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FINITE NONNILPOTENT GROUPS WITH FEW HEIGHTS OF NONNORMAL SUBGROUPS

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Abstract. The number of prime factors of the order of a group G (multiplicities counted) is said to be the *height* of G and denoted by $\mathrm{n}_\lambda(G)$. We classify the nonnilpotent groups G with $\mathrm{n}_\lambda(G) = 2$ and nonsolvable groups G with $\mathrm{n}_\lambda(G) \in \{3, 4\}$.

This note supplements [B1, §2 and §3].

A group is said to be *Dedekindian* if all its subgroups are normal. It is natural to consider groups having a few nonnormal subgroups as close to Dedekindian. There is a number of papers about groups with few nonnormal subgroups (see [B1, P, S2, S3, Z]). For example, O. Schmidt [S2, S3] has studied the groups with at most two conjugate classes of nonnormal subgroups.

Let G be a group (only finite groups are considered and we use the same notation as in [B1]), n a natural number. Let $\lambda(n)$ be the number of prime factors of n (multiplicities counted). For example, $\lambda(1) = 0$, $\lambda(48) = 5$. Set $\lambda(G) = \lambda(|G|)$, where $|G|$ is the order of G. The number $\lambda(G)$ we call the *height* of G. Write

$$\mathcal{N}_\lambda(G) = \{\lambda(H) \mid H \text{ is not normal subgroup of G }\}, \ \ n_\lambda(G) = |\mathcal{N}_\lambda(G)|.$$

We have

$$\begin{aligned} &\lambda(\mathrm{S}_4) = 4, \quad \mathcal{N}_\lambda(\mathrm{S}_4) = \{1, 2, 3\}, \quad n_\lambda(\mathrm{S}_4) = 3, \\ &n_\lambda(\mathrm{SL}(2, 5)) = 4 = \mathrm{n}_\lambda(\mathrm{PSL}(2, 7)) = \mathrm{n}_\lambda(\mathrm{PGL}(2, 7)) = \mathrm{n}_\lambda(\mathrm{PGL}(2, 5)) \\ &\qquad\qquad\qquad = \mathrm{n}_\lambda(\mathrm{PSL}(2, 3^2)) = \mathrm{n}_\lambda(\mathrm{PSL}(2, 3^3)), \\ &\mathrm{n}_\lambda(\mathrm{SL}(2, 7)) = 5. \end{aligned}$$

The group G is Dedekindian if and only if $\mathrm{n}_\lambda(G) = 0$.

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Let $\Delta(G)$ be the set of orders of nonnormal subgroups of G. For example, $\Delta(\mathrm{S}_4) = \{2,3,4,6,8\}$, $\Delta(\mathrm{SL}(2,5)) = \{3,5,4,6,10,8,12,20,24\}$. Obviously, $|\Delta(G)| \geq \mathrm{n}_\lambda(G)$. If G is a p-group then $|\Delta(G)| = \mathrm{n}_\lambda(G)$ (I think that there are few nonnilpotent groups G satisfying $|\Delta(G)| = \mathrm{n}_\lambda(G)$).

The groups G with $\mathrm{n}_\lambda(G) = 1$ were classified in [B1, Proposition 2.1, Theorems 2.5 and 2.6]; see also [P] and [Z] for another approach. The classification of such p-groups is fairly nontrivial (see above three papers). Next, in [B1, Theorem 3.1] the nonnilpotent groups G with $\mathrm{no}(G) = |\{|H| \mid H \text{ is not normal in } G\}| = 2$ are classified. In this note we classify the non primary groups G with $\mathrm{n}_\lambda(G) = 2$ (a group G is said to be *primary* if its order is a power of a prime); see Proposition 2 and Theorem 3. We have noticed that, as a rule, $\mathrm{n}_\lambda(G) < \mathrm{no}(G)$, so Theorem 3 is a proper generalization of [B1, Theorem 3.1]. We also classify the nonsolvable groups G with $\mathrm{n}_\lambda(G) \in \{3,4\}$ (see Theorems 4, 8); in the proof of Theorem 8 we use the classification of finite simple groups.

Classification of p-groups G such that $\mathrm{n}_\lambda(G) = 2$, is not obtained yet. If $|G| = p^n$, $n > 2$, then $\mathrm{n}_\lambda(G) \leq n-2$. If G is a p-group of maximal class and order p^n, then $\mathrm{n}_\lambda(G) = n-2$, unless $G \cong \mathrm{Q}_{2^n}$ where $\mathrm{n}_\lambda(G) = n-3$. As Passman [P] proved, only few p-groups G satisfy $\mathrm{n}_\lambda(G) < n-2$.

Let G be extraspecial of order p^5. If G has a subgroup $\cong \mathrm{E}_{p^3}$, then $\mathrm{n}_\lambda(G) = 2$, and if G has no subgroup $\cong \mathrm{E}_{p^3}$, then $\mathrm{n}_\lambda(G) = 1$ (in the last case, by Blackburn's Theorem [Bla, Theorem 4.3], we must have $p = 2$). Indeed, $p \in \Delta(G)$. If $H < G$ is a nonnormal subgroup of order p^2, then $G' \not\leq H$ so $H \cong \mathrm{E}_{p^2}$; in that case, $H \times G' \cong \mathrm{E}_{p^3}$. Since G has no abelian subgroup of index p, $p^3 \notin \Delta(G)$.

In what follows, p, q are distinct primes and r is a prime which may be equal or not to p or q. We write $A \cdot B$ to denote the semidirect product of A and B with kernel B.

If G is a minimal nonnilpotent group, then (O. Schmidt [S1], Y. Gol'fand [G] and L. Redei [R])

(MNN) $G = P \cdot Q$, where $P \in \mathrm{Syl}_p(G)$ is cyclic, $G' = Q \in \mathrm{Syl}_q(G)$, $\mathrm{Z}(G) = \mho_1(P)\Phi(Q) = \Phi(G)$, $G/\Phi(Q)$ is minimal nonabelian. In particular, G is minimal nonabelian if and only if Q is abelian. We have $|Q| = q^{b+c}$, where b is the order q modulo p and $c \leq b/2$ (if b is odd, then $c = 0$). If $q > 2$, then $\exp(Q) = q$.

Such G we call an $\mathrm{S}(p,q)$-*group* or S-*group*. If, in addition, Q is abelian, then G is minimal nonabelian; in that case, such G is called an $\mathrm{A}(p,q)$-group or A-group.

Theorem 1 ([B1, Proposition 2.1]). *If G is a nonnilpotent group with $\mathrm{n}_\lambda(G) = 1$, then one of the following holds:*

(a) G is a minimal nonabelian $\{p,q\}$-group with $|G'| = q$.

(b) G is a minimal nonabelian group of order pq^2 with $|G'| = q^2$.

In Proposition 2 we classify the groups G decomposable in nontrivial direct product and such that $\mathrm{n}_\lambda(G) = 2$,

Proposition 2. *Suppose that a group $G = A \times B$, where $\mathrm{n}_\lambda(G) = 2$, $\mathrm{n}_\lambda(A) > 0$ and $B > \{1\}$. Then $\mathrm{n}_\lambda(A) = 1$ and $|B| = r$. More precisely, one and only one of the following holds:*

(a) $A = \mathrm{M}_{p^n}$, *r is arbitrary,* $\Delta(G) = \{p, pr\}$.
(b) $A = P \cdot Q$ *is minimal nonabelian of order* $p^a q$ *with* $|A'| = q$. *If* $a > 1$, *then* $r \neq q$; *then* $\Delta(G) = \{p^a, p^a r\}$. *If* $a = 1$, *then r is arbitrary; then* $\Delta(G) = \{p, q, pq\}$ *provided* $r = q$ *and* $\Delta(G) = \{p, pr\}$ *provided* $r \neq q$.
(c) $A = P \cdot Q$ *is minimal nonabelian of order* pq^2 *with* $|A'| = q^2$, *r is arbitrary,* $\Delta(G) = \{p, q.pr, qr\}$.

Proof. Recall that p, q, r are primes and $p \neq q$. It is easy to check that groups (a)–(c) satisfy the hypothesis. Let us prove that if G satisfies the hypothesis, then it is one of groups (a)–(c).

Let $A_0 < A$ be nonnormal and $\{1\} < L \leq B$. Then $A_0 \times L$ is not G-invariant since $A \cap (A_0 \times L) = A_0$. It follows that $|B| = r$ and $\mathrm{n}_\lambda(A) = 1$. The structure of A is known (see Theorem 1 and [B1, Theorem 2.4]). In particular, if A is not nilpotent, it is an $\mathrm{A}(p, q)$-group. If A is a p-group, then $A \cong \mathrm{M}_{p^n}$ and G satisfy the hypothesis for arbitrary r.

Now let $A = P \cdot Q$ be minimal nonabelian of order $p^a q$. If $r = q$, then G has a nonnormal subgroup of order q. Let us prove this. The group G has exactly $q + 1 > 2$ subgroups of order q. If $Q_1 < G$ is of order q, $A' \neq Q_1 \neq B$, then Q_1 is not normal in G. Indeed, if this is false, then $\mathrm{C}_G(Q_1) \geq AB = G$, and so $BQ_1 \leq \mathrm{Z}(G)$. In that case, G is abelian, a contradiction. Besides, G has a nonnormal subgroups P of order p^a and $P \times B$ of order $p^a q$. It follows from $\mathrm{n}_\lambda(G) = 2$ that then $a = 1$ so G is as in (b).

If $A = P \cdot Q$ is minimal nonabelian of order pq^2, then r is arbitrary and $\Delta(G) = \{p, q, pr, qr\}$ so G satisfies the hypothesis. □

In view of Proposition 2, one can confine in the sequel to nonnilpotent groups which are not decomposed in nontrivial direct product.

Note that if G is an $\mathrm{S}(p, 2)$-group, then $G' \ncong \mathrm{D}_8$. Indeed, assume that this is false; then G' has a characteristic subgroup L of order 4. If $P \in \mathrm{Syl}_p(G)$, then P centralizes L and G'/L so P centralizes G'. It follows that G is nilpotent, a contradiction.

Suppose that all proper subgroups of a nonsolvable group G are solvable. Let $N < G$ be maximal normal subgroup of G and $A < G$ be maximal in G. Then AN is solvable since A and N are solvable. It follows that $AN < G$ so $N < A$, and we conclude that $N = \Phi(G)$. Now it is clear that G/N is nonabelian simple.

Recall that if N is a normal abelian p-subgroup of G, $N \le P \in \mathrm{Syl}_p(G)$ and N is complemented in P, then N is complemented in G (Gaschütz; [H, Hauptsatz I.17.4(a)]).

In what follows, we use the above three facts freely.

Now we are ready to prove our main result.

THEOREM 3. *Let G be a nonnilpotent group which has no nontrivial direct factor. Let $\mathrm{n}_\lambda(G) = 2$. Then one of the following holds:*

(a) *G is an $\mathrm{A}(p,q)$-group of order pq^3, $\Delta(G) = \{p, q, q^2\}$.*
(b) *G is an $\mathrm{A}(p,q)$-group of order p^2q^2, $\Delta(G) = \{q, p^2, pq\}$.*
(c) *G is an $\mathrm{S}(p,q)$-group of order pq^3 with nonabelian subgroup of order q^3 and exponent $q > 2$, $\Delta(G) = \{p, q, pq.q^2\}$.*
(d) *$G = \mathrm{SL}(2,3)$ is an $\mathrm{S}(3,2)$-group of order $3 \cdot 2^3$, $\Delta(G) = \{3, 4, 6\}$.*
(e) *$G = P \cdot Q$ is an $\mathrm{S}(3,2)$-group of order $3^2 \cdot 2^3$, $G/\mho_1(P) \cong \mathrm{SL}(2,3)$, $\Delta(G) = \{4, 9, 12, 18\}$.*
(f) *$G = (P \times Q) \cdot R$ is a Frobenius group of order pqr^s which kernel R of order r^s is a minimal normal subgroup, $s \le 3$, $p \ne r \ne q$, where $|P| = p$, $|Q| = q$. If $s = 3$, then PR and QR are minimal nonabelian and $\Delta(G) = \{p, q, r, r^2, pq\}$. If $s = 1$, then $\Delta(G) = \{p, q, pq\}$. If $s = 2$, then $\mathcal{N}_\lambda(G) = \{1, 2\}$ (there are few possibilities for $\Delta(G)$).*
(g) *$G = P \cdot Q$ is a Frobenius group of order p^2q^s with minimal normal subgroup of order q^s, $s = 1, 2, 3$. If $s = 3$, then the subgroup $\mho_1(P)Q$ of order pq^3 is minimal nonabelian. If $s = 1$, then $\Delta(G) = \{p, p^2\}$. If $s = 2$, then $\mathcal{N}_\lambda(G) = \{1, 2\}$. If $s = 3$, then $\Delta(G) = \{p, p^2, q, q^2\}$.*
(h) *$G = P \cdot Q$, where $P \cong \mathrm{Q}_8$, $|Q| = q$, G contains exactly one cyclic subgroup of index 2, $\Delta(G) = \{4, 8\}$.*
(i) *$G = P \cdot (Q \times R)$ of order p^aqr, $P \in \mathrm{Syl}_p(G)$ is cyclic of order p^a, $|Q| = q$, $|R| = r$, $p \ne r \ne q$, $\mho_1(P) = \mathrm{Z}(G)$, $G/\mathrm{Z}(G)$ is a Frobenius group of order pqr, $\Delta(G) = \{p^a, p^aq, p^ar\}$.*
(j) *$G = P \cdot Q$ is a Frobenius group of order pq^2 with minimal normal subgroup of order q, $\mathcal{N}_\lambda(G) = \{1, 2\}$.*
(k) *$G = P \cdot Q$ is of order p^aq, P is cyclic of order p^a, $a > 1$, $G/\mathrm{Z}(G)$ is a Frobenius group of order p^2q, $\Delta(G) = \{p^{a-1}, p^a\}$.*
(l) *$G = P \cdot Q$ is of order p^aq^2, $a > 1$, $\mho_1(P) = \mathrm{Z}(G)$, $G/\mho_1(P)$ is a Frobenius group of order pq^2, all subgroups of order q are G-invariant, $\Delta(G) = \{p^a, p^aq\}$.*

PROOF. Write $\mathcal{N}_\lambda(G) = \{m.n\}$, $m < n$.

(i) Let $G = P \cdot Q$ be an $\mathrm{S}(p,q)$-group as in (MNN).

(i1) Suppose that P is not maximal in G. Then Q is nonabelian so special. If $\{1\} < Z \le \mathrm{Z}(Q)$, then PZ is not normal in G, so $|\mathrm{Z}(Q)| = q$, and we have $|P| = p^m$ and $n = m + 1$. Next, $|Q/\mathrm{Z}(Q)| = q^b$, where $b > 1$ is the order of q (mod p). Therefore, if $\mathrm{Z}(Q) < Q_1 < Q$, then Q_1 and $\mho_1(P)Q_1$ are not normal in G. Since $\mathrm{n}_\lambda(G) = 2$, then $b \le 3$. Since G has a nonnormal subgroup of order q^2, we get $m \leq 2$. Since G has a nonnormal subgroup of order q^b, we get $b \leq n$.

If $m = 1$, then $n = 2$ so $b = 2$ and $|G| = pq^3$, where Q is nonabelian of order q^3, $Q \in \{\mathrm{Q}_8, \mathrm{S}(p^3)\}$, where $\mathrm{S}(p^3)$ is a group of order p^3 and exponent $p > 2$. If $Q \cong \mathrm{Q}_8$, then $p = 3$, $\Delta(G) = \{3, 4, 6\}$ and $G \cong \mathrm{SL}(2, 3)$ is as in (d). If $q > 2$, then $\Delta(G) = \{p, q, pq, q^2\}$ and G is as in (c).

Now let $m = 2$. If $b = 3$, then Q has a non-G-invariant subgroups of orders q, q^2 and q^3 so $\mathrm{n}_\lambda(G) > 2$, a contradiction. Thus, $b = 2$ and so $|G| = p^2q^3$. If $q > 2$, then Q has, in addition, a nonnormal subgroup of order q so $\mathcal{N}_\lambda(G) = \{1, 2, 3\}$, a contradiction. Thus, $q = 2$, $Q \cong Q_8$, $p = 3$ so $|G| = 3^2 2^3$, $\Delta(G) = \{4, 9, 12, 18\}$ and G is as in (e).

(i2) Suppose that P is maximal in G. Then Q is minimal normal in G and G is minimal nonabelian. Set $|Q| = q^b$; then $b > 1$ (otherwise, $\mathrm{n}_\lambda(G) = 1$). Since G has a nonnormal subgroup of order q, we get $m = 1$. Since $\mathrm{n}_\lambda(G) = 2$, we get $b \leq 3$. Therefore, if $|P| = p$, then $b = 3$, $|G| = pq^3$, $\Delta(G) = \{p, q, q^2\}$, and G is as in (a).

Now let $|P| > p$; then $|P| = p^n$. If $Q_1 < Q$ is of order q, then $\Omega_1(P)Q_1$ is not normal in G so $n = \lambda(\Omega_1(P)Q_1) = 2$ and $\mathcal{N}_\lambda(G) = \{1, 2\}$. Then $b \leq 3$. If $b = 3$ and $Q_2 < Q$ is of order q^2, then $\Omega_1(P)Q_2$ is not normal in G so $3 \in \mathcal{N}_\lambda(G)$, a contradiction. Thus, $b = 2$, $|G| = p^2q^2$, $\Delta(G) = \{q, p^2, pq\}$ so G is as in (b).

Next we assume that G is not an S-group.

(ii) Let us prove that G is solvable. Assume that G is a counterexample of minimal order. If $H < G$, then $\mathrm{n}_\lambda(H) \leq \mathrm{n}_\lambda(G) \leq 2$ so all proper subgroups of G are solvable, by induction. If $N \lhd G$, then $\mathrm{n}_\lambda(G/N) \leq \mathrm{n}_\lambda(G) \leq 2$ so all proper epimorphic images of G are solvable. It follows that G is nonabelian simple and $\lambda(H) \leq 2$ for all $H < G$. Let p be a minimal prime divisor of G and $P \in \mathrm{Syl}_p(G)$; then $|P| > p$ (Burnside). If $|P| > p^2$, then G contains subgroups of orders p, p^2 and p^3 so $\mathrm{n}_\lambda(G) > 2$, a contradiction. Thus, $|P| = p^2$ and $\mathcal{N}_\lambda(G) = \{1, 2\}$. By Burnside, $P < \mathrm{N}_G(P)$ and so $\lambda(\mathrm{N}_G(P)) > 2$, a contradiction.

Since G is solvable, there is in G a Carter subgroup K. Recall that K is nilpotent and $\mathrm{N}_G(K) = K$ (so $\mathrm{Z}(G) < K$), and all Carter subgroups are conjugate in G, and, whenever $K \leq H \leq G$, then $\mathrm{N}_G(H) = H$ [H, Satz VI.12.3]. By hypothesis, $K < G$. We conclude that K is either maximal or second maximal in G.

(iii) Assume that K is not primary. Then $K = P \times Q$, where $P \in \mathrm{Syl}_p(K)$ is not G-invariant and $Q > \{1\}$. Since $\mathrm{n}_\lambda(G) = 2$, K is maximal in G and $m = \lambda(P)$, $n = \lambda(K)$. Since all proper subgroups of P are G-invariant, it follows that P is cyclic. If $|Q|$ is not prime and $\{1\} < Q_0 < Q$, then PQ_0 is not G-invariant so $\mathrm{n}_\lambda(G) > 2$, a contradiction. Thus, $|Q| = q$ so K is cyclic, $|K| = p^mq$, $n = m + 1$ and $\mathcal{N}_\lambda(G) = \{m, m + 1\}$.

(iii1) Let $m > 1$; then $Q \lhd G$ since $\lambda(Q) = 1 < m$. In that case, consideration of G-invariant subgroup $\mathrm{C}_G(Q) \geq K$ shows that $Q \leq \mathrm{Z}(G)$. Similarly, $\mho_1(P) \leq \mathrm{Z}(G)$ so $|K : \mathrm{Z}(G)| = p$. We have $K \cap K^x = \mathrm{Z}(G)$ for all $x \in G - K$, and so $\bar{G} = G/\mathrm{Z}(G) = \bar{K} \cdot \bar{N}$ is a Frobenius group with kernel $\bar{N}$ of order r^b ($r \neq p$) and index p. Since K is maximal in G, $\bar{N}$ is minimal normal subgroup of $\bar{G}$ so $\bar{G}$ is $\mathrm{A}(p, r)$-group. If $b > 1$, then $\bar{N}$ has a non-G-invariant subgroup $\bar{F}$ of order r^{b-1}; then $\lambda(F) = \lambda(\mathrm{Z}(G)) + b - 1 = m + b - 1 \leq n = m + 1$, and we conclude that $b = 2$. If $r \neq q$ and $b = 2$, then G has a nonnormal subgroup of order r, and we get $\mathrm{n}_\lambda(G) > 2$ since $\lambda(r) = 1 < m$, a contradiction. Thus, if $b = 2$, then $r = q$.

Suppose that $r = q$. Let $Q_0 \in \mathrm{Syl}_q(G)$; then $|Q_0| = q^{b+1}$ and $Q_0/Q \cong \mathrm{E}_{q^b}$; next, $Q_0 \lhd G$ since $N \lhd G$ and Q_0 is a direct factor of N. The subgroup Q_0 is noncyclic (otherwise, by (MNN), G has no S-subgroup so nilpotent). Since $m > 1$ and $\mathcal{N}_\lambda(G) = \{m, m+1\}$, all subgroups of order q are normal in G. Assume that $\exp(Q_0) = q$. If $|Q_0| = q^2$, then Q is a direct factor of G (Maschke's Theorem), contrary to the hypothesis. Thus, $|Q_0| > q^2$ so G has a nonnormal subgroup of order q^2, and we conclude that $m = 2$; then $\mathcal{N}_\lambda(G) = \{2, 3\}$. By the previous paragraph, $b = 2$ so $|Q_0| = q^3$. All subgroups of order q are G-invariant so $\exp(Q_0) = q^2$. If $q > 2$, then $\Omega_1(Q_0)(\cong \mathrm{E}_{q^2})$ is normal in G so $\bar{N}$ is not minimal normal subgroup of $\bar{G}$, a contradiction. Thus, $q = 2$ so that $Q_0 \cong \mathrm{Q}_8$; then $p = 3$. In that case, $|P| = 3^2$, $|G| = 3^2 2^3$, $G/\mho_1(P) \cong \mathrm{SL}(2, 3)$ and $\Delta(G) = \{4, 9, 12, 18\}$ so G is as in (e).

Now let $r \neq q$. Then, by the first paragraph of this part, $b = 1$ and $G = K \cdot R = (P \times Q) \cdot R = (P \cdot R) \times Q$, contrary to the hypothesis.

(iii2) Now let $m = 1$ so $|K| = pq$; then $\mathcal{N}_\lambda(G) = \{1, 2\}$. Since K is maximal in G, we get $\mathrm{N}_G(P) = K$ so $P \in \mathrm{Syl}_p(G)$.

Assume that Q is not normal in G; then $\mathrm{N}_G(Q) = K$ so K is a $\{p, q\}$-Hall subgroup of G. By Burnside's Normal p-Complement Theorem applied twice, $G = K \cdot H$, where H is a minimal normal subgroup of G which is a $\{p, q\}'$-Hall subgroup of G. It follows that $|H| = r^s$, where $r \in \pi(G) - \{p, q\}$. We have $s \leq n + 1 = 3$. We see that G is a Frobenius group with kernel H. Assume that $s = 3$. Then the subgroups PH and QH are minimal nonabelian. Indeed, assume that, for example, PH is not minimal nonabelian. Then, by Maschke's Theorem, it contains a non-G-invariant subgroup of order pr^2 so $3 \in \mathcal{N}_\lambda(G)$, a contradiction. In that case, $\Delta(G) = \{p, q, r, pq, r^2\}$ so $\mathrm{n}_\lambda(G) = 2$. If $s = 1$, then $\Delta(G) = \{p, q, pq\}$. If $s = 2$, then this and previous two groups are as in (f) (for $\Delta(G)$ we have few possibilities, among of them $\{p, q, pq, r\}, \{p, q, pq, pr, r\}, \{p, q, pq, qr, r\}, \{p, q, pq, pr, qr, r\}$).

Now suppose that $Q \lhd G$. Then, as above, $Q = \mathrm{Z}(G)$, $G/Q = \bar{G} = \bar{K} \cdot \bar{N}$ is a Frobenius group with kernel $\bar{N}$, where $|\bar{K}| = p$ and $|\bar{N}| = r^b$, $r \neq p$, $\bar{N}$ is a minimal normal subgroup of $\bar{G}$. Since G has no nontrivial direct factor, we get $r = q$ so $G = P \cdot N$ and $N \in \mathrm{Syl}_q(G)$. In that case, N is noncyclic (otherwise, by (MNN), G has no S-subgroup so nilpotent). Since Q is not a direct factor of G, we get $b > 1$ (Fitting's Lemma). Since $\bar{G}$ has a nonnormal subgroup $\bar{Q}_1$ of order q^{b-1}, then $b = \lambda(Q_1) = 2$ (recall that 2 is the maximal member of the set $\mathcal{N}_\lambda(G)$); then $|N| = q^3$. Since N/Q is minimal normal subgroup of G/Q, we get $|\Omega_1(N)| \neq q^2$. If $|\Omega_1(N)| = q$, then $q = 2$, $N \cong \mathrm{Q}_8$ and $G \cong \mathrm{SL}(2,3)$ is as in (d). If $|\Omega_1(N)| = q^3$, then N is nonabelian (otherwise, Q is a direct factor of G, by Fitting's Lemma) so $N \cong \mathrm{S}(q^3)$, and G is as in (c).

(iv) Let K be a p-subgroup with non-G-invariant maximal subgroup, say K_1; then $K \in \mathrm{Syl}_p(G)$ is maximal in G, $\mathcal{N}_\lambda(G) = \{m, m+1\}$, $|K| = p^{m+1}$, $m \geq 1$. In that case, if $L < K$ is of index $> p$, then $L \lhd G$ and so K is not generated by subgroups of index p^2. Then one of the following holds:

1) K is cyclic;
2) K is abelian of type (p^m, p);
3) $K \cong \mathrm{Q}_8$.

Note that the group $\mathrm{M}_{p^{m+1}}$ is also not generated by subgroups of index p^2, however, if $K \cong \mathrm{M}_{p^{m+1}}$, $m > 1$, then K has a nonnormal subgroup of order p which is impossible since $\lambda(p) = 1 < m$.

(iv1) Let K be abelian of order p^{m+1}; then G has a normal p-complement H (Burnside), and H is a minimal normal subgroup of G since K is maximal in G. Set $|H| = q^b$. If K is abelian of type (p,p), then G is not a Frobenius group so $\mathrm{C}_G(P_1) = G$ for some $P_1 < P$ of order p. In that case, P_1 is a direct factor of G, a contradiction.

Suppose that $b = 1$. If K is cyclic, then $\Delta(G) = \{p^m, p^{m+1}\}$, $G/\mathrm{Z}(G)$ is a Frobenius group of order p^2q. Now let K be abelian of type (p^m, p). In that case, as we have proved, $m > 1$. Then all subgroups of order p are G-invariant so lie in $\mathrm{Z}(G)$. Therefore, if $K = U \times V$, where $|V| = p$, then V is a direct factor of G (Gaschütz), a contradiction. Thus, K is cyclic so G is as in (k).

Next we assume that $b > 1$. Since G has nonnormal subgroups of order q^i, $i = 1, \ldots, b-1$, we get $b \leq 3$ and $m = 1$, $|K| = p^2$. By the above, K is cyclic. In that case, G is a Frobenius group of order p^2q^b, $1 < b \leq 3$. If $b = 2$, then $\Delta(G) = \{p, q, p^2\}$ and G is as in (g). Let $b = 3$. Then $\mho_1(K)H$ must be minimal nonabelian (otherwise, G contains a subgroup of order pq^2, by Maschke's Theorem), which is q-closed so non-G-invariant, and we get $p, p^2, pq^2 \in \mathcal{N}_\lambda(G)$ so $\mathrm{n}_\lambda(G) > 2$, a contradiction. Thus, G is as in (g).

(iv2) Let $K \cong \mathrm{Q}_8$; then $\mathcal{N}_\lambda(G) = \{2, 3\}$. We claim that then $G = K \cdot Q$ is 2-nilpotent, where $|Q| = q^b$. Indeed, at least two maximal subgroups of K are nonnormal in G, the subgroup $L < K$ of order 2 is G-invariant. By Burnside, $QL/L \lhd G/L$. Since $|QL| = 2|Q|$, Q is characteristic in QL so $Q \lhd G$. Since K is maximal in G, Q is minimal normal subgroup of G. If $|Q| > q$, there is a nonnormal subgroup of order q in Q so $1 \in \mathcal{N}_\lambda(G)$, a contradiction. Thus, $b = 1$. In that case, $\mathrm{C}_G(Q)$ is cyclic of index 2 in G so G is as in (h).

(v) Suppose that all maximal subgroups of K are normal in G. Then K is a cyclic p-subgroup, and we conclude that $K \in \mathrm{Syl}_p(G)$ since $\mathrm{N}_G(K) = K$, and hence $G = K \cdot H$, where H is a normal p'-Hall subgroup of G (Burnside). Then $|H|$ is not a prime (otherwise, $\mathrm{n}_\lambda(G) = 1$). Since $\mathrm{C}_G(\mho_1(K)) \geq KH = G$, we get $\mho_1(K) = \mathrm{Z}(G)$. Write $\bar{G} = G/\mho_1(G)$; then $\bar{G} = \bar{K} \cdot \bar{H}$ is a Frobenius group so $\bar{H}$ is nilpotent, by theorem of Witt [BZ, Theorem 10.7]. Since $\bar{H} \cong H$, the subgroup H is also nilpotent.

(v1) Suppose that $|\pi(H)| > 1$; then $\pi(H) = \{q, r\}$ (otherwise, $K < L < M < G$ and L, M are not normal in G so $\mathrm{n}_\lambda(G) > 2$ (here we use Hall's theorem on solvable groups). In that case, $\{p, q\}$- and $\{p, r\}$-Hall subgroups of G, say U and V, respectively, are not normal in G. It follows that $\lambda(U) = \lambda(V)$. Obviously, U and V are maximal in G (otherwise, $\mathrm{n}_\lambda(G) > 2$). Let $Q \in \mathrm{Syl}_q(U)$ and $R \in \mathrm{Syl}_r(V)$; then $H = Q \times R$ and Q, R are minimal normal subgroups of G (for example, KQ is maximal in $(KQ) \cdot R = G$ so R is a minimal normal subgroup of G). We have $|K| = p^m$ and $\lambda(KQ) = n$.

Let $|Q| > q$. Then G has a nonnormal subgroup of order q so $\lambda(K) = m = 1$ (otherwise, $1, \lambda(K), \lambda(KQ) \in \mathcal{N}_\lambda(G)$ so $\mathrm{n}_\lambda(G) > 2$). In that case, G has a nonnormal subgroup of order qr so $p, qr, |KQ| \in \Delta(G)$ and $\mathrm{n}_\lambda(G) > 2$, a contradiction. Thus, $|Q| = q$ so $|H| = qr$, $|G| = p^m qr$, $\bar{G} = G/\mho_1(K)$ is a Frobenius group of order pqr with kernel $\bar{H}$ of order qr, and we get $\Delta(G) = \{p^m, p^m q, p^m r\}$, and G is as in (i).

In what follows we assume that $H \in \mathrm{Syl}_q(G)$.

(v2) Suppose that $K < M < G$, where M is maximal in G; then K is maximal in M (otherwise, $\mathrm{n}_\lambda(G) > 2$). We have $M = K \cdot Q$ (Burnside) and $\mho_1(K) = \mathrm{Z}(G)$. Set $|Q| = q^b$, $|G : M| = q^s = |H : Q|$, $|K| = p^m$. Next, Q is a minimal normal subgroup of M and so of G (in fact, $\mathrm{N}_G(Q) > M$ whence $Q \triangleleft G$).

(v2.1) Suppose that $m > 1$. Then $b = 1$ (otherwise, there is in G the following non-G-invariant subgroups: K of order $p^m > p$, KQ of order $p^m q^b > p^m$ and a subgroup of Q of order q so $\mathrm{n}_\lambda(G) > 2$).

Let, in addition, $s > 1$; then $m = 2$. Indeed, since H/Q contains a non-G-invariant subgroup Q_1/Q of order q, we get $\lambda(Q_1) = 2 < \lambda(KQ)$ so $2 = \lambda(Q_1) = \lambda(K) = m$. Since $p^2, p^2 q \in \Delta(G)$ and $\mathrm{n}_\lambda(G) = 2$, we must have $\mathcal{N}_\lambda(G) = \{2, 3\}$. It follows that $s \leq 3$ (otherwise, H/Q contains a non-G-invariant subgroup Q_2/Q of order q^3 so $\lambda(Q_2) = 4 > 3$) and all subgroups of order q must be G-invariant since $1 \notin \mathcal{N}_\lambda(G)$.

Assume that $s = 3$. Then $|H| = q^4$ and H has a non-G-invariant subgroup H_1 of order q^3. Then $\mho_1(K) \times H_1$ of order pq^3 is not G-invariant, a contradiction since $\lambda(\mho_1(K)H_1) = 4 > 3$. Thus, $s < 3$.

Assume that $s = 2$. Then $|H| = q^3$. It follows that $\exp(H) = q^2$ (otherwise, $H = \Omega_1(H)$ is elementary abelian and all subgroups of order q in H are G-invariant, which is not the case since $H/Q \cong \mathrm{E}_{q^2}$ is minimal normal subgroup of G/Q). Then K is contained in (non-G-invariant) subgroup of order $p^2 q^2$, a contradiction since $4 \notin \mathcal{N}_\lambda(G)$.

It remains to consider case $s = 1$ (by assumption, $m > 1$); then $|H| = q^2$, $|G| = p^m q^2$, $\mathcal{N}_\lambda(G) = \{m, m+1\}$. All subgroups of G of order q are G-invariant, $G/\mho_1(K)$ is a Frobenius group of order pq^2, $\Delta(G) = \{p^m, p^m q\}$. Since H is cyclic or abelian of type (p,p), we get two nonisomorphic groups. Our G are as in (l).

(v2.2) Let $m = 1$. Then G is a Frobenius group and $b \leq 2$ (indeed, if $b > 2$, then G has a nonnormal subgroups of orders p, q^2 and pq^b so $\mathrm{n}_\lambda(G) > 2$, a contradiction).

Suppose that $b = 1$. Then $p, pq \in \Delta(G)$ so $\mathcal{N}_\lambda(G) = \{1, 2\}$ hence $s \leq 2$ (if $s > 2$, then G has a non-G-invariant subgroup of order q^3 so $\mathrm{n}_\lambda(G) > 2$). If $s = 1$, then G (of order pq^2) is as in (j). Let $s = 2$. Then G has no subgroup of order pq^2 (otherwise, $\mathrm{n}_\lambda(G) > 2$). It follows that H is either $\cong \mathrm{Q}_8$ (then $G \cong \mathrm{SL}(2,3)$) or $\exp(H) = p$. In the first case, K is not a Carter subgroup since it centralized by Q, a contradiction. In the second case, $q \equiv 1 \pmod p$ since KQ is nonnilpotent so H/Q is not a minimal normal subgroup of G/Q (take in G/Q a minimal nonnilpotent subgroup!), a contradiction.

Suppose that $b = 2$. Then K and KQ are not G-invariant so $\mathcal{N}_\lambda(G) = \{1, 3\}$ and $s \leq 2$ (otherwise, G has a nonnormal subgroup of order q^4 so $4 \in \mathcal{N}_\lambda(G)$, a contradiction. If $s = 1$, then the G-invariant subgroup $[H, Q] < Q$ so $Q \leq \mathrm{Z}(H)$ since Q is a minimal normal subgroup of G, and we conclude that H is abelian. If $\exp(H) = q^2$, then $\{1\} < \Phi(H) < Q$, a contradiction since Q is a minimal normal subgroup of G. Thus, $H \cong \mathrm{E}_{q^3}$. Since G/Q is nonabelian of order pq, we get $q \equiv 1 \pmod p$. Then Q is not minimal normal subgroup of KQ (indeed, by (MNN), KQ contains a proper S-subgroup) so Q is not minimal normal subgroup of G since H is abelian, a final contradiction. ∎

It follows from the proof of Theorem 3 that if $\mathrm{n}_\lambda(G) = 2$, then $|\Delta(G)| \leq 6$, and this estimate is attained. Therefore, Theorem 3 is a very strong generalization of [B1, Theorem 3.1].

It is easy to deduce from Theorems 1 and 3 classification of nonnilpotent groups without three nonnormal subgroups of pairwise distinct orders and which are not nontrivial direct products. We get the following groups of Theorems 1 and 3: 1(a), 1(b), 3(g) of order p^2q, 3(h), 3(j), 3(k), 3(l) (compare with [B1, Theorem 3.1]). Note that O. Schmidt has classified the groups with one [S1] and two [S2] non-invariant classes of conjugate subgroups (the proof of the last result is not full; see [B1, §2, §3]).

In what follows we use freely the following facts. If $\lambda(G) \leq 3$, then G is solvable. If G is nonsolvable, there is $H < G$ with $\lambda(H) \geq 3$. Let us prove the second assertion using induction on $|G|$. Then all proper subgroups of G are solvable so $G/\Phi(G)$ is nonabelian simple. By induction, $\Phi(G) = \{1\}$. Let p be the minimal prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$; then P is noncyclic (Burnside). By assumption, $\lambda(P) = 2$ so P is abelian. Again, by Burnside,

$N_G(P) > P$. If $H/P \le N_G(P)$ is of prime order, then $\lambda(H) = 3$ so G is not a counterexample.

We also use freely the description of subgroups of the simple group $PSL(2, p^n)$ [D].

Theorem 4. *If G is a nonsolvable group and $n_\lambda(G) = 3$, then one of the following holds:*

(a) $G \cong PSL(2, 5)$.
(b) $G \cong PSL(2, p)$, $\lambda(p \pm 1) \le 3$, $p^2 \not\equiv 1 \pmod 5$, $|G| \not\equiv 0 \pmod 8$.

Proof. (i) Suppose that G is a nonabelian simple. Note that a nonsolvable group contains a subgroup S with $\lambda(S) = 3$. It follows that G has no proper subgroup H with $\lambda(H) > 3$. Thus, all proper subgroups H of G are solvable and $\lambda(H) \le 3$. In what follows we do not use the Odd Order Theorem. Let p be the minimal prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$; then P is noncyclic (Burnside). By what has just has been said, $p^2 \le |P| \le p^3$.

(i1) Let $|P| = p^3$. Let $H < G$ be an $S(q, p)$-subgroup (H exists, by Frobenius Normal p-Complement Theorem). Then $\lambda(H) = 3$ so $p = 2$ and $H \cong A_4$, the alternating group of degree 4. Let $P_1 \in \mathrm{Syl}_p(H)$. One may assume that that $P_1 < P$. Then $N_G(P_1) \ge \langle P, H\rangle$ so $\lambda(N_G(P_1)) > 3$, contrary to the previous paragraph.

(i2) Let $|P| = p^2$; then $p = 2$. In that case, $G \cong PSL(2, p)$, by [W]. Since all subgroups of G are known [D], it follows that G is as in (b).

Next we assume that G is not simple. Let $M < G$ be a maximal normal subgroup. Then G/M is simple.

(ii) Suppose that M is solvable; then $\bar{G} = G/M$ is nonabelian simple so, by Theorem 3, $n_\lambda(\bar{G}) = 3$. Let $\bar{H} < \bar{G}$ and $\bar{F} < \bar{G}$, where $\lambda(\bar{H}) = 1$ and $\lambda(\bar{F}) = 3$. Since all nonidentity subgroups of $\bar{F}$ are not $\bar{G}$-invariant, it follows that $\mathcal{N}_\lambda(G) = \{\lambda(M) + 1, \lambda(M) + 2, \lambda(M) + 3\}$. Then all proper subgroups of H are normal in G so H is a cyclic q-subgroup for prime $q = |\bar{H}|$. In particular, M is a cyclic q-subgroup. As q one can choose every prime from $\pi(\bar{G})$. Since $|\pi(\bar{G})| > 2$ (Burnside), we get a contradiction.

(iii) Suppose that M is nonsolvable. Then $n_\lambda(M) > 2$ (Theorem 3) so $n_\lambda(M) = 3$ since $n_\lambda(M) \le n_\lambda(G) = 3$. It follows that all subgroups of the simple group G/M are normal so $|G : M| = q$, a prime. By induction, M is a group from conclusion. It follows that $\mathcal{N}_\lambda(M) = \{1, 2, 3\} = \mathcal{N}_\lambda(G)$. Let $P \in \mathrm{Syl}_2(M)$. Then, by Frattini's Lemma, $G = MN_G(P)$. Since $N_M(P) > P$, we get $\lambda(N_G(P)) > 3$, and this is a final contradiction since all subgroups of $N_G(P)$, containing P, are not normal in G. ◻

I do not know if the number of groups of Theorem 4(b) is infinite.

Let $n_{s\lambda}(G) = |\{\lambda(H) \mid H < G \text{ is nonnormal and solvable }\}|$. Arguing as in the proof of Theorem 4, one can prove that if $n_{s\lambda}(G) = 3$, then a nonsolvable group G is as in Theorem 4 but in (b) the condition $p^2 \not\equiv 1 \pmod 5$ must be omitted.

REMARK 5. Here we consider a nonnilpotent group G all of whose nonnormal subgroups are cyclic.[1] Suppose that G has a noncyclic Sylow subgroup. By hypothesis, all noncyclic Sylow subgroups are G-invariant. If $P \in \mathrm{Syl}_p(G)$ is noncyclic, then $P \triangleleft G$ and G/P is Dedekindian. It follows that G is solvable (this is also true if P does not exist, by Burnside). Since G is nonnilpotent, P is the unique noncyclic Sylow subgroup of G. Since all Sylow subgroups of the Dedekindian group G/P are cyclic, G/P is itself cyclic, and we get $G' \leq P$. Let K be a Carter subgroup of G. Then K is cyclic and maximal in G since, if $K < M < G$, then M is noncyclic so normal in G, which is impossible. It follows that $G = KP$ so $P_0 = K \cap P \triangleleft G$. Since $K \leq \mathrm{C}_G(P_0) \triangleleft G$, we conclude that $P_0 \leq \mathrm{Z}(G) < K$. Write $\bar{G} = G/\mathrm{Z}(G)$. Then $\bar{G} = \bar{K} \cdot \bar{P}$ is a Frobenius group with cyclic complement $\bar{K}$ and kernel $\bar{P}$ which is a minimal normal subgroup of $\bar{G}$. Since $\bar{P}$ is elementary abelian, it has no proper subgroup of order p^2 so $|\bar{P}| \leq p^2$. Suppose that $|\bar{P}| = p^2$ and $P_0 > \{1\}$. If $P_1/P_0 < P/P_0$ is of order p, then P_1 is cyclic since it is not normal in G. It follows that P has $p+1$ distinct cyclic subgroups of index p; note that $\mathrm{Z}(G)$ is cyclic. Since P is noncyclic, we get $P \cong \mathrm{Q}_8$ hence $|P_0| = 2$, $|\bar{K}| = 3$. If Z_0 is a subgroup. of index 2 in $\mathrm{Z}(G)$, then $G/Z_0 \cong \mathrm{SL}(2,3)$. As it is easy to see, such G satisfies the hypothesis. Now let $P_0 = \{1\}$. Then $P \cong \mathrm{E}_{p^2}$, $G = K \cdot P$. We have $K_G = \mathrm{Z}(G)$. Write $\bar{G} = G/\mathrm{Z}(G)$. Since K is cyclic and maximal in G, the group G satisfies the hypothesis if and only if for every $\bar{H}/\bar{P}$ of prime order, $\bar{H}$ is minimal nonabelian. If $|\bar{P}| = p$, then G satisfies the hypothesis.

REMARK 6. Suppose that all subgroups of prime order $p > 2$ are G-invariant. Then G' is p-nilpotent. Indeed, all normal cyclic subgroups of G centralize G'. It follows from (MNN) that G' has no $\mathrm{S}(q,p)$-subgroups so p-nilpotent, by Frobenius' Normal p-Complement Theorem. In particular, if all subgroups of odd prime orders are G-invariant, then G'/P is nilpotent for $P \in \mathrm{Syl}_2(G)$.

REMARK 7. In the proof of Theorem 8 we use the following fact: If all subgroups of G of order 4 are normal, then G is solvable. Isaacs in his letter at 4/08/07 noticed that, under this condition, either G is 2-nilpotent or its Sylow 2-subgroup is normal and elementary abelian and has proved this. Below we offer another proof of Isaacs' assertion. Assume that G is not 2-nilpotent. Let $P_1 \in \mathrm{Syl}_2(G)$. By Frobenius' Normal p-Complement Theorem, there is in G an $\mathrm{S}(q,2)$-subgroup $S = Q \cdot P$; then $\exp(P) \leq 4$, by (MNN). If $L < P$ is cyclic of order 4, then L centralizes $G' > P$, a contradiction since $\exp(\mathrm{Z}(P)) = 2$. Thus, $\exp(P) = 2$; then $P \cap \mathrm{Z}(S) = \{1\}$. If $|P| > 4$, it has two subgroups

[1]The p-groups all of whose nonnormal subgroups are cyclic, are almost classified in [P, Theorem 2.9].

A, B of order 4 such that $A\cap B$ is of order 2. Then $\{1\} < A\cap B \leq P\cap \mathrm{Z}(S)$, a contradiction. Thus, $|P| = 4$ so P is minimal normal subgroup in S. Assume that $P < U \leq P_1$, where $|U : P| = 2$. Then U contains a subgroup $X \neq P$ of order 4. In that case, $P\cap X$ is of order 2 and contained in $\mathrm{Z}(S)\cap P$, a contradiction. Thus, if G is not 2-nilpotent, its Sylow 2-subgroup is normal in G and $\cong \mathrm{E}_4$.

THEOREM 8. *Let G be a nonsolvable group with $\mathrm{n}_\lambda(G) \leq 4$. Then one of the following holds:*

(a) $G \cong \mathrm{PSL}(2,p)$, where $\lambda(p\pm 1) \leq 4$.
(b) $G \cong \mathrm{PSL}(2,8)$.
(c) $G \cong \mathrm{PSL}(2,3^2) \cong \mathrm{A}_6$.
(d) $G \cong PSL(2,3^3)$.
(e) $G = G_1 \times C$, where G_1 is a group of Theorem 4 and $|C|$ prime.
(f) $G \cong \mathrm{PGL}(2,p)$, where either $p = 5,7$ or p is as in Theorem 4(b).
(g) $G \cong \mathrm{SL}(2,p)$, where p is such as in Theorem 4.

PROOF. (i) (This part is proved by Kazarin) Suppose that G is nonabelian simple; then $\lambda(H) \leq 4$ for all solvable $H < G$. Let $\{1\} < R < G$ be primary. Set $N = \mathrm{N}_G(R)$. Assume that N is nonsolvable. Then there is a solvable $F/R < N/R$ with $\lambda(F/R) > 2$ (Theorem 4). Since all subgroups of (the solvable subgroup) F are nonnormal in G, it follows that $\lambda(R) = 1$ and $\lambda(F) = 4$. Then $N/R \cong PSL(2,p)$ is as in Theorem 4 so $\{1,2,3,4,\lambda(N)\} \subseteq \mathcal{N}_\lambda(G)$, a contradiction since $\lambda(N) > 4$. Thus, all local subgroups of G are solvable. By Thompson's Theorem [T], G is isomorphic with one of the following groups:

$$(1) \quad PSL(2,p^n),\ Sz(2^{2m+1}),\ PSL(3,3),\ PSU(3,3),\ A_7,\ M_{11},\ {}^2F_4(2)'.$$

Since Sylow 2-subgroups of G have order at most 16, this excludes groups $\mathrm{Sz}(2^{2m+1})$ and ${}^2F_4(2)'$. Note, that if the order of a Sylow 2-subgroup S of G is 16, then S is maximal in G. Hence M_{11}, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$ and A_7 does not satisfy the hypothesis. Now it remains to discuss groups $\mathrm{PSL}(2,p^n)$ only. Note that $\mathrm{PSL}(2,p^n)$ contains the subgroup $\cong \mathrm{PSL}(2,p)$.

(i1) Let $G \cong \mathrm{PSL}(2,2^n)$, $n > 1$. Then G has a solvable subgroup H of order $2^n(2^n-1)$. Since $4 \geq \lambda(H) = n + \lambda(2^n-1)$, we get $n \leq 3$. If $n = 2$, then $G \cong \mathrm{PSL}(2,4) \cong \mathrm{A}_5$; then $\mathrm{n}_\lambda(G) = 3$. If $n = 3$, then $G \cong \mathrm{PSL}(2,8)$ and $\mathrm{n}_\lambda(G) = 4$.

(i2) Let $G \cong \mathrm{PSL}(2,p^n)$, $n > 1$, $p > 2$. Then G has a solvable subgroup H of order $\frac{1}{2}(p^n-1)p^n$. We have $n - 1 + \lambda(p^n-1) = \lambda(H) \leq 4$.

Let $p = 3$. Since $\lambda(3^n-1) \geq 2$, we get $n \leq 3$. If $n = 2$, then $G \cong \mathrm{PSL}(2,3^2) \cong \mathrm{A}_6$. If $n = 3$, then $G \cong \mathrm{PSL}(2,3^3)$. Both these groups satisfy the hypothesis.

Now let $p > 3$, $n > 1$. Since $\lambda(\frac{1}{2}(p^n-1)) \geq 2$, we must have $n = 2$. Then $\lambda(\frac{1}{2}(p^2-1)) \geq 3$ so $\lambda(\frac{1}{2}p^n(p^n-1)) = n + \lambda(\frac{1}{2}(p^2-1)) > 4$, contrary to the hypothesis.

(i3) Let $G \cong \mathrm{PSL}(2,p)$. If $F < G$ is nonsolvable, then $F \cong PSL(2,5)$ so $\lambda(F) = 4$. If $H < G$ is solvable, then $\lambda(H) \le 4$. Since all subgroups of G are known, we get $\lambda(p \pm 1) \le 4$. The case where G is simple, is complete.

(ii) Suppose that the Fitting subgroup $\mathrm{F}(G) > \{1\}$. Let R be the solvable radical of G; then G/R has no nonidentity solvable normal subgroup. Write $a = \lambda(R)$. If $H/R < G/R$ is solvable, then $\lambda(H/R) \le 4$ (otherwise, $\mathrm{n}_\lambda(G/R) > 4$) and all nonidentity subgroups of H/R are not G-invariant.

Let $H/R < G/R$ with $\lambda(H/R) = 3$ (such H exists, by Theorem 4). Then $a+1, a+2, a+3 \in \mathcal{N}_\lambda(G)$. By Remarks 6 and 7, G has a nonnormal subgroups of orders $r > 2$ and 4. It follows that $a = 1$ and so $\mathcal{N}_\lambda(G) = \{1,2,3,4\}$ so $|R| = p$, a prime. In that case, $\mathrm{n}_\lambda(G/R) = 3$ so G/R is a group of Theorem 4; in particular, G is nonabelian simple. Then $\mathrm{C}_G(R) = G$ so $R = \mathrm{Z}(G)$. If $R < G'$, then R is a subgroup of the Schur multiplier of the group G/R so $G \cong \mathrm{SL}(2,p)$, where either $p = 5$ or p is as in Theorem 4(b) (Schur [Sc]). If $R \not\le G'$, then $G = G_1 \times R$, where G_1 is as in Theorem 4.

(iii) Now suppose that $\mathrm{F}(G) = \{1\}$ and G is not simple; then G has no nonidentity solvable normal subgroup. By assumption, $\lambda(H) \le 4$ for all solvable $H < G$. Let N be a maximal normal subgroup of G. Then, by assumption, N is nonsolvable.

Assume that G/N is nonabelian simple. Let $P \in \mathrm{Syl}_2(N)$ and $K = \mathrm{N}_G(P)$. We have $|P| \ge 4$ and all nonidentity subgroups of P and K/P are not G-invariant. It follows that K/P has no solvable subgroup F/K such that $\lambda(F/K) > 2$. Then, by Theorem 4, K/P is solvable so $\lambda(K) \le 4$. However, $NK = G$ (Frattini's Lemma) so K cover the nonsolvable group G/N, and we conclude that K is nonsolvable, a contradiction.

Thus, $|G/N| = p$, a prime, so $G' \le N$. Assume that $G' < N$; then $\lambda(G/G') \ge 2$. Let $P \in \mathrm{Syl}_2(G')$; then $\lambda(\mathrm{N}_{G'}(P)) \ge 3$ (Burnside) and $\mathrm{N}_{G'}(P)$ is solvable (here we use the Odd Order Theorem) so $\lambda(\mathrm{N}_G(P)) \ge 5$. Since $\mathrm{N}_G(P)$ is solvable (Frattini's Lemma), we get $\mathrm{F}(G) > \{1\}$, a contradiction. Thus, $|G : G'| = p$ so that $G' = N$. The same argument shows that $G'' = G'$, i.e., G' is the last member of the derived series of G. Let R be a minimal normal subgroup of G. Then $R \le G'$ and, as above, G/R is solvable so $R = G'$. It follows that $R = R_1 \times \cdots \times R_k$, where $R_1 \cong \ldots \cong R_k$ are nonabelian simple. In that case, R has a solvable subgroup H such that $4 \ge \lambda(H) = 3k$ so $k = 1$ since $\mathrm{F}(G) = \{1\}$. Thus, R is nonabelian simple. If $\lambda(H) \le 3$ for all solvable $H < R$, then R is a group of Theorem 4, and then $G \cong \mathrm{PGL}(2,p)$, where either $p = 5$ or p is as in Theorem 4(b) (recall that in the case under consideration, $\mathrm{Aut}(R) \cong \mathrm{PGL}(2,p)$).

In what follows we assume that there is a solvable $H < R$ with $\lambda(H) = 4$. Since the normalizer of every nonidentity solvable subgroup of R is solvable, R is one of groups of list (1). As in (i), we have only to check the case where $R \cong \mathrm{PSL}(2,p^n)$. If $p = 2$, then $R \cong \mathrm{PSL}(2,8)$. In that case, if $P \in \mathrm{Syl}_2(R)$, then $\mathrm{N}_R(P) = C \cdot P$ is of order $2^3 \cdot 7$ so $\mathrm{N}_G(P) = 2^3 \cdot 7p$ (in fact, $p = 3$ since

$|\mathrm{Aut}(\mathrm{PSL}((2,8):\mathrm{PSL}(2,8)|=3)$. Since $\mathrm{N}_G(P)$ is solvable and $\lambda(\mathrm{N}_G(P))=5$, we get $\mathrm{F}(G)>\{1\}$, a contradiction. If $p=3$, then $R\cong\mathrm{PSL}(2,3^n)$, $n\in\{2,3\}$. Let $Q\in\mathrm{Syl}_3(\mathrm{R})$. Assume that $n=2$. Then $\lambda(\mathrm{N_R(Q)})=4$ so $\lambda(\mathrm{N_G(Q)})=5$ and, as above, we get a contradiction since $\mathrm{N}_G(Q)$ is solvable. Now let $n=3$. Then $\lambda(\mathrm{N_R(Q)})=4$ so $\lambda(\mathrm{N_G(Q)})=5$, and we again get a contradiction.

Thus, $p>3$. In that case, $n=1$ and $\mathrm{Aut}(R)\cong\mathrm{PGL}(2,p)$ so $G\cong\mathrm{PGL}(2,p)$. Assume that $H<R$ is dihedral of order $p\pm1$ with $\lambda(p\pm1)=4$. Then $\lambda(\mathrm{N}_G(H))=5$, $\mathrm{N}_G(H)$ is solvable so $\mathrm{F}(G)>\{1\}$, a contradiction. Thus, p is such as in Theorem 4, completing the proof. □

I think that if $|\Delta(G)|=\mathrm{n}_\lambda(G)$, then $|\pi(G)|$ must be small.

Let $\mathrm{no}(G)$ be as above. Then, if G is nonsolvable, then $\mathrm{no}(G)\geq\lambda(G)+|\pi(G)|\geq7$ [B2].

Problems

1. Classify the p-groups G satisfying $\mathrm{n}_\lambda(G)=2$.
2. Classify the nonnilpotent groups G satisfying $\mathrm{n}_\lambda(G)=3$.
3. Classify the nonsolvable groups G satisfying $\mathrm{n}_\lambda(G)=5$.
4. Let $\mathrm{n}_{s\lambda}(G)$ is defined in the paragraph preceding Remark 5. Classify the nonsolvable groups G satisfying $\mathrm{n}_{s\lambda}(G)\in\{4,5\}$.
5. Classify the groups G such that $\mathrm{n}_{s\lambda}(G)\leq|\pi(G)|$.
6. Classify the groups all of whose minimal nonabelian subgroups (S-subgroups) have the same order.
7. Classify the groups with $|\Delta(G)|=\mathrm{n}_\lambda(G)$.
8. Classify the p-groups all of whose nonnormal subgroups are metacyclic (abelian).
9. Find $\mathrm{n}_\lambda(\mathrm{A}_n)$, $\mathrm{n}_\lambda(\mathrm{S}_n)$.

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