = k-1$, $M = 2$ and

$$l = \frac{27}{64}\frac{|k-1|}{|k-1|-2}, \qquad L = \frac{27}{64}(|k-1|-2)^2,$$
$$p = \sqrt{\frac{|k-1|+3}{|k-1|-2}}, \qquad P = 128(|k-1|+3).$$

The condition $L > 1$ of Theorem 4.1 is satisfied, because $L > 0.43(|k|-3)^2$ and $|k| > 5$. So, we conclude that

$$\max\left\{\left|\theta_1^{(1)} - \frac{2y}{2x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} > c|2x|^{-\lambda}, \tag{4.1}$$

where

$$\lambda = 1 + \frac{\log P}{\log L}, \qquad c^{-1} = 4pP(\max\{1, 2l\})^{\lambda-1}.$$

If we assume that $|k| > 14$, then $\max\{1, 2l\} = 1$ and $c^{-1} = 4pP$. Further, according to Lemma 4.2, we have

$$\max\left\{\left|\theta_1^{(1)} - \frac{y}{x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} \leq \frac{1}{4}\frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|x|^2},$$

and (4.1) implies

$$\frac{1}{4}\sqrt{\frac{|k-1|-2}{|k-1|+3}} \cdot \frac{1}{128(|k-1|+3)}|2x|^{-\lambda} < \frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|2x|^2}.$$

Hence,

$$|2x|^{2-\lambda} \leq 2^9 \frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}. \tag{4.2}$$

Proposition 4.3. *Let $k \in \mathbb{Z}[i]$ and $|k| \geq 350$. Then all solutions of the system of equations (2.2) and (2.3) are given by $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ and $(x, y, z) = (\pm(4k^2 - 2k - 1), \pm(4k^2 + 2k - 1), \pm(8k^2 - 1))$.*

Proof. We use the estimate for x, $|x| \geq (4|k| - 3)^{4|k|-3}$ (from Lemma 3.5), and after taking a logarithm in (4.2), we obtain

$$(2-\lambda)(\log 2 + (4|k|-3)\log(4|k|-3)) \leq \log\left(2^9 \frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}\right), \tag{4.3}$$

where log denotes the natural logarithm. This gives us an inequality for k, since

$$\lambda = 1 + \frac{7\log 2 + \log(|k-1|+3)}{\log\frac{27}{64} + 2\log(|k-1|-2)}.$$

Finally, let us assume that $|k| \geq 350$. Then $2 - \lambda > 0.01$. The right side of (4.3) satisfies the following inequality

$$\log\left(2^9 \frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}\right) \quad \leq \quad \log(3|k|) + 7.$$

On the other hand, we obtain that the left side of (4.3) satisfies

$$\begin{aligned}(2-\lambda)(\log 2 + (4|k|-3)\log(4|k|-3)) &> 0.01(\log 2 + (4|k|-3)\log(4|k|-3)) \\ &> \log(3|k|) + 7,\end{aligned}$$

and that is a contradiction. ∎

5. Linear forms in three logarithms

In this section, we study the case where $k \in \mathbb{Z}[i]$ and $5 < |k| < 350$. We will apply a method similar to those used in [2].

Let $x = v_m = w_n$ for some $m, n \in \mathbb{N}$. By solving the recurrences (2.14) and (3.1) for (v_m) and (w_n), we obtain

$$\begin{aligned} x &= \frac{\sqrt{k-1}+\sqrt{k+1}}{2\sqrt{k+1}}(k+\sqrt{k^2-1})^m \\ &\quad -\frac{\sqrt{k-1}-\sqrt{k+1}}{2\sqrt{k+1}}(k-\sqrt{k^2-1})^m, \\ x &= \frac{\sqrt{k-1}+2\sqrt{k}}{4\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n \\ &\quad -\frac{\sqrt{k-1}-2\sqrt{k}}{4\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n. \end{aligned}$$

From now on, let us assume that $\mathrm{Re}(k) > 0$. Besides that, we will discuss the case where $\mathrm{Re}(k) = 0$ and $\mathrm{Im}(k) > 0$. The other two cases ($\mathrm{Re}(k) < 0$ and $\mathrm{Re}(k) = 0$, $\mathrm{Im}(k) < 0$) can be avoided by taking a quadruple $\{-k+1, -k-1, -4k, -d\}$ instead of $\{k-1, k+1, 4k, d\}$, because $\{k-1, k+1, 4k, d\}$ is a Diophantine quadruple in $\mathbb{Z}[i]$ if and only if $\{-k+1, -k-1, -4k, -d\}$ is a Diophantine quadruple.

Let

$$P = \frac{\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}}(k+\sqrt{k^2-1})^m, \tag{5.1}$$

$$Q = \frac{\sqrt{k-1}+2\sqrt{k}}{2\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n. \tag{5.2}$$

The equation $v_m = w_n$ implies that

$$P + \frac{2}{k+1}P^{-1} = Q + \frac{3k+1}{4k}Q^{-1}. \tag{5.3}$$

Further, according to (5.3) and the trivial estimates $|P| > 5$ and $|Q| > 9$, we have

$$||P|-|Q|| \le |P-Q| \le \left|\frac{3k+1}{4k}\right| |Q|^{-1} + \left|\frac{2}{k+1}\right| |P|^{-1} < 0.2.$$

Hence, $|P| \le |Q| + 0.2 \le 1.03|Q|$, i.e. $|Q|^{-1} \le 1.03|P|^{-1}$ and

$$\left|\frac{P-Q}{P}\right| \le \left|\frac{3k+1}{4k}\right| |Q|^{-1}|P|^{-1} + \left|\frac{2}{k+1}\right| |P|^{-2} < 1.33|P|^{-2} < 0.06.$$

Finally, we obtain that

$$\begin{aligned}\left|\log\frac{|P|}{|Q|}\right| &= \left|\log\left(1-\frac{|P|-|Q|}{|P|}\right)\right| < 1.33|P|^{-2}+(1.33|P|^{-2})^2\\ &< 1.5|P|^{-2} = 1.5\left|\frac{\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}}\right|^{-2}|k+\sqrt{k^2-1}|^{-2m}\\ &< 1.5\left(\frac{1}{5}\right)^{2m} < 16^{-m}.\end{aligned}$$

The above expression can be written as a linear form in three logarithms:

$$\begin{aligned}&\Big|m\log|k+\sqrt{k^2-1}|-n\log|2k-1+2\sqrt{k^2-k}|\\ &\quad+\log\left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}(\sqrt{k-1}+2\sqrt{k})}\right|\Big| < 16^{-m}\end{aligned} \tag{5.4}$$

and it is valid for all $k\in\mathbb{Z}[i]$ such that $\mathrm{Re}(k)>0$ and $|k|>5$.

We use the following theorem of Baker and Wüstholz ([3, p. 20]) to obtain an upper bound for m.

THEOREM 5.1. *Let Λ be a nonzero linear form in logarithms of l algebraic numbers $\alpha_1,\dots,\alpha_l$ with rational integer coefficients $b_1,\dots,b_l$. Then*

$$\log\Lambda \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B=\max(|b_1|,\dots,|b_l|)$ and where d is the degree of the number field generated by $\alpha_1,\dots,\alpha_l$ over the rationals.

Here

$$h'(\alpha)=\max(h(\alpha),\frac{1}{d}|\log\alpha|,\frac{1}{d}),$$

where $h(\alpha)$ denotes the standard logarithmic Weil height of α ([3, p. 22]).

Let

$$\Lambda = |m\log\alpha_1 - n\log\alpha_2+\log\alpha_3|, \tag{5.5}$$

where

$$\begin{aligned}\alpha_1 &= |k+\sqrt{k^2-1}|,\\ \alpha_2 &= |2k-1+2\sqrt{k^2-k}|,\\ \alpha_3 &= \left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}(\sqrt{k-1}+2\sqrt{k})}\right|.\end{aligned}$$

First, let us verify that the condition $\Lambda\neq 0$ in Theorem 5.1 is satisfied or equivalently $|P|\neq|Q|$. This condition is not trivially satisfied and it will be proved in the following lemma.

LEMMA 5.2. *If $v_m=w_n$, then $|P|\neq|Q|$ for all $k\in\mathbb{Z}[i]\backslash\{0,\pm1\}$.*

PROOF. Assume that $|P|=|Q|$. If $P=Q$, then (5.3) imply that $3k^2-4k+1=0$. The only solution of this equation in $\mathbb{Z}[i]$ is $k=1$, so we conclude that $P\neq Q$.

Let us denote

$$\alpha=\sqrt{\frac{k-1}{k+1}},\qquad \beta=\sqrt{\frac{k-1}{k}}.$$

According to (5.1) and (5.2), we have

$$P=a+b\alpha,\qquad Q=c+d\beta,$$

where $a,b,c,d\in\mathbb{Q}(i)$. Since $v_m=a$ and $w_n=c$, the assumption $v_m=w_n$ implies $a=c$. Further, we have

$$|P|^2 = p+u\alpha+\overline{u\alpha}+q|\alpha|^2, \tag{5.6}$$

$$|Q|^2 = r+v\beta+\overline{v\beta}+s|\beta|^2, \tag{5.7}$$

where $p,q,r,s\in\mathbb{Q}$ and $u,v\in\mathbb{Q}(i)$.

At the moment, let us point out several facts that are crucial for our proof:

i) The complex numbers α,β are algebraic numbers of degree 2 over $\mathbb{Q}(i)$, for $k\in\mathbb{Z}[i]\setminus\{0,\pm1\}$.

ii) The basis for $\mathbb{Q}(i)(\alpha,\overline{\alpha})$ (considered as a vector space over $\mathbb{Q}(i)$) is $B_\alpha=\{1,\alpha,\overline{\alpha},|\alpha|^2\}$ and, analogously, the basis for $\mathbb{Q}(i)(\beta,\overline{\beta})$ is $B_\beta=\{1,\beta,\overline{\beta},|\beta|^2\}$.

iii) The set $B_{\alpha,\beta}=\{1,\alpha,\overline{\alpha},|\alpha|^2,\beta,\overline{\beta},|\beta|^2\}$ is linearly independent over $\mathbb{Q}(i)$.

Obviously, $|P|^2$ is an element of the algebraic extension field $\mathbb{Q}(i)(\alpha,\overline{\alpha})$ and is uniquely represented in (5.6). Analogously, $|Q|^2\in\mathbb{Q}(i)(\beta,\overline{\beta})$ is uniquely represented in (5.7). Since $B_{\alpha,\beta}$ is a linearly independent set, the assumption $|P|^2=|Q|^2$ implies that $u=q=v=s=0$. Therefrom, it follows $b=d=0$. Hence, we have $P=a$ and $Q=c$, i.e. $P=Q$, a contradiction.

In what follows, we will prove the statements i), ii) and iii).

PROOF OF i): Let us assume conversely that $\alpha\in\mathbb{Q}(i)$. Then $(k+1)/(k-1)$ is a perfect square in $\mathbb{Q}(i)$. So, there exist $\rho,A,B\in\mathbb{Z}[i]$ such that

$$k+1=\rho A^2,\qquad k-1=\rho B^2.$$

Therefrom it follows that $2=\rho(A^2-B^2)$. Using the facts that $\mathbb{Z}[i]$ is a ring with unique factorization with units ±1, $\pm i$ and that $a+bi\in\mathbb{Z}[i]$ is a prime if and only if one of a,b is zero and the other is a prime congruent to 3 (mod 4) or both a,b are nonzero and a^2+b^2 is prime, we get that only finitely many cases may occur: $(\rho,A^2-B^2)\in\{(\pm2,\pm1),(\pm2i,\mp i),(\pm1,\pm2),(\pm i,\mp2i),(\pm(1+i),\pm(1-i)),(\pm(1-i),\pm(1+i))\}$. This implies that $k\in\{\pm1,0\}$. In the same way, the assumption that $\beta\in\mathbb{Q}(i)$, i.e. that $(k-1)/k$ is a perfect square in $\mathbb{Q}(i)$ implies that $k\in\{0,1\}$.

PROOF OF ii): If $\gamma\in\mathbb{Q}(i)(\alpha,\overline{\alpha})$, then $\gamma=\sum q_{ij}\alpha^i\overline{\alpha}^j$, where $q_{ij}\in\mathbb{Q}(i)$. But, $\alpha^2,\overline{\alpha}^2\in\mathbb{Q}(i)$ and $\alpha\overline{\alpha}=|\alpha|^2$ imply that $\gamma=q_0+q_1\alpha+q_2\overline{\alpha}+q_3|\alpha|^2$

for some $q_0, q_1, q_2, q_3 \in \mathbb{Q}(i)$. Hence, the set B_α spans $\mathbb{Q}(i)(\alpha, \overline{\alpha})$. Next we have to show that B_α is linearly independent. Suppose that the set $\{1, \alpha, \overline{\alpha}\}$ is linearly dependent. Hence, there exist $A, B \in \mathbb{Q}(i)$ such that

$$\overline{\alpha} = A + B\alpha.$$

By squaring the previous equation, we obtain

$$\overline{\alpha}^2 - A^2 - B^2\alpha^2 = 2AB\alpha$$

and, therefrom, we get that $\alpha \in \mathbb{Q}(i)$, a contradiction. So, $\{1, \alpha, \overline{\alpha}\}$ is linearly independent.

Further, if we assume that the set $\{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is linearly dependent, then

$$|\alpha|^2 = A + B\alpha + C\overline{\alpha} \tag{5.8}$$

for some $A, B, C \in \mathbb{Q}(i)$. Multiplication by α gives us

$$C|\alpha|^2 = -B\alpha^2 - A\alpha + \alpha^2\overline{\alpha}. \tag{5.9}$$

Suppose $C = 0$. Then $|\alpha|^2 = A + B\alpha$ and by squaring we get that $2AB\alpha \in \mathbb{Q}(i)$, a contradiction. Since $C \neq 0$, according to (5.8) and (5.9), it follows that

$$A + B\alpha + C\overline{\alpha} = -\frac{B}{C}\alpha^2 - \frac{A}{C}\alpha + \frac{1}{C}\alpha^2\overline{\alpha},$$

and because $\{1, \alpha, \overline{\alpha}\}$ is linearly independent, we have

$$A = -\frac{B}{C}\alpha^2, \qquad B = \frac{A}{C}, \qquad C = \frac{1}{C}\alpha^2.$$

Therefore, $C^2 = \alpha^2$, but this is a contradiction, because α^2 is not a perfect square in $\mathbb{Q}(i)$.

Proof of iii): It suffices to prove that β, $\overline{\beta}$ and $|\beta|^2$ are not elements of $\mathbb{Q}(i)(\alpha, \overline{\alpha})$. Suppose that β can be represented as

$$\beta = A + B\alpha + C\overline{\alpha} + D|\alpha|^2,$$

for some $A, B, C, D \in \mathbb{Q}(i)$. By squaring, we get

$$\begin{aligned}\beta^2 \quad = \quad & A^2 + B^2\alpha^2 + C^2\overline{\alpha}^2 + D^2|\alpha|^4 \\ & +2AB\alpha + 2AC\overline{\alpha} + 2AD|\alpha|^2 + 2BC|\alpha|^2 + 2BD\alpha^2\overline{\alpha} + 2CD\overline{\alpha}^2\alpha.\end{aligned}$$

Since $\beta^2, \alpha^2, \overline{\alpha}^2 \in \mathbb{Q}(i)$ and set $\{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is linearly independent over $\mathbb{Q}(i)$, we obtain that

$$AB + CD\overline{\alpha}^2 = 0, \tag{5.10}$$

$$AC + BD\alpha^2 = 0, \tag{5.11}$$

$$AD + BC = 0. \tag{5.12}$$

Now, (5.10) and (5.12) imply that $A^2 = C^2\overline{\alpha}^2$, then (5.11) and (5.12) imply that $A^2 = B^2\alpha^2$ and (5.10) and (5.11) imply that $A^2 = D^2|\alpha|^4$. Hence,

$\beta^2 = 4A^2$, a contradiction. Similarly, we can obtain that $\overline{\beta}$ and $|\beta|^2$ are not in $\mathbb{Q}(i)(\alpha, \overline{\alpha})$. ∎

The next step is to determine B from Theorem 5.1, which is in our case $B = \max(m, n)$. The result is given in the following lemma.

LEMMA 5.3. *Let $k \in \mathbb{Z}[i]$ such that $|k| > 5$ and $Re(k) > 0$. If $v_m = w_n$, then $n \leq m$.*

PROOF. We have already shown that $|Q| \leq |P| + 0.2$ and, therefrom, $|Q| < 1.04|P|$, i.e.

$$\left|1 + \frac{\sqrt{k-1}}{2\sqrt{k}}\right| \alpha_2^n < 1.04 \left|1 + \frac{\sqrt{k-1}}{\sqrt{k+1}}\right| \alpha_1^m.$$

After taking logarithms we get

$$n < \log\left(\frac{1.04\left|1 + \frac{\sqrt{k-1}}{\sqrt{k+1}}\right|}{\left|1 + \frac{\sqrt{k-1}}{2\sqrt{k}}\right|}\right)\frac{1}{\log \alpha_2} + m\frac{\log \alpha_1}{\log \alpha_2}. \tag{5.13}$$

Observe that we have

$$\left|1 + \frac{\sqrt{k-1}}{2\sqrt{k}}\right| > 1, \qquad 1.04\left|1 + \frac{\sqrt{k-1}}{\sqrt{k+1}}\right| \leq 2.4, \qquad \log \alpha_2 > 2,$$

and these inequalities imply that

$$\log\left(\frac{1.04\left|1 + \frac{\sqrt{k-1}}{\sqrt{k+1}}\right|}{\left|1 + \frac{\sqrt{k-1}}{2\sqrt{k}}\right|}\right)\frac{1}{\log \alpha_2} < \frac{1}{2}.$$

Since,

$$\frac{\alpha_1}{|k|} = \left|1 + \sqrt{1 - \frac{1}{k^2}}\right| < 2.02, \qquad \frac{\alpha_2}{|k|} \geq \left(2\left|1 + \sqrt{1 - \frac{1}{k}}\right| - \frac{1}{|k|}\right) > 3.5,$$

we have

$$\frac{\log \alpha_1}{\log \alpha_2} < 1.$$

Applying the above inequalities to (5.13), we obtain

$$n < \frac{1}{2} + m,$$

and therefore $n \leq m$. ∎

Note that $B(m, n) = m > 1$, since $v_1 \neq w_1$.

Our next aim is to determine standard logarithmic Weil height of α_i, $i = 1, 2, 3$. For that purpose we need minimal polynomials of these algebraic numbers. The minimal polynomials of the algebraic numbers $\beta_1 = k + \sqrt{k^2 - 1}$, $\beta_2 = 2k - 1 + 2\sqrt{k^2 - k}$ and $\beta_3 = \dfrac{2\sqrt{k}(\sqrt{k-1} + \sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1} + 2\sqrt{k})}$ were given in [6],

$$
\begin{aligned}
q_1(x) &= x^2 - 2kx + 1,\\
q_2(x) &= x^2 - 2(2k-1)x + 1,\\
q_3(x) &= (9k^4 + 24k^3 + 22k^2 + 8k + 1)x^4 - 16k(3k^3 + 7k^2 + 5k + 1)x^3\\
&\quad +48k^2(k^2 + 4k + 3)x^2 - 128k^2(k+1)x + 64k^2.
\end{aligned}
$$

According to the proof of Theorem 9.11 in [15], we can determine the minimal polynomials of the algebraic numbers $\beta_i\overline{\beta_i} = \alpha_i^2$, $i = 1, 2, 3$ (and clearly of α_i, too).

The minimal polynomials of α_1, α_2 are, respectively,

$$
\begin{aligned}
p_1(x) &= x^8 - 4(\mu^2 + \nu^2)x^6 + (8\mu^2 - 8\nu^2 - 2)x^4 - 4(\mu^2 + \nu^2)x^2 + 1,\\
p_2(x) &= x^8 - 4(4(\mu^2 + \nu^2 - \mu) + 1)x^6 + (32(\mu^2 - \nu^2 - \mu - 2) + 6)x^4\\
&\quad -4(4(\mu^2 + \nu^2 - \mu) + 1)^2x^2 + 1,
\end{aligned}
$$

where $k = \mu + i\nu$. The minimal polynomial of α_3 is of degree 32,

$$
p_3(x) = \sum_{i=0}^{16} a_i x^{2i}
$$

and it was derived by the help of the program package *Mathematica*. We list only few of its coefficients a_i (because its coefficients are huge rational functions in μ, ν).

$$
\begin{aligned}
a_0 &= -\frac{2^{48}(\mu^2 + \nu^2)^8}{((1+\mu)^2 + \nu^2)^8((1+3\mu)^2 + 3\nu^2)^8},\\
a_1 &= \frac{2^{50}(\mu^2 + \nu^2)^8}{((1+\mu)^2 + \nu^2)^7((1+3\mu)^2 + 3\nu^2)^8},\\
a_2 &= -\frac{3 \cdot 2^{45}(\mu^2 + \nu^2)^8}{((1+\mu)^2 + \nu^2)^7((1+3\mu)^2 + 3\nu^2)^8}(21 + 46\mu + 13(\mu^2 + \nu^2)),\\
a_3 &= \frac{2^{44}(\mu^2 + \nu^2)^7}{((1+\mu)^2 + \nu^2)^6((1+3\mu)^2 + 3\nu^2)^8}(3 - 26\mu + 247\mu^2 + 300\mu^3\\
&\quad +36\mu^4 + 191\nu^2 + 300\mu\nu^2 + 72\mu^2\nu^2 + 36\nu^4),
\end{aligned}
$$

$$\begin{aligned}
a_4 &= -\frac{2^{38}(\mu^2+\nu^2)^6}{((1+\mu)^2+\nu^2)^6((1+3\mu)^2+3\nu^2)^8}(-1+52\mu+14\mu^2-3196\mu^3 \\
&\quad +20299\mu^4+48864\mu^5+40476\mu^6+9648\mu^7+324\mu^8+170\nu^2 \\
&\quad -2780\mu\nu^2+32854\mu^2\nu^2+86112\mu^3\nu^2+91452\mu^4\nu^2+28944\mu^5\nu^2 \\
&\quad +1296\mu^6\nu^2+13867\nu^4+37248\mu\nu^4+61476\mu^2\nu^4+28944\mu^3\nu^4 \\
&\quad +1944\mu^4\nu^4+10500\nu^6+9648\mu\nu^6+1296\mu^2\nu^6+324\nu^8), \\
&\vdots \\
a_{14} &= -\frac{1536(\mu^2+\nu^2)^2}{((1+\mu)^2+\nu^2)((1+3\mu)^2+3\nu^2)^2}(21+46\mu+13\mu^2+13\nu^2), \\
a_{15} &= \frac{256(\mu^2+\nu^2)}{((1+\mu)^2+\nu^2)}, \\
a_{16} &= -1.
\end{aligned}$$

Further, for the purpose of determining the heights $h(\alpha_i)$, we have to find all roots of the minimal polynomials $p_i(x)$ or, if it is not possible, we have to bound them. With some algebraic manipulation, we can get all roots of $p_1(x)$ and $p_2(x)$. The roots of $p_1(x)$ are

$$\begin{aligned}
x_1, x_2 &= \pm|k+\sqrt{k^2-1}| = \pm\alpha_1, \\
x_3, x_4 &= \pm|k-\sqrt{k^2-1}|, \\
x_5, x_6 &= \pm\sqrt{|k|^2-|k^2-1|-\sqrt{(|k|^2-|k^2-1|)^2-1}}, \\
x_7, x_8 &= \pm\sqrt{|k|^2-|k^2-1|+\sqrt{(|k|^2-|k^2-1|)^2-1}}.
\end{aligned}$$

It can be shown that $|x_3| = |x_4| \le 1$ and that $|x_i| = 1$ for $i = 5, 6, 7, 8$. So,

$$h(\alpha_1) \le \frac{1}{8}\log(|x_1|\cdot|x_2|) = \frac{1}{4}\log|k+\sqrt{k^2-1}| \le \frac{1}{4}\log(2|k|+1) \le 1.64\ .$$

The roots of $p_2(x)$ are

$$\begin{aligned}
x_1, x_2 &= \pm\alpha_2, \\
x_3, x_4 &= \pm|2k-1-2\sqrt{k^2-k}|, \\
x_5, x_6 &= \pm\sqrt{|2k-1|^2-4|k^2-k|+\sqrt{(|2k-1|^2-4|k^2-k|)^2-1}}, \\
x_7, x_8 &= \pm\sqrt{|2k-1|^2-4|k^2-k|-\sqrt{(|2k-1|^2-4|k^2-k|)^2-1}}.
\end{aligned}$$

As in the previous case, we obtain that

$$h(\alpha_2) \le \frac{1}{8}\log(|2k-1+2\sqrt{k^2-k}|^2) \le \frac{1}{4}\log(4|k|+3) \le 1.82.$$

The estimate for $h(\alpha_3)$ will be less accurate than those for $h(\alpha_1)$ and $h(\alpha_2)$, because most roots of $p_3(x)$ cannot be found analytically. By calculation, we obtain following 8 roots:

$$\begin{aligned}
x_1, x_2 &= \pm\alpha_3, \\
x_3, x_4 &= \pm\left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1}-2\sqrt{k})}\right|, \\
x_5, x_6 &= \pm\left|\frac{2k(\sqrt{k+1}-\sqrt{k-1})+\sqrt{2k(k-1)(\sqrt{k^2-1}-k)}}{\sqrt{k+1}(3k+1)}\right|, \\
x_7, x_8 &= \pm\left|\frac{2k(\sqrt{k+1}-\sqrt{k-1})-\sqrt{2k(k-1)(\sqrt{k^2-1}-k)}}{\sqrt{k+1}(3k+1)}\right|.
\end{aligned}$$

We estimate the remaining roots on the following way:

$$|x_i| \le 32 \cdot \max\{|a_j|, 0 \le j \le 16\}, \quad i = 9, 10, \ldots, 32,$$

where a_i are the coefficients of $p_3(x)$. We give as an example the estimate for $|a_4|$,

$$\begin{aligned}
|a_4| &= \frac{2^{38}(\mu^2+\nu^2)^6}{((1+\mu)^2+\nu^2)^6(1+3\mu)^2+3\nu^2)^8}|p(\mu,\nu)| \\
&\le \frac{2^{38}(\mu^2+\nu^2)^6}{(\mu^2+\nu^2)^6(9(\mu^2+\nu^2))^8}\sum|b_{ij}|(\mu^2+\nu^2)^4 \le 5.5\cdot 10^6,
\end{aligned}$$

where

$$p(\mu,\nu) = \sum_{0\le i+j\le 8} b_{ij}\mu^i\nu^j = -1 + 52\mu + 14\mu^2 + \cdots + 1296\mu^2\nu^6 + 324\nu^8.$$

All coefficients are bounded by:

$$\max\{|a_j|, 0 \le j \le 16\} \le |a_8| < 1.65 \cdot 10^8.$$

It can be seen that $|x_i| < 1$ for $i = 5, 6, 7, 8$. So, we have that

$$h(\alpha_3) \le \frac{1}{32}\log(a_n' \alpha_3^2 |x_3||x_4|(32 \cdot 1.65 \cdot 10^8)^{24}),$$

where $a_n' = ((1+\mu)^2+\nu^2)^8((1+3\mu)^2+(3\nu)^2)^8$ represents the leading coefficient of the minimal polynomial of α_3 with integer coefficients. By using the estimates

$$\begin{aligned}\alpha_3^2 x_3 x_4 &= 16\cdot\left|\frac{k}{k+1}\right|^2\cdot\left|\frac{2k+2\sqrt{k^2-1}}{3k+1}\right|^2\\ &\le 16\left(1+\frac{1}{|k|-1}\right)^2 4\left(\frac{2|k|+1}{3|k|-1}\right)^2 < 62,\\ a_n' &\le (1+2|k|+|k|^2)^8(1+6|k|+9|k|^2)^8 < 6.5\cdot 10^{52},\end{aligned}$$

we get

$$h(\alpha_3) < \frac{1}{32}\log(6.5\cdot 10^{52}\cdot 62\cdot(32\cdot 1.65\cdot 10^8)^{24}) < 20.72.$$

Now, we have to estimate d, where d is the degree of the number field generated by $\alpha_1,\alpha_2,\alpha_3$. We obtain

$$d \le [\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3):\mathbb{Q}(\alpha_1,\alpha_2)][\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}(\alpha_1)][\mathbb{Q}(\alpha_1):\mathbb{Q}] \le 32\cdot 8\cdot 8.$$

Finally, we apply Theorem 5.1 to the form (5.5). Combining (5.4) and the above estimates, we find that

$$\begin{aligned}-m\log 16 &\ge \log\Lambda\\ &\ge -18\cdot 4!\cdot 3^4(32\cdot 2048)^5\cdot 1.64\cdot 1.82\cdot 18.69\cdot\log(6\cdot 2048)\log m\\ &> -2.5\cdot 10^{31}\log m.\end{aligned}$$

Therefore, we obtain

$$m \le 2.5\cdot 10^{31}\log m. \tag{5.14}$$

The inequality (5.14) is not valid for $m \ge 2\cdot 10^{33}$. Therefore, we have

$$|m\theta - n + \beta| < \alpha\cdot 16^{-m}, \qquad 1 < m < 2\cdot 10^{33}, \tag{5.15}$$

for $\theta = \log\alpha_1/\log\alpha_2$, $\beta = \log\alpha_3/\log\alpha_2$, $\alpha = 1/\log\alpha_2$.

In the case of $k = i\nu$, $5 < \nu < 350$, the same conclusion, i.e. (5.15), can be obtained. The only difference is that we take

$$Q = \frac{\sqrt{k-1}-2\sqrt{k}}{2\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n$$

in (5.2). We omit further details.

Finally, we have to carry out the procedure described in Sections 4 and 5 for the case $x = v_m = w_n'$. By solving the recurrence (3.2) for (w_m'), we obtain

$$x = \frac{2\sqrt{k}-\sqrt{k-1}}{4\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n + \frac{2\sqrt{k}+\sqrt{k-1}}{4\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n.$$

If $\mathrm{Re}(k) > 0$, then instead of (5.2) we take

$$Q = \frac{2\sqrt{k}-\sqrt{k-1}}{2\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n,$$

and related algebraic numbers are α_1, α_2, α_3', where

$$\alpha_3' = \left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1}-2\sqrt{k})}\right|.$$

If $\mathrm{Re}(k)=0$ and $\mathrm{Im}(k)>0$, then we put

$$Q = \frac{2\sqrt{k}+\sqrt{k-1}}{2\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n,$$

and we deal with α_1, α_2', α_3, where

$$\alpha_2' = \left|2k-1-2\sqrt{k^2-k}\right|.$$

All estimates remain valid, so we omit further details.

6. The reduction method

Our next step is reducing the upper bound of the solution of (5.15). We will use the reduction method similar to one described in [7, Lemma 4a)] (and originally introduced in [2]).

Lemma 6.1 ([7]). *Let θ, β, α, a be positive real numbers and let M be a positive integer. Let p/q be a convergent of the continued fraction expansion of θ such that $q > 6M$. Furthermore, let $\varepsilon = \|\beta q\| - M \cdot \|\theta q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then the inequality*

$$|m\theta - n + \beta| < \alpha a^{-m}, \tag{6.1}$$

has no integer solutions m and n such that

$$\log(\alpha q/\varepsilon)/\log a \le m \le M.$$

We apply Lemma 6.1 to (5.15) for each $k \in \mathbb{Z}[i]$, $5 < |k| < 350$ such that $\mathrm{Re}(k) > 0$ or $\mathrm{Re}(k) = 0$, $\mathrm{Im}(k) > 0$. The reduction gives us a new bound $M_0 = 33$, in all cases. Another application of the reduction in all cases gives us that $m \le 6$. By checking all the possibilities $0 < n \le m \le 6$, we conclude that the equation $v_m = w_n$ has only trivial solution $v_0 = w_0 = 1$ and the equation $v_m = w_n'$ has only solutions $v_2 = w_2' = 4k^2 - 2k - 1$, $v_0 = w_0' = 1$.

Therefore, we proved

Proposition 6.2. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$, $Re(k) > 0$ or $Re(k) = 0$ and $Im(k) > 0$. The only solutions of the system (2.2) and (2.3) are $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ and $(x, y, z) = (\pm(4k^2 - 2k - 1), \pm(4k^2 + 2k - 1), \pm(8k^2 - 1))$, with mixed signs.*

7. The case $1 < |k| \le 5$

This case is interesting, because there are some extra fundamental solutions of (2.2) and (2.3) for certain values of the parameter k. As before, we will assume that $\mathrm{Re}(k) > 0$ or $\mathrm{Re}(k) = 0$ and $\mathrm{Im}(k) > 0$. These fundamental solutions of (2.2) also appear (besides $(x_0, y_0) = (\pm 1, \pm 1)$):

- $(x_0, y_0) = (0, \pm(1+i))$ for $k = 1+i$,
- $(x_0, y_0) = (0, \pm(1-i))$ for $k = 1-i$,
- $(x_0, y_0) = (0, \pm i)$ for $k = 3$.

For (2.3), we obtain these fundamental solutions (besides $(x_1, z_1) = (\pm 1, \pm 1)$):

- $(x_1, z_1) = (0, \pm(1+2i))$ for $k = 1+i$,
- $(x_1, z_1) = (0, \pm(1-2i))$ for $k = 1-i$,
- $(x_1, z_1) = (\pm(1+i), \pm(2+3i))$ for $k = 3$,
- $(x_1, z_1) = (0, \pm 2i)$ for $k = 5$,
- $(x_1, z_1) = (\pm i, \pm 3i)$ for $k = 5$,
- $(x_1, z_1) = (\pm 2, \pm 4)$ for $k = 5$.

For each above k the fundamental solutions $(x_0, y_0) = (\pm 1, \pm 1)$ and $(x_1, z_1) = (\pm 1, \pm 1)$ lead to the equations $v_m = w_n$ and $v_m = w'_n$, where the sequences (v_m), (w_n) and (w'_n) are given by (2.14), (2.12) and (2.13), respectively. We apply methods given in Sections 5 and 6, and obtain that the only solution of $v_m = w_n$ is $m = n = 0$ and solutions of $v_m = w'_n$ are $m = n = 0$ and $m = n = 2$.

Now, each remaining case will be treated separately.

Case: $k = 1+i$

In Section 2, we showed that all solutions of (2.2) are given by recurrence sequences (2.10) and (2.11). Precisely, according to (2.10) the fundamental solution $(x_0, y_0) = (0, 1+i)$ generates this recurrence sequence

$$u_0 = 0, \qquad u_1 = -1+i, \qquad u_{m+2} = 2(1+i)u - m + 1 - u_m, \quad m \in \mathbb{N}_0,$$

and according to (2.11) we obtain the sequence $(-u_m)$. The fundamental solution $(x_0, y_0) = (1, 1)$ generates the sequence

$$v_0 = 1, \qquad v_1 = 1+2i, \qquad v_{m+2} = 2(1+i)v_{m+1} - v_m, \quad m \in \mathbb{N}_0.$$

So, if (x, y) is a solution of (2.2), then $x = \pm u_m$ and $x = \pm v_m$. Further, (2.12) and (2.13) imply that the fundamental solution $(x_1, z_1) = (0, 1+2i)$ leads to the sequence

$$q_0 = 0, \qquad q_1 = -2+i, \qquad q_{n+2} = 2(1+2i)q_{n+1} - q_n, \quad n \in \mathbb{N}_0,$$

and the fundamental solution $(x_1, z_1) = (1, 1)$ leads to

$$w_0 = 1, \qquad w_1 = 1+3i, \qquad w_{n+2} = 2(1+2i)w_{n+1} - w_n,$$

$$w'_0 = 1, \qquad w'_1 = 1+i, \qquad w'_{n+2} = 2(1+2i)w'_{n+1} - w'_n, \quad n \in \mathbb{N}_0.$$

Hence, if (x, z) is a solution of (2.3), then x is given by $(\pm q_n)$, $(\pm w_n)$ and $(\pm w'_n)$ and one of the following cases occur:

a) $v_m = \pm q_n$,
b) $u_m = \pm w_n$ or $u_m = \pm w'_n$,
c) $u_m = \pm q_n$.

a) Suppose that $v_m = \pm q_n$ for $m, n \in \mathbb{N}_0$. We apply the congruence method from Section 3 on sequences $(v_m \mod \delta)$ and $(q_n \mod \delta)$, where $\delta \in \{-1+2i, -4-4i\}$, and get that $v_{3m+1} = \pm q_n$, $m, n \in \mathbb{N}_0$, because

$$(v_m \bmod (-1+2i)) = (-1+i, 0, 2i, 2i, 0, -1+i, -1+i, 0, \ldots),$$
$$(q_n \bmod (-1+2i)) = (0, 0, 0, \ldots).$$

The following sequences

$$(v_{3m+1} \bmod (-4-4i)) = (-3-2i, -7, -7, -3-2i, -3-2i, -7, \ldots),$$
$$(q_n \bmod (-4-4i)) = (0, -2+i, -4-2i, -2-i, -4, -6+i, -4+2i, -6-i, 0, \ldots),$$

imply that $v_m \neq \pm q_n$, for all $m, n \in \mathbb{N}_0$.

b) Similarly, as in the case a), by applying the congruence method we obtain that there is no solution in this case.

c) By applying the congruence method as in the case a), we obtain that

$$u_{6m} = \pm q_{4n}, \quad m, n \in \mathbb{N}_0.$$

By repeating the procedures described in Sections 5 and 6, we conclude that the above equation has only the trivial solution $u_0 = q_0 = 0$. The meaning of this unexpected solution is that the Diophantine triple $\{i, 1+i, 2+i\}$ is extended by the element $d = i$, but such extension is not considered as a proper extension since i is already an element of the starting triple.

Case: $k = 1 - i$

By conjugating, this case becomes the same as the previous one.

Case: $k = 3$

The fundamental solutions of (2.2) are $(x_0, y_0) = (0, i)$ and $(x_0, y_0) = (1, 1)$. They generate two recurrence sequences

$$u_0 = 0, \qquad u_1 = 2i, \qquad u_{m+2} = 6u_{m+1} - u_m,$$
$$v_0 = 1, \qquad v_1 = 5, \qquad v_{m+2} = 6v_{m+1} - v_m.$$

The fundamental solutions of (2.3), $(x_1, z_1) = (1, 1)$ and $(x_1, z_1) = (1+i, 2+3i)$, generate following sequences

$$w_0 = 1, \qquad w_1 = 4, \qquad u_{n+2} = 10w_{n+1} - w_n,$$
$$w'_0 = 1, \qquad w_1 = 3, \qquad u_{n+2} = 10w_{n+1} - w_n,$$
$$q_0 = 1+i, \qquad q_1 = 9+11i, \qquad q_{n+2} = 6q_{n+1} - q_n,$$
$$q'_0 = 1+i, \qquad q'_1 = 1-i, \qquad q'_{n+2} = 6q_{n+1} - q_n.$$

Note that $q'_n = \overline{q_{n-1}}$. So, following cases have to be analyzed:

a) $v_m = \pm q_n$ or $v_m = \pm \overline{q_n}$,
b) $u_m = \pm w_n$ or $u_m = \pm w'_n$,
c) $u_m = \pm q_n$ or $u_m = \pm \overline{q_n}$.

By congruence method we obtain that the cases a), b) and c) have no solution.

CASE: $k = 5$

As in the previous case, the solutions can be obtained only from $v_m = w_n$ or $v_m = w'_n$.

Therefore, we proved

PROPOSITION 7.1. *Let $k \in \mathbb{Z}[i]$ and $1 < |k| \le 5$, $Re(k) > 0$ or $Re(k) = 0$ and $Im(k) > 0$. Solutions of the system (2.2) and (2.3) are $(x, y, z) = (\pm 1, \pm 1, \pm 1)$, $(x, y, z) = (\pm(4k^2 - 2k - 1), \pm(4k^2 + 2k - 1), \pm(8k^2 - 1))$, with mixed signs. If $k \ne 1 \pm i$, then these solutions are the only solutions of the system (2.2) and (2.3). If $k = 1 + i$ or if $k = 1 - i$, then there exist the extra solutions $(x, y, z) = (0, \pm(1 + i), \pm(1 + 2i))$ and $(x, y, z) = (0, \pm(1 - i), \pm(1 - 2i))$, respectively.*

8. THE CASE $k = i$

The main difference between this case and the previous ones is that Lemma 2.2 cannot be applied. Hence, we rewrite the original problem (2.1) as

$$y^2 + ix^2 = i + 1, \tag{8.1}$$

$$z^2 - (2 - 2i)x^2 = -1 + 2i. \tag{8.2}$$

The advantage of the above equations is that the solutions can be given immediately by using [11]. So, all solutions of (8.1) are given by

$$y_m^{(j)} + x_m^{(j)} \sqrt{-i} = \rho_j (i + (1 - i)\sqrt{-i})^m,\ m \in \mathbb{N}_0,\ j = 1, 2, 3, 4,$$

where $\rho_1 = 1 + \sqrt{-i}$, $\rho_2 = -1 + \sqrt{-i}$, $\rho_3 = -\rho_1$, $\rho_4 = -\rho_2$, and all solutions of (8.2) are

$$z_n^{(k)} + \tilde{x}_n^{(k)} \sqrt{2 - 2i} = \sigma_j (-1 + 2i + (1 - i)\sqrt{2 - 2i})^n,\ n \in \mathbb{N}_0,\ k = 1, 2, 3, 4,$$

where $\sigma_1 = 1 + \sqrt{2 - 2i}$, $\sigma_2 = 1 - \sqrt{2 - 2i}$, $\sigma_3 = -\sigma_1$, $\sigma_4 = -\sigma_2$.

Hence, our problem of solving the system of equation is reduced to

$$x_m^{(j)} = \tilde{x}_n^{(k)}, \quad m, n \in \mathbb{N}_0,$$

where $j, k \in \{1, 2, 3, 4\}$. Note that some of these equations have solution for $m = n = 0$, which implies that our system of Pellian equations has a trivial solution $(x, y, z) = (\pm 1, \pm 1, \pm 1)$. These solutions correspond to $d = 0$.

Further, it can be shown that $x_{m+1}^{(1)} = x_m^{(2)} = -x_{m+1}^{(3)} = -x_m^{(4)} = x_m$ and that $\tilde{x}_n^{(3)} = -\tilde{x}_n^{(1)}$ and $\tilde{x}_n^{(4)} = -\tilde{x}_n^{(2)}$. So, it remains to study the four cases

$$x_m = \pm \tilde{x}_n^{(j)},\ m, n \in \mathbb{N},\ j \in \{1, 2\}. \tag{8.3}$$

Solutions x_m and $\tilde{x}_n^{(k)}$ $(k=1,2)$ satisfy following recursions

$$x_{m+1}^{(j)} = 2ix_m^{(j)} - x_{m-1}^{(j)},\ m \geq 1,$$

$$\tilde{x}_{n+1}^{(k)} = 2(-1+2i)\tilde{x}_n^{(k)} - \tilde{x}_{n-1}^{(k)},\ n \geq 1.$$

By solving these recursions we obtain the formulas

$$x_m = (\frac{1}{2} + \frac{1+i}{2\sqrt{2}})(i(1+\sqrt{2}))^m + (\frac{1}{2} - \frac{1+i}{2\sqrt{2}})(i(1-\sqrt{2}))^m,$$

$$\tilde{x}_n^{(1)} = \frac{2-\sqrt{1+i}}{4}(-1+2i+2i\sqrt{1+i})^n + \frac{2+\sqrt{1+i}}{4}(-1+2i-2i\sqrt{1+i})^n,$$

$$\tilde{x}_n^{(2)} = -\frac{2+\sqrt{1+i}}{4}(-1+2i+2i\sqrt{1+i})^n - \frac{2-\sqrt{1+i}}{4}(-1+2i-2i\sqrt{1+i})^n.$$

Each equation in (8.3) should be treated separately. By applying the methods given in Section 6 and 7, we obtain that the only solution is $x = x_2 = \tilde{x}_2^{(1)} = -5-2i$. This implies that the system (8.1) and (8.2) has solutions $(x,y,z) = (\pm(-5-2i), \pm(-5+2i), \pm 9)$ (which corresponds to $d = -20i$, i.e. to $16k^3 - 4k$ for $k = i$). Therefore, we proved

Proposition 8.1. *Let $k = i$. The all solutions of the system (2.1) are $(x,y,z,d) = (\pm 1, \pm 1, \pm 1, 0)$ and $(x,y,z,d) = (\pm(-5-2i), \pm(-5+2i), \pm 9, -20i)$, with mixed signs.*

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