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THE ZERO-DIVISOR GRAPH WITH RESPECT TO IDEALS OF A COMMUTATIVE SEMIRING

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Abstract. In a manner analogous to a commutative ring, the ideal-based zero-divisor graph of a commutative semiring R can be defined as the undirected graph $\Gamma_I(R)$ for some ideal I of R. The properties and possible structures of the graph $\Gamma_I(R)$ are studied.

1. Introduction

Throughout all semirings are assumed to be commutative semirings with non-zero identity. The zero-divisor graph of a semiring is the (simple) graph whose vertex set is the set of non-zero zero-divisors, and an edge is drawn between two distinct vertices if their product is zero. This definition is the same as that introduced by D. F. Anderson and P. S. Livingston in [1]. In [5], Beck introduced the concept of a zero-divisor graph of a commutative ring. However, he let all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. In recent years, the study of zero-divisor graphs has grown in various directions. At the heart is the interplay between the ring-theoretic properties of a ring and the graph-theoretic properties of its zero-divisor graph, begun in [1] and continued in [2, 13]. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g. [1, 2, 11, 13].

Let R be a commutative semiring with non-zero identity. We can define the zero-divisor graph $\Gamma(R)$ as above. We know (at least as far as I am aware) of no systematic study of zero-divisors in the semiring context. The bulk of this paper is devoted to stating and proving analogues to several well-known results of the ideal-based zero-divisor graph in the theory of rings. In fact,

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the main object of this paper is to study the interplay of semiring-theoretic properties of R with graph-theoretic properties of $\Gamma_I(R)$ for some ideal I of R.

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring, we mean a commutative semigroup $(R, \cdot)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$, both are connected by ring-like distributivity. A commutative semiring R is said to be a semidomain if $ab = 0$ $(a, b \in R)$, then either $a = 0$ or $b = 0$. A semifield is a commutative semiring in which the non-zero elements form a group under multiplication. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$ (so $\{0\}$ is a k-ideal of R). The k-closure $\mathrm{cl}(K)$ of K is defined by $\mathrm{cl}(K) = \{a \in R : a + c = d$ for some $c, d \in K\}$ is an ideal of R satisfying $K \subseteq \mathrm{cl}(K)$ and $\mathrm{cl}(\mathrm{cl}(K)) = \mathrm{cl}(K)$. So an ideal K of R is a k-ideal if and only if $K = \mathrm{cl}(K)$. A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. If I is an ideal of R, then the radical of I, denoted by $\mathrm{rad}(I)$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, contains I, and is the intersection of all the prime ideals of R that contain I [3]. A primary ideal I of R is a proper ideal of R such that, if $xy \in I$ and $x \notin I$, then $y^n \in I$ for some positive integer n. If I is primary, then $\mathrm{rad}(I) = P$ is a prime ideal of R by [4, Theorem 38]. In this case, we also say that I is a P-primary ideal of R. A proper ideal I of R is said to be maximal (resp. k-maximal) if J is an ideal (resp. k-ideal) in R such that $I \subsetneqq J$, then $J = R$. A non-zero element a of R is said to be semi-unit in R if there exist $r, s \in R$ such that $1 + ra = sa$. R is said to be a local semiring if and only if R has a unique maximal k-ideal.

Let R be a commutative semiring with non-zero identity. We use the notation A^* to refer to the non-zero elements of A. For two distinct vertices a and b in a graph Γ, the distance between a and b, denoted $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a, b) = \infty$. The diameter of a graph Γ is

$$\mathrm{diam}(\Gamma) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } \Gamma\}.$$

We will use use the notation $\mathrm{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z^*(R)$. A graph is said to be connected if there exists a path between any two distinct vertices, and a graph is complete if it is connected with diameter at most one.

2. Definition and basic properties of $\Gamma_I(R)$

Let I be an ideal of a commutative semiring R with non-zero identity. We define an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This definition is the same as that introduced by S. P. Redmond in [13].

Compare the next result with [13, Proposition 2.2 and Theorem 2.4].

Lemma 2.1. *Let I be an ideal of a semiring R. Then:*

(i) If $I = (0)$, then $\Gamma_I(R) = \Gamma(R)$.

(ii) If $I \neq (0)$, then I is a prime ideal of R if and only if $\Gamma_I(R) = \emptyset$.

(iii) $\Gamma_I(R)$ is connected with $\mathrm{diam}(\Gamma_I(R)) \leq 3$.

Proof. The proof is straightforward. □

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that $R = \cup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is the zero element of R/I. Clearly, if R is commutative, then so is R/I (see [9, 10]).

Theorem 2.2. *Let I be a proper Q-ideal of a semiring R. Then $\Gamma_I(R) = \emptyset$ if and only if $\Gamma(R/I) = \emptyset$.*

Proof. By Lemma 2.1 (ii) and [7, Theorem 2.6], $\Gamma_I(R) = \emptyset$ if and only if R/I is a semidomain, and so the proof is complete. □

Lemma 2.3. *Let I be a proper Q-ideal of a semiring R. Then:*

(i) If $q_0 \in Q$ and $q_0 + I$ is the zero in R/I, then $q_0 \in I$.

(ii) If $q \in I \cap Q$ and $q_0 + I$ is the zero in R/I, then $q = q_0$.

Proof. (i) By [4, Lemma 36], we must have $q_0 + I = I$; hence $q_0 \in I$ since every Q-ideal is a k-ideal of R by [10, Lemma 2].

(ii) Since $q + q_0 \in (q_0 + I) \cap (q + I)$, we must have $q_0 + I = q + I$, as required. □

The next theorem investigates the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$ (compare the next result with [13, Proposition 2.5]).

Theorem 2.4. *Let I be a proper Q-ideal of a semiring R and let $x = q_1 + a$, $y = q_2 + b \in R - I$, where $q_1, q_2 \in Q$ and $a, b \in I$. Then:*

(i) If $q_1 + I$ is adjacent to $q_2 + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.

(ii) If x is adjacent to y in $\Gamma_I(R)$ and $q_1 \neq q_2$, then $q_1 + I$ is adjacent to $q_2 + I$ in $\Gamma(R/I)$.

(iii) If x is adjacent to y in $\Gamma_I(R)$ and $q_1 = q_2$, then $x^2, y^2 \in I$.

PROOF. (i) First note that $q_1, q_2 \notin I$. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I and let $(q_1 + I) \odot (q_2 + I) = q_0 + I$, where $q_1q_2 + I \subseteq q_0 + I$. So it follows from Lemma 2.3 that $q_1q_2 + c = q_0 + d \in I$ for some $c, d \in I$; hence $q_1q_2 \in I$ since I is a k-ideal. Therefore, $xy = q_1q_2 + aq_2 + bq_1 + ab \in I$; so x is adjacent to y in $\Gamma_I(R)$.

(ii) By assumption, $q_1q_2 \in I$. Let q_3 be the unique element of Q such that $(q_1 + I) \odot (q_2 + I) = q_3 + I$ and $q_1q_2 + I \subseteq q_3 + I$. It follows from Lemma 2.3 (i) that $q_1q_2 + q_0 = q_3 + e$ for some $e \in I$; hence $q_3 \in I$. Now the assertion follows from Lemma 2.3 (ii).

(iii) By hypothesis, $q_1^2 = q_2^2 \in I$; hence $x^2, y^2 \in I$. □

Compare the next result with [13, Corollary 2.6].

COROLLARY 2.5. *Let I be a proper Q-ideal of a semiring R and let $x = q_1 + a$, $y = q_2 + b \in R - I$, where $q_1, q_2 \in Q$ and $a, b \in I$. If x is adjacent to y in $\Gamma_I(R)$ and $q_1 \neq q_2$, then all distinct elements of $q_1 + I$ and $q_2 + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $q_1 + I$ are adjacent in $\Gamma_I(R)$.*

PROOF. This follows from Theorem 2.4. □

Let G be a graph. We say that $\{G_i\}_{i \in J}$ is a collection of disjoint subgraphs of G if all the vertices and edges of each G_i are contained in G and no two of these G_i contain a common vertex (compare the next result with [13, Corollary 2.7]).

THEOREM 2.6. *Let I be a proper Q-ideal of a semiring R. Then $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$.*

PROOF. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I and let $Z(R/I) - \{q_0 + I\} = \bigcup_{q_i \in Q}\{q_i + I\}$, and if $i \neq j$, then $(q_i + I) \cap (q_j + I) = \emptyset$. For each $k \in I$, define a graph G_k with vertices $\{q_i + k : q_i \in Q\}$, where $q_i + k$ is adjacent to $q_j + k$ in G_k whenever $q_i + I$ is adjacent to $q_j + I$ in $\Gamma(R/I)$. Then Corollary 2.5 gives G_k is a subgraph of $\Gamma_I(R)$. Also, each $G_k \cong \Gamma(R/I)$, and G_k and G_j contain no common vertices if $k \neq j$. □

A vertex x of a connected graph G is a cut-point of G if there are vertices y, z of G such that x is in every path from y to z (and $x \neq y$, $x \neq z$). Equivalently, for a connected graph G, x is a cut-point of G if $G - \{x\}$ is not connected (compare the next result with [13, Theorem 3.2]).

THEOREM 2.7. *Let I be a non-zero proper Q-ideal of a semiring R. Then $\Gamma_I(R)$ has no cut-points.*

PROOF. Suppose that x is a cut-point of $\Gamma_I(R)$. Then there are $y, z \in R - I$ such that x lies on every path from y to z. By Lemma 2.1, the shortest path from y to z is of length 2 or 3. Let $x = q_1 + a$, $y = q_2 + b$ and $z = q_3 + c$ where $q_1, q_2, q_3 \in Q$ and $a, b, c \in I$. We divide the proof into two cases.

CASE 1. Suppose $y - x - z$ is a path of shortest length from y to z. If $q_1 = q_2$, then Corollary 2.5 gives x is adjacent z implies y is adjacent to z. Similarly, if $q_1 = q_3$, then y is adjacent to z. So suppose that $q_1 \neq q_2$ and $q_1 \neq q_3$. Suppose that $0 \neq e \in I$; we show that $x + e \neq y$. Let us assume the opposite. Since $x + e = q_1 + a + e = q_2 + b \in (q_1 + I) \cap (q_2 + I)$, we must have $q_1 = q_2$, which is a contradiction. Thus $x + e \neq y$. Similarly, $x + e \neq z$. Then $xy, xz \in I$ imply $y(x + e), z(x + e) \in I$. Hence $y - (x + e) - z$ is a path in $\Gamma_I(R)$. Therefore, in all cases we get a contradiction.

CASE 2. Suppose (without loss of generality) $y - x - w - z$ is a path of shortest length from y to z and let $w = q_4 + f$, where $q_4 \in Q$ and $f \in I$. If $q_1 = q_4$, then y is adjacent to x implies y is adjacent to w, therefore $y - w - z$ is a path. If $q_1 \neq q_4$, then let $0 \neq e \in I$. As above, y and w are adjacent to x means that y and w are also adjacent to $x + e$. Hence $y - (x + e) - w - z$ is a path. Thus in all cases we get a contradiction. □

The connectivity of a graph G, denoted by $\mathrm{k}(G)$, is defined to be the minimum number of vertices it is necessary to remove from G in order to produce a disconnected graph. Let I be a Q-ideal of a semiring R. We call the subset $q_i + I$ (for some $q_i \in Q$) a column of $\Gamma_I(R)$. If $q_i^2 \in I$, then we call $q_i + I$ a connected column of $\Gamma_I(R)$ (compare the next result with [13, Theorem 3.3]).

THEOREM 2.8. *Let I be a proper Q-ideal of a semiring R. Then:*

(i) If $\Gamma(R/I)$ is the graph on one vertex, then $\mathrm{k}(\Gamma_I(R)) = |I| - 1$.

(ii) If $\Gamma(R/I)$ has at most two vertices, then $2 \leq \mathrm{k}(\Gamma_I(R)) \leq |I| \cdot \mathrm{k}(\Gamma(R/I))$.

PROOF. (i) By assumption, $\Gamma_I(R)$ consists of a single connected column; so it is a complete graph on $|I|$ vertices by Theorem 2.6.

(ii) Since the graph $\Gamma_I(R)$ is connected, we must have $1 \leq \mathrm{k}(\Gamma_I(R))$. Assume that $\mathrm{k}(\Gamma(R/I)) = m$ and let $q_1 + I, q_2 + I, ..., q_m + I$ (for some $q_i \in Q$) be vertices of $\Gamma(R/I)$ which, once removed, give a disconnected graph. Define G to be the graph obtained from $\Gamma_I(R)$ by removing the columns corresponding to $q_1 + I, ..., q_m + I$ (this means the removal of $m \cdot |I|$ vertices); we show that G is disconnected. By hypothesis, there exist vertices $q + I$ and $q' + I$ $(q, q' \in Q)$ of $\Gamma(R/I)$ such that $q + I$ is not connected to $q' + I$ after $q_1 + I, ..., q_m + I$ are removed from $\Gamma(R/I)$. Then q and q' are vertices of G. Suppose $q - a_1 - ... - a_n - q'$ is a path in G. There are elements $s_1, ..., s_n \in Q$ and $c_1, ..., c_n \in I$ such that $a_i = s_i + c_i$ for each $1 \leq i \leq n$. It is easy to see that for $i \neq j$, $a_i a_j \in I$ if and only if $s_i s_j \in I$ since I is a k-ideal. Also, $qa_i \in I$ if and only if $qs_i \in I$. Similarly, for q'. Then $q - s_1 - ... - s_n - q'$ is a path in G. By Corollary 2.5, we may assume that $s_j + I \neq s_{j+1} + I$ for $1 \leq j \leq n$. Therefore, $q + I - s_1 + I - ... - s_n + I - q' + I$ is a path in $\Gamma(R/I)$ after $q_1 + I, ..., q_m + I$ have been removed. This is a contradiction. Hence G must be disconnected. □

3. Girth

Let R be a semiring. The grith of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle, otherwise, $\mathrm{gr}(\Gamma) = \infty$. We will use the notation $\mathrm{gr}(\Gamma(R))$ to denote the grith of the graph of $Z^*(R)$ (compare the Lemma 3.1, Proposition 3.2 and Theorem 3.3 with [13, Lemma 5.1, Lemma 5.2 and Theorem 5.5]).

Lemma 3.1. *Let I be a proper Q-ideal of a semiring R. Then $\mathrm{gr}(\Gamma_I(R)) \leq \mathrm{gr}(\Gamma(R/I))$. In particular, if $\Gamma(R/I)$ contains a cycle, then so does $\Gamma_I(R)$.*

Proof. We may assume that $\mathrm{gr}(\Gamma(R/I)) = n < \infty$. Let

$$q_1 + I - q_2 + I - \cdots - q_n + I - q_1 + I$$

be a cycle in $\Gamma(R/I)$ through n distinct vertices, where $q_i \in Q$ for $1 \leq i \leq n$. Then Theorem 2.4 gives $q_1 - q_2 - ... - q_n - q_1$ is a cycle in $\Gamma_I(R)$ of length n, as needed. □

Proposition 3.2. *Let I be a proper Q-ideal of a semiring R. Then:*

(i) If $|I| \geq 3$ and $\Gamma_I(R)$ contains a connected column, then $\mathrm{gr}(\Gamma_I(R)) = 3$.
(ii) Assume that $I \neq (0)$ and $\Gamma(R/I)$ has only one vertex. Then $\mathrm{gr}(\Gamma_I(R)) = 3$ if $|I| \geq 3$, and $\mathrm{gr}(\Gamma_I(R)) = \infty$ if $|I| = 2$.
(iii) If I has two elements, $\Gamma(R/I)$ has at least two vertices, and $\Gamma_I(R)$ has at least one connected column, then $\mathrm{gr}(\Gamma_I(R)) = 3$.

Proof. (i) Let $q_1 + I$ be a connected column of $\Gamma_I(R)$, where $q \in Q$. Then $q^2 \in I$. Let $x = q_1 + a$ and $y = q_2 + b$ be non-zero elements of I, where $q_1, q_2 \in Q$ and $a, b \in I$; so $q_1, q_2 \in I$ since I is a k-ideal. Then $q - (q + q_1) - (q + q_2) - q$ is a cycle of length 3 in $\Gamma_I(R)$ (note that $\mathrm{gr}(G) \geq 3$ for any graph G).

(ii) If $\Gamma(R/I)$ has only one vertex, then (i) gives that $\Gamma_I(R)$ consists of a single connected column. Thus it is a complete graph, and therefore has a cycle of length 3 unless $\Gamma_I(R)$ has only two vertices.

(iii) Let $q + I$ be a connected column of $\Gamma_I(R)$, where $q \in Q$. Then $q^2 \in I$. Let $q' + I$ be a vertex adjacent to $q + I$. Suppose that $0 \neq x \in I$. Then $x = q_1 + b$ for some $q_1 \in I \cap Q$ and $b \in I$ since I is a k-ideal. Then $q' - q - (q + q_1) - q'$ is a cycle of length 3 in $\Gamma_I(R)$. □

Theorem 3.3. *Let I be a non-zero proper Q-ideal of a semiring R that is not a prime ideal. Then* $\mathrm{gr}(\Gamma_I(R)) = \infty$ *if* $\Gamma(R/I)$ *has only one vertex and* $|I| = 2$, $\mathrm{gr}(\Gamma_I(R)) = 4$ *if* $\mathrm{gr}(\Gamma(R/I)) > 3$ *and* $\Gamma_I(R)$ *has no connected columns, and* $\mathrm{gr}(\Gamma_I(R)) = 3$ *otherwise.*

Proof. By Proposition 3.2, it is enough to show that if $I \neq (0)$, $\Gamma_I(R)$ has no connected columns, and $\mathrm{gr}(\Gamma(R/I)) > 3$, then $\mathrm{gr}(\Gamma_I(R)) = 4$. By assumption, $\Gamma(R/I)$ must have at least two vertices. It follows from Lemma 3.1 that $\mathrm{gr}(\Gamma_I(R)) \leq 4$. Let $x = q_1 + a$, $y = q_2 + b$, and $z = q_3 + c$ (where $q_1, q_2, q_3 \in Q$, $a, b, c \in I$) be such that $x - y - z - x$ is a cycle in $\Gamma_I(R)$ of length 3 and we provide a contradiction. As $\mathrm{gr}(\Gamma(R/I)) > 3$, $q_1 + I - q_2 + I - q_3 + I - q_1 + I$ cannot be a cycle in $\Gamma(R/I)$. Therefore, we must have either $q_1 = q_2 = q_3$ or $q_1 = q_3$. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. If $q_1 = q_2$, then $(q_1 + I) \odot (q_2 + I) = q_0 + I$, where $q_1^2 + I \subseteq q_0 + I$. So $q_1^2 \in I$; hence $q_1 + I$ is a connected column of $\Gamma_I(R)$, which is a contradiction. We get a similar contradiction if $q_2 = q_3$ or $q_1 = q_3$. Hence, $\mathrm{gr}(\Gamma_I(R)) = 4$. ∎

Let R be a semiring. R is called cancellative if whenever $ac = bc$ for some elements a, b and c of R with $a \neq 0$, then $b = c$. Also, we define the Jacobson radical of R, denoted by $\mathrm{Jac}(R)$, to be the intersection of all the maximal k-ideals of R. Then by [14, Corollary 2.2], the Jacobson radical of R always exists and by [7, Lemma 2.12], it is a k-ideal.

Lemma 3.4. *Let R be a semiring and let $r \in R$. Then:*

(i) If r is a nilpotent element of R, then it is not a semi-unit.

(ii) If $r \in \mathrm{Jac}(R)$, then for every $a \in R$, the element $1 + ra$ is a semi-unit of R.

Proof. (i) We may assume that $r \neq 0$. Suppose not. Then $1 + rs = tr$ for some $s, t \in R$. Let $n \geq 2$ be an integer such that $r^n = 0$, but $r^{n-1} \neq 0$. Then $(1 + rs)^n = 1 + ur = 0$ for some $u \in R$; hence $r^{n-1} + ur^n = r^{n-1} = 0$, which is a contradiction. Thus r is not a semi-unit.

(ii) Suppose that $r \in \mathrm{Jac}(R)$. Suppose that, for some $a \in R$, it is the case that $b = 1 + ra$ is not a semi-unit of R. Then $1 + t_1 b = t_2 b$ holds for no $t_1, t_2 \in R$. So, $1 \notin \mathrm{cl}(Rb)$ yields that $\mathrm{cl}(Rb)$ is a proper k-ideal of R. By [13, Corollary 2.2], there exists a maximal k-ideal P of R such that $1 + ra \in \mathrm{cl}(Rb) \subseteq P$. But $r \in P$ by definition of $\mathrm{Jac}(R)$, and so $1 \in P$, a contradiction. ∎

Lemma 3.5. *Let R be a semiring. Then:*

(i) Let I be an ideal in R that is maximal among all annihilators of non-zero elements of R. Then I is prime.

(ii) Let $P_1, \ldots, P_n$ be prime k-ideals and let J be an ideal of R contained in $\bigcup_{i=1}^{n} P_i$. Then $J \subseteq P_i$ for some i.

PROOF. (i) Let $I = (0 : x)$ for some $x \in R$. Given $ab \in I$, we must prove that $a \in I$ or $b \in I$. Assume $a \notin I$. Then $ax \neq 0$ and $I \subseteq (0 : ax)$. By hypothesis, it can not be properly larger. Thus $I = (0 : ax)$; hence $b \in I$. The proof of the part (ii) is straightforward and can be found in [8]. $\square$

Let I be an ideal of a semiring R. A prime ideal P of R is called an associated prime ideal of I if P is the annihilator $(0 : x)$ of some $x \in I$ (so P is a k-ideal by [6, Lemma 2.1]). If R is a semiring, then R is Noetherian (resp. Artinian) if any non-empty set of k-ideals of R has a maximal member (resp. minimal member) with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on k-ideals of R.

LEMMA 3.6. *Let R be a Noetherian semiring (in particular, R could be a finite semiring) and I a non-zero ideal of R. Then:*

(i) Every maximal element of the family of k-ideals $A = \{(0 : x) : 0 \neq x \in I\}$ is an associated prime of I.
(ii) The set of zero-divisors for I is the union of all the associated primes of I.

PROOF. (i) follows from Lemma 3.5 (i). To see (ii), assume that $ax = 0$ for some $x \neq 0$. Then $a \in (0 : x) \in A$, and by (i) there is an associated prime of I containing $(0 : x)$, as needed. $\square$

PROPOSITION 3.7. *Let R be an Artinian cancellative semiring. Then:*

(i) Every element of R is either a semi-unit or a nilpotent element.
(ii) R is a local semiring.

PROOF. (i) Let x be a non-zero element of R which is not nilpotent; we show that x is a semi-unit. Consider the sequence of k-ideals $\mathrm{cl}(Rx) \supseteq \mathrm{cl}(Rx^2) \supseteq \ldots.$ By the descending chain condition, there must be elements $r, s \in R$ such that $x^n(1 + rx) = x^n(sx)$ for some integer n. Since R is a cancellative semiring and $x^n \neq 0$, we may cancel x^n, and hence $1 + rx = sx$, as required.

(ii) It is enough to show that the set of non-semi-units P of R is a k-maximal ideal. By (i), P is the set of nilpotent elements of R: so it is an ideal of R. It remains to show that P is a unique maximal k-ideal. Suppose that $a, a + b \in P$; we show that $b \in P$. Let m be smallest integer with $a^m = 0$ and let n be such that $(a + b)^n = 0$. We may assume that $m \leq n$. $(a + b)^n = 0$ gives $a^{m-1}b^n = 0$. Suppose that b is not nilpotent. Then by (i), b is a semi-unit; hence $1 + rb = sb$. It follows from the equality $a^{m-1}b^n = 0$ that $a^{m-1}b^{n-1} = 0$. Similary, we get $a^{m-1}b = 0$; hence $a^{m-1} = 0$, which is a contradiction. Thus b is nilpotent; hence $b \in P$. Since 1 is a semi-unit of R, we must have $P \neq R$. As R is not trivial, it has at least one maximal k-ideal, let J be one such ideal. Therefore, $J \subseteq P \subsetneqq R$. Thus $J = P$ since J

is k-maximal. We have thus shown that R has at least one maximal k-ideal, and for any maximal k-ideal of R must be equal to P. □

Proposition 3.8. *Let R be an Artinian cancellative semiring with a unique maximal k-ideal P. Then $P = (0 : x)$ for some $0 \neq x \in P$.*

Proof. By Proposition 3.7, we must have $P \subseteq Z(P)$. By Lemma 3.6, $Z(P)$ is the set-theoretic union of all the associated primes $P_1, ..., P_{n-1}$ and P_n of P (note that they are k-ideals). Now Lemma 3.5 gives $P \subseteq P_i$ for some i; hence $P = P_i$, as needed. □

Theorem 3.9. *Let R be an Artinian cancellative semiring (in particular, R could be a finite cancellative semiring) with a unique maximal k-ideal P. If $\Gamma(R)$ contains a cycle, then $\mathrm{gr}(\Gamma(R)) = 3$.*

Proof. Suppose $\Gamma(R)$ contains a cycle. Then $P = (0; x)$ for some $x \in P$ by Proposition 3.8. If there are $y, z \in P^* - \{x\}$ with $yz = 0$, then $y - x - z - y$ is a is a triangle. Otherwise, $\Gamma(R)$ contains no cycle, a contradiction. Therefore, $\mathrm{gr}(\Gamma(R)) = 3$. □

4. Primary ideals

In this section, we will investigate the ideal-based zero-divisor graph with respect to primary ideals of a semiring.

Proposition 4.1. *Let I be an ideal of a semiring R. Then I is a P-primary ideal of R if and only if $\Gamma_I(R) = P - I$.*

Proof. Suppose that I is a P-primary ideal of R; we show that $\Gamma_I(R) = P - I$. Let $r \in \Gamma_I(R)$. Then $r \notin I$ and $ra \in I$ for some $a \notin I$. Thus I primary gives $r \in P - I$; hence $\Gamma_I(R) \subseteq P - I$. For the reverse inclusion, assume that $b \in P - I$. Since $b \notin I$, there must be an integer $n \geq 2$ such that $b^{n-1} = s \notin I$ and $bs \in I$. Then $b \in \Gamma_I(R)$, and so we have equality.

Conversely, assume that $\Gamma_I(R) = P - I$ and let $cx \in I$ with $x \notin I$; we show that $c \in P$. Let us suppose the opposite. Then $c \notin I$; hence $c \in \Gamma_I(R) = P - I$, which is a contradiction. Thus I is a P-primary ideal of R. □

Theorem 4.2. *Let I be an ideal of a semiring R. Then I is a primary ideal of R if and only if $\Gamma_I(R) \cup I$ is an (prime) ideal of R.*

Proof. This follows from Proposition 4.1. □

Theorem 4.3. *Let I and J be P-primary ideals of a semiring R. Then $\Gamma_I(R) = \Gamma_J(R)$ if and only if $I = J$.*

Proof. Since $I \subseteq P$ and $J \subseteq P$, the result follows from Proposition 4.1. □

Lemma 4.4. *If I is a P-primary k-ideal of a semiring R, then P is a k-ideal of R.*

PROOF. Let $a, a+b \in P$; we show that $b \in P$. There is an integer n such that $a^n \in I$ and $(a+b)^n = a^n + c + b^n \in I$ for some $c \in P$, so $c + b^n \in I$ since I is a k-ideal. If $c \in I$, then the result is clear. So suppose that $c \notin I$. Let $m \geq 2$ be a positive integer such that $c, c^2, ..., c^{m-1} \notin I$, but $c^m \in I$. By assumption, $c^m + c^{m-1}b^n \in I$. Thus it follows that $c^{m-1}b^n \in I$ with $c^{m-1} \notin I$; hence $b^{nk} \in I$ for some k. Thus $b \in P$, as required. □

THEOREM 4.5. *Assume that I is a P-primary Q-ideal of a semiring R and let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. Then $\Gamma(R/I) \cup \{q_0 + I\}$ is a prime k-ideal of R/I.*

PROOF. Suppose that I is a P-primary ideal of R. It follows from Lemma 4.4 and [7, Proposition 2.2 and Theorem 2.5] that P/I is a prime k-ideal of R/I. It is enough to show that $\Gamma(R/I) \cup \{q_0 + I\} = P/I$. Let $q + I \in \Gamma(R/I) \cup \{q_0 + I\}$, where $q \in Q$. If $q + I = q_0 + I$, then we are done. So we may assume that $q + I \neq q_0 + I$. Then there is an element $q_0 + I \neq q_1 + I \in R/I$ such that $(q + I) \odot (q_1 + I) = q_0 + I$, where $q_1 \in Q$ and $q_1 q + I \subseteq q_0 + I = I$, so $q_1 q \in I$ with $q_1 \notin I$ by Lemma 2.3 and the fact that I is a k-ideal. Then $q \in P \cap Q$ since I is a P-primary ideal; hence $q + I \in P/I$ by [7, Proposition 2.2]. Therefore, $\Gamma(R/I) \cup \{q_0 + I\} \subseteq P/I$. For the other containment, suppose that $q + I \in P/I$, where $q \in Q \cap P$. We may assume that $q_0 + I \neq q + I$. Then $q \notin I$ and there is a positive integer $n \geq 2$ such that $s = q^{n-1} \notin I$ and $qs \in I$ since I is primary. There are $q_1 \in Q$ and $a \in I$ such that $s = q_1 + a$; so $qs = qq_1 + aq$. Hence $qq_1 \in I$ since I is a k-ideal. There is a unique element q_2 of Q with $(q + I) \odot (q_1 + I) = q_2 + I$ and $qq_1 + I \subseteq q_2 + I$. Then $qq_1 + c = q_2 + d$ for some $c, d \in I$; so $q_2 \in I$. Hence $q_2 = q_0$ by Lemma 2.3. Thus $q + I \in \Gamma(R/I)$, and so we have equality. □

REMARK 4.6. Let R be a Noetherian semiring. Then:

(1) Every proper k-ideal of R is a finite intersection of irreducible k-ideals (an ideal of R is irreducible if it is not a finite intersection of k-ideals of R that properly contain it) (see [12, Lemma 2] and [8]).
(2) Every irreducible k-ideal of R is primary (see [12, Lemma 3] and [8]).
(3) Let P be a prime ideal of R, and let $Q_1, ..., Q_n$ be P-primary k-ideals of R. Then $\bigcap_{i=1}^n Q_i$ is also a P-primary k-ideal (see [4, Theorem 41] and [7, Lemma 2.12]).
(4) By (1) and (2), every proper k-ideal of R is a finite intersection of primary k-ideals. Therefore, by (3), every proper k-ideal (so 0) of R has a minimal primary decomposition.
(5) Let $I_1, ..., I_n$ be ideals of R and let P be a prime ideal containing $\bigcap_{i=1}^n I_i$. Then $I_i \subseteq P$ for some i. If $P = \bigcap_{i=1}^n I_i$, then $P = I_i$ for some i [8].
(6) Let Q be a P-primary ideal of R, and let $a \in R$. Clearly, if $a \in Q$, then $(Q : a) = R$. Suppose that $a \notin Q$; we show that $(Q : a)$ is

P-primary. If $x \in (Q : a)$, then $x \in P$; hence $Q \subseteq (Q : a) \subseteq P$, so that $P \subseteq \sqrt{(Q : a)} \subseteq P$ by [4, Corollary 25]. Hence $\sqrt{(Q : a)} = P$. Now suppose that $c, d \in R$ are such that $cda \in Q$, but $d \notin P$. Then Q primary gives $c \in (Q : a)$.

(7) Assume that $0 = \bigcap_{i=1}^{n} Q_i$ with $\sqrt{Q_i} = P_i$ is a minimal primary decomposition of 0, and let $P \in \mathrm{Ass}(R)$. Then $P = (0 : a)$ for some $a \in R$. Then we must have $(0 : a) = \bigcap_{i=1}^{n}(Q_i : a)$. By (6), we have $(Q_i : a) = R$ if $a \in Q_i$, while $(Q_i : a)$ is P_i-primary if $a \notin Q_i$. Let $i_1, ..., i_s$ be such that $a \notin Q_{i_1}, ..., Q_{i_s}$. Hence by [4, Corollary 25], we see that $P = \sqrt{(0 : a)} = \bigcap_{j=1}^{s} P_{i_j}$; hence $P = P_{i_k}$ for some k. Therefore, we must have $\mathrm{Ass}(R)$ is a finite set.

Theorem 4.7. *Let I be a P-primary Q-ideal of a Noetherian semiring R. Then* $\mathrm{diam}(\Gamma(R/I)) \leq 2$.

Proof. Let q_0 be the unique element in Q such that $q_0 + I$ is the zero in R/I. By Theorem 4.5, $\Gamma(R/I) \cup \{q_0 + I\} = P/I$ is a prime ideal of R/I. It follows from Lemma 3.6 (ii) that P/I is the union of all the associated primes of R/I; hence $P/I \in \mathrm{Ass}(R/I)$. It follows from Lemma 3.5 (ii) and Remark 4.6 (7) that $P/I = (0 : \bar{p})$ for some $\bar{p} \in \Gamma(R/I)$; hence $\mathrm{diam}(\Gamma(R/I)) \leq 2$. □

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References

[1] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative rings*, J. Algebra **217** (1999), 434-447.

[2] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, The zero-divisor graph of a commutative ring, II, Lecture Notes in Pure and Appl. Math. **220**, Dekker, New York, 2001.

[3] P. Allen, Ideal theory in semirings, Dissertation, Texas Christian University, 1967.

[4] P. J. Allen, J. Neggers and H. S. Kim, *Ideal theory in commutative A-semirings*, Kyungpook Math. J **46** (2006), 261-271.

[5] I. Beck, *Coloring of a commutative ring*, J. Algebra **116** (1988), 208-226.

[6] S. Ebrahimi Atani, *On k-weakly primary ideals over semirings*, Sarajevo J. Math. **3(15)** (2007), 9-13.

[7] S. Ebrahimi Atani, *The ideal theory in quotients of commutative semirings*, Glas. Math. Ser. III **42(62)** (2007), 301-308.

[8] S. Ebrahimi Atani and R. Ebrahimi Atani, *Ideal theory in commutative semirings*, Bul. Acad. Stiinte Repub. Mold. Mat. **2(57)** (2008), 14-23.

[9] V. Gupta and J. N. Chaudhari, *Some remarks on semirings*, Rad. Mat. **12** (2003), 13-18.

[10] V. Gupta and J. N. Chaudhari, *Right π-regular semirings*, Sarajevo J. Math. **2** (2006), 3-9.

[11] H. R. Maimani, M. R. Pournaki and S. Yassemi, *Zero-divisor graph with respect to an ideal*, Comm. Algebra **34** (2006), 923-929.

[12] J. R. Mosher, *Generalized quotients of hemirings*, Compositio Math. **22** (1970), 275-281.

[13] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), 4425-4443.

[14] M. K. Sen and M. R. Adhikari, *On maximal k-ideals of simirings*, Proc. Amer. Math. Soc. **118** (1993), 699-703.

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