

THE STRUCTURES OF STANDARD (\mathfrak{g}, K) -MODULES OF $SL(3, \mathbf{R})$

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ABSTRACT. We describe explicitly the structures of standard (\mathfrak{g}, K) -modules of $SL(3, \mathbf{R})$.

1. INTRODUCTION

For an admissible representation of a real reductive Lie group, the (\mathfrak{g}, K) -module structure is a fundamental data of those. As far as we know, for some ‘small’ reductive Lie groups G , the (\mathfrak{g}, K) -module structures of standard representations are completely described. For example, the description of them for $SL(2, \mathbf{R})$ is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. $SU(n, 1)$ by Kraljević [5] and $Spin(1, 2n)$ by Thieleker [12]. Moreover, in recent years, many authors give the explicit description of degenerate principal series representations of several groups, e.g. Fujimura [2], Howe and Tan [4], Lee [6], Lee and Loke [7]. However, for standard representations of Lie groups of higher rank, there are few references as far as the author knows. It seems to be difficult to describe the whole (\mathfrak{g}, K) -module structures of those representations, since their K -types are not multiplicity free. In the paper [11], the (\mathfrak{g}, K) -module structures of principal series representations of $Sp(2, \mathbf{R})$ are described by Oda. In a former paper [10], we extend the result for principal series representations of $Sp(3, \mathbf{R})$. The method in these papers is applicable to study of standard representations of other groups. In this paper, we use this method to study standard (\mathfrak{g}, K) -modules of $SL(3, \mathbf{R})$.

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Before describing the case of $SL(3, \mathbf{R})$, let us explain the problem in a more precise form for a general real semisimple Lie group G with its Lie algebra \mathfrak{g} . Fix a maximal compact subgroup K of G . Since standard (\mathfrak{g}, K) -modules are realized as subspaces of $L^2(K)$ as a K -module, we can investigate those K -module structure by Peter-Weyl's theorem. In order to describe the action of \mathfrak{g} or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of \mathfrak{p} or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. To study the action of $\mathfrak{p}_{\mathbf{C}}$, we compute the linear map $\Gamma_{\tau, i}$ defined as follows.

Let (π, H_{π}) be a standard representation of G with its subspace $H_{\pi, K}$ of K -finite vectors. For a K -type (τ, V_{τ}) of π and a K -homomorphism $\eta: V_{\tau} \rightarrow H_{\pi, K}$, we define a linear map $\tilde{\eta}: \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_{\tau} \rightarrow H_{\pi, K}$ by $X \otimes v \mapsto X \cdot \eta(v)$. Then $\tilde{\eta}$ is a K -homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K . Let $V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq \bigoplus_{i \in I} V_{\tau_i}$ be the decomposition into a direct sum of irreducible K -modules and fix ι_i an injective K -homomorphism from V_{τ_i} to $V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$ for each i . We define a linear map $\Gamma_{\tau, i}: \text{Hom}_K(V_{\tau}, H_{\pi, K}) \rightarrow \text{Hom}_K(V_{\tau_i}, H_{\pi, K})$ by $\eta \mapsto \tilde{\eta} \circ \iota_i$. These linear maps $\Gamma_{\tau, i}$ ($i \in I$) characterize the action of $\mathfrak{p}_{\mathbf{C}}$. The goal of this paper is to give explicit expressions of ι_i and $\Gamma_{\tau, i}$ for any standard representation π of $G = SL(3, \mathbf{R})$. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_{\pi}[\tau]$ and $H_{\pi}[\tau_i]$. Here $H_{\pi}[\tau]$ is τ -isotypic component of H_{π} . These are described in Proposition 3.2, Theorem 4.5 and 5.5. We remark that R. Howe give another description of $\Gamma_{\tau, i}$ in [3] when π is a principal series representation of $GL(3, \mathbf{R})$.

As an application, we can utilize the contiguous relations to obtain the explicit formulae of some spherical functions. In the paper [8], Manabe, Ishii and Oda give the explicit formulae of Whittaker functions for principal series representations of $SL(3, \mathbf{R})$ to solve the holonomic system of differential equations characterizing those functions, which is derived from the Capelli elements and the contiguous relations around minimal K -type. We can obtain the holonomic systems characterizing Whittaker functions for standard representations of $SL(3, \mathbf{R})$ induced from the maximal parabolic subgroup by using the result of this paper. We give the explicit formulae of Whittaker functions by solving this system in [9]. On the other hand, if we have the explicit formula of Whittaker function with a certain K -type, then we can give those with another K -type by using contiguous relations.

We give the contents of this paper. In Section 2, we recall the structure of $SL(3, \mathbf{R})$ and define standard representations. In Section 3, we introduce the standard basis of a finite dimensional irreducible representation of K and give explicit expressions of $\iota_i: V_{\tau_i} \rightarrow V_{\tau} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}$. In Section 4, we introduce the general setting of this paper and give matrix representations of $\Gamma_{\tau, i}$ for principal series representations in Theorem 4.5. In Section 5, we give the matrix representations of $\Gamma_{\tau, i}$ for standard representations of $SL(3, \mathbf{R})$ induced

from the maximal parabolic subgroup in Theorem 5.5. In Section 6, we give explicit expressions of the action of $\mathfrak{p}_{\mathbf{C}}$ in Proposition 6.2.

2. PRELIMINARIES

2.1. *Groups and algebras.* We denote by \mathbf{Z} , \mathbf{R} and \mathbf{C} the ring of rational integers, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq 0}$ be the set of non-negative integers, 1_n the unit matrix of size n and $O_{m,n}$ the zero matrix of size $m \times n$ and E_{ij} the matrix of size 3 with 1 at (i, j) -th entry and 0 at other entries. We denote by δ_{ij} the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For a Lie algebra \mathfrak{l} , we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of \mathfrak{l} .

Let G be the special linear group $SL(3, \mathbf{R})$ of degree three and \mathfrak{g} its Lie algebra. We define a Cartan involution θ of G by $G \ni g \mapsto {}^t g^{-1} \in G$. Here ${}^t g$ and g^{-1} means the transpose and the inverse of g , respectively. Then the maximal compact subgroup of G is given by

$$K = \{g \in G \mid \theta(g) = g\} = SO(3).$$

If we denote the differential of θ again by θ , then we have $\theta(X) = -{}^t X$ for $X \in \mathfrak{g}$. Let \mathfrak{k} and \mathfrak{p} be the $+1$ and the -1 eigenspaces of θ in \mathfrak{g} , respectively, that is,

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid {}^t X = -X\} = \mathfrak{so}(3), \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid {}^t X = X\}.$$

Then \mathfrak{k} is the Lie algebra of K and \mathfrak{g} has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Put $\mathfrak{a}_0 = \{\text{diag}(t_1, t_2, t_3) \mid t_i \in \mathbf{R} (1 \leq i \leq 3), t_1 + t_2 + t_3 = 0\}$. Then \mathfrak{a}_0 is a maximal abelian subalgebra of \mathfrak{p} . For each $1 \leq i \leq 3$, we define a linear form e_i on \mathfrak{a}_0 by $\mathfrak{a}_0 \ni \text{diag}(t_1, t_2, t_3) \mapsto t_i \in \mathbf{C}$. The set Σ of the roots for $(\mathfrak{a}_0, \mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_0, \mathfrak{g}) = \{e_i - e_j \mid 1 \leq i \neq j \leq 3\}$, and the subset $\Sigma^+ = \{e_i - e_j \mid 1 \leq i < j \leq 3\}$ forms a positive root system. For each $\alpha \in \Sigma$, we denote the root space by \mathfrak{g}_{α} and choose a root vector E_{α} in \mathfrak{g}_{α} by $E_{e_i - e_j} = E_{ij} (1 \leq i \neq j \leq 3)$.

If we put $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{\alpha}$, then \mathfrak{g} has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$. Also we have $G = N_0 A_0 K$, where $N_0 = \exp(\mathfrak{n}_0)$ and $A_0 = \exp(\mathfrak{a}_0)$.

Let $\mathfrak{n}_1, \mathfrak{n}_2$ be subalgebras of \mathfrak{n}_0 defined by $\mathfrak{n}_1 = \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_1 - e_3}$, $\mathfrak{n}_2 = \mathfrak{g}_{e_1 - e_3} \oplus \mathfrak{g}_{e_2 - e_3}$. We take a basis $\{H_1, H_2\}$ of \mathfrak{a}_0 by

$$H_1 = \text{diag}(1, 0, -1), \quad H_2 = \text{diag}(0, 1, -1),$$

and set $H^{(1)} = 2H_1 - H_2$, $H^{(2)} = H_1 + H_2$. we define subalgebras $\mathfrak{a}_1, \mathfrak{a}_2$ of \mathfrak{a}_0 by $\mathfrak{a}_1 = \mathbf{R} \cdot H^{(1)}$, $\mathfrak{a}_2 = \mathbf{R} \cdot H^{(2)}$. The group G has three non-trivial standard parabolic subgroups P_0, P_1, P_2 with Langland decompositions $P_i = N_i A_i M_i (0 \leq i \leq 2)$ where

$$M_0 = \{\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2) \mid \varepsilon_i \in \{\pm 1\} (1 \leq i \leq 2)\},$$

$$\begin{aligned}
 M_1 &= \left\{ \left(\begin{array}{cc} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{array} \right) \middle| h \in SL^\pm(2, \mathbf{R}) \right\}, \\
 M_2 &= \left\{ \left(\begin{array}{cc} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{array} \right) \middle| h \in SL^\pm(2, \mathbf{R}) \right\}, \\
 A_i &= \exp(\mathfrak{a}_i) \quad N_i = \exp(\mathfrak{n}_i) \quad (i = 1, 2).
 \end{aligned}$$

Here $SL^\pm(2, \mathbf{R}) = \{g \in GL(2, \mathbf{R}) \mid \det(g) = \pm 1\}$. For $i = 1, 2$, let \mathfrak{m}_i be a Lie algebra of M_i .

2.2. *Definition of the P_i -principal series representations of G .* For $0 \leq i \leq 2$, in order to define the P_i -principal series representation of G , we prepare the data (ν_i, σ_i) as follows.

For $\nu_0 \in \text{Hom}_{\mathbf{R}}(\mathfrak{a}_0, \mathbf{C})$, we define a coordinate $(\nu_{0,1}, \nu_{0,2}) \in \mathbf{C}^2$ by $\nu_{0,i} = \nu_0(H_i)$ ($i = 1, 2$). Then the half sum $\rho_0 = e_1 - e_3$ of the positive roots has coordinate $(\rho_{0,1}, \rho_{0,2}) = (2, 1)$. We define a quasicharacter $e^{\nu_0} : A_0 \rightarrow \mathbf{C}^\times$ by

$$e^{\nu_0}(a) = a_1^{\nu_{0,1}} a_2^{\nu_{0,2}}, \quad a = \text{diag}(a_1, a_2, a_3) \in A_0.$$

We fix a character σ_0 of M_0 . σ_0 is realized by $(\sigma_{0,1}, \sigma_{0,2}) \in \{0, 1\}^{\oplus 2}$ such that

$$\sigma_0(\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}}, \quad \varepsilon_1, \varepsilon_2 \in \{\pm 1\}.$$

For each $i = 1, 2$, we identify $\nu_i \in \text{Hom}_{\mathbf{R}}(\mathfrak{a}_i, \mathbf{C})$ with a complex number $\nu_i(H^{(i)}) \in \mathbf{C}$. Let ρ_i ($i = 1, 2$) be the half sums of positive roots whose root spaces are contained in \mathfrak{n}_i , i.e. $\rho_1 = \frac{1}{2}(2e_1 - e_2 - e_3)$, $\rho_2 = \frac{1}{2}(e_1 + e_2 - 2e_3)$. Then both ρ_1 and ρ_2 are identified with 3. We fix a discrete series representation σ_i of $M_i \simeq SL^\pm(2, \mathbf{R})$ for $i = 1, 2$.

DEFINITION 2.1. *For $0 \leq i \leq 2$, we define the P_i -principal series representation $\pi_{(\nu_i, \sigma_i)}$ of G by*

$$\pi_{(\nu_i, \sigma_i)} = \text{Ind}_{P_i}^G(1_{N_i} \otimes e^{\nu_i + \rho_i} \otimes \sigma_i),$$

i.e. $\pi_{(\nu_i, \sigma_i)}$ is the right regular representation of G on the space $H_{(\nu_i, \sigma_i)}$ which is the completion of

$$H_{(\nu_i, \sigma_i)}^\infty = \left\{ f : G \rightarrow V_{\sigma_i} \text{ smooth} \middle| \begin{array}{l} f(namx) = e^{\nu_i + \rho_i}(a)\sigma_i(m)f(x) \\ \text{for } n \in N_i, a \in A_i, m \in M_i, x \in G \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_K \|f(k)\|_{\sigma_i}^2 dk.$$

Here V_{σ_i} is a representation space of σ_i and $\|\cdot\|_{\sigma_i}$ is its norm.

REMARK 2.2. *The P_i -principal series representations are also called standard representations or generalized principal representations.*

3. REPRESENTATIONS OF $K = SO(3)$

3.1. *The spinor covering.* To describe the finite dimensional representations of $SO(3)$, the simplest way seems to be the one utilizing the double covering $\varphi: SU(2) = Spin(3) \rightarrow SO(3)$. We use the following realization introduced in [8].

We define $\varphi: SU(2) \rightarrow SO(3)$ by

$$\varphi(x) = \begin{pmatrix} p^2 + q^2 - r^2 - s^2 & -2(ps - qr) & 2(pr + qs) \\ 2(ps + qr) & p^2 - q^2 + r^2 - s^2 & -2(pq - rs) \\ -2(pr - qs) & 2(pq + rs) & p^2 - q^2 - r^2 + s^2 \end{pmatrix}$$

for $x = \begin{pmatrix} p + \sqrt{-1}q & r + \sqrt{-1}s \\ -r + \sqrt{-1}s & p - \sqrt{-1}q \end{pmatrix} \in SU(2)$ ($p, q, r, s \in \mathbf{R}$). Then φ is surjective homomorphism whose kernel is given by $\{\pm 1_2\}$.

The differential $d\varphi: \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ of φ is an isomorphism and it maps the basis

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

to $-2K_{23}, 2K_{13}, -2K_{12}$. Here $K_{ij} = E_{ij} - E_{ji}$ ($1 \leq i < j \leq 3$).

3.2. *Representations of $SU(2)$.* The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor product τ_l ($l \in \mathbf{Z}_{\geq 0}$) of the representation $SU(2) \ni g \mapsto (v \mapsto g \cdot v) \in GL(\mathbf{C}^2)$. We use the following realizations of those which are introduced in [8].

Let V_l be the subspace consisting of degree l homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) = f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $\mathfrak{su}(2)$, the derivation of τ_l , denoted by same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k} \mid 0 \leq k \leq l\}$ of V_l and the basis $\{u_1, u_2, u_3\}$ of $\mathfrak{su}(2)$. Namely, we have

$$\tau_l(H)v_k = (l - 2k)v_k, \quad \tau_l(E)v_k = -kv_{k-1}, \quad \tau_l(F)v_k = (k - l)v_{k+1}.$$

Here $\{E, H, F\}$ is \mathfrak{sl}_2 -triple defined by

$$H = -\sqrt{-1}u_1, \quad E = \frac{1}{2}(u_2 - \sqrt{-1}u_3), \quad F = -\frac{1}{2}(u_2 + \sqrt{-1}u_3) \in \mathfrak{su}(2)_{\mathbf{C}}.$$

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect to $\varphi: SU(2) \rightarrow SO(3)$ is that $\tau_l(-1_2) = (-1)^l = 1$, i.e. l is even. For $l \in \mathbf{Z}_{\geq 0}$, we denote the irreducible representation of $SO(3)$ induced from (τ_{2l}, V_{2l}) again by (τ_{2l}, V_{2l}) .

3.3. *The adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$.* It is known that $\mathfrak{p}_{\mathbf{C}}$ becomes a K -module via the adjoint action of K . Concerning this, we have the following lemma.

LEMMA 3.1. *Let $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_4, V_4) and $\{X_j \mid 0 \leq j \leq 4\}$ be a basis of $\mathfrak{p}_{\mathbf{C}}$ defined as follows:*

$$\begin{aligned} X_0 &= H_2 - \sqrt{-1}(E_{23} + E_{32}), & X_1 &= -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) + (E_{13} + E_{31})\}, \\ X_2 &= -\frac{1}{3}(2H_1 - H_2), & X_3 &= -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) - (E_{13} + E_{31})\}, \\ X_4 &= H_2 + \sqrt{-1}(E_{23} + E_{32}). \end{aligned}$$

Then via the isomorphism between V_4 and $\mathfrak{p}_{\mathbf{C}}$ as K -modules we have the identification $w_j = X_j$ ($0 \leq j \leq 4$).

PROOF. By direct computation, we have Table. 1 of the adjoint actions of the basis $\{d\varphi(E), d\varphi(H), d\varphi(F)\}$ of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$. Comparing the actions in the above with the actions in Subsection 3.2, we obtain the assertion. \square

TABLE 1. The adjoint actions of $\mathfrak{k}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$.

	X_0	X_1	X_2	X_3	X_4
$d\varphi(H)$	$4X_0$	$2X_1$	0	$-2X_3$	$-4X_4$
$d\varphi(E)$	0	$-X_0$	$-2X_1$	$-3X_2$	$-4X_3$
$d\varphi(F)$	$-4X_1$	$-3X_2$	$-2X_3$	$-1X_4$	0

3.4. *Clebsch-Gordan coefficients for the representations of $\mathfrak{sl}(2, \mathbf{C})$ with respect to standard basis.* For later use, we consider the irreducible decomposition of $V_l \otimes_{\mathbf{C}} V_4$ as $\mathfrak{sl}(2, \mathbf{C}) = \mathfrak{su}(2)_{\mathbf{C}}$ -modules for arbitrary non-negative integer l .

Generically, the tensor product $V_l \otimes_{\mathbf{C}} V_4$ has five irreducible components $V_{l+4}, V_{l+2}, V_l, V_{l-2}$ and V_{l-4} . Here some components may vanish. We give an explicit expression of a nonzero $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism from each irreducible component to $V_l \otimes_{\mathbf{C}} V_4$ as follows.

PROPOSITION 3.2. *Let $\{v_k^{(l)} \mid 0 \leq k \leq l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\geq 0}$. We put $v_k^{(l)} = 0$ when $k < 0$ or $k > l$.*

If V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish, then we define linear maps $I_{2m}^l: V_{l+2m} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ ($-2 \leq m \leq 2$) by

$$I_{2m}^l(v_k^{(l+2m)}) = \sum_{i=0}^4 A_{[l, 2m; k, i]} \cdot v_{k+2-m-i}^{(l)} \otimes w_i.$$

Here the coefficients $A_{[l, 2m; k, i]} = a(l, 2m; k, i)/d(l, 2m)$ are defined by following formulae.

FORMULA 1: The coefficients of $I_4^l: V_{l+4} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 4; k, 0) &= (l+4-k)(l+3-k)(l+2-k)(l+1-k), \\ a(l, 4; k, 1) &= 4(l+4-k)(l+3-k)(l+2-k)k, \\ a(l, 4; k, 2) &= 6(l+4-k)(l+3-k)k(k-1), \\ a(l, 4; k, 3) &= 4(l+4-k)k(k-1)(k-2), \\ a(l, 4; k, 4) &= k(k-1)(k-2)(k-3), \\ d(l, 4) &= (l+4)(l+3)(l+2)(l+1). \end{aligned}$$

FORMULA 2: The coefficients of $I_2^l: V_{l+2} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 2; k, 0) &= (l+2-k)(l+1-k)(l-k), \\ a(l, 2; k, 1) &= -(l+2-k)(l+1-k)(l-4k), \\ a(l, 2; k, 2) &= -3(l+2-k)(l-2k+2)k, \\ a(l, 2; k, 3) &= -(3l-4k+8)k(k-1), \\ a(l, 2; k, 4) &= -k(k-1)(k-2), \quad d(l, 2) = (l+2)(l+1)l. \end{aligned}$$

FORMULA 3: The coefficients of $I_0^l: V_l \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, 0; k, 0) &= (l-k)(l-1-k), & a(l, 0; k, 1) &= -2(l-k)(l-2k-1), \\ a(l, 0; k, 2) &= (l^2 - 6kl + 6k^2 - l), & a(l, 0; k, 3) &= 2(l-2k+1)k, \\ a(l, 0; k, 4) &= k(k-1), & d(l, 0) &= l(l-1). \end{aligned}$$

FORMULA 4: The coefficients of $I_{-2}^l: V_{l-2} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, -2; k, 0) &= (l-k-2), & a(l, -2; k, 1) &= -(3l-4k-6), \\ a(l, -2; k, 2) &= 3(l-2k-2), & a(l, -2; k, 3) &= -(l-4k-2), \\ a(l, -2; k, 4) &= -k, & d(l, -2) &= l-2. \end{aligned}$$

FORMULA 5: The coefficients of $I_{-4}^l: V_{l-4} \rightarrow V_l \otimes_{\mathbf{C}} V_4$ are given as follows:

$$\begin{aligned} a(l, -4; k, 0) &= 1, & a(l, -4; k, 1) &= -4, & a(l, -4; k, 2) &= 6, \\ a(l, -4; k, 3) &= -4, & a(l, -4; k, 4) &= 1, & d(l, -4) &= 1. \end{aligned}$$

Then I_{2m}^l is a generator of $\text{Hom}_{\mathfrak{sl}(2, \mathbf{C})}(V_{l+2m}, V_l \otimes_{\mathbf{C}} V_4)$, which is unique up to scalar multiple.

PROOF. We have

$$\begin{aligned} &(\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)}) \\ &= \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot (\tau_l(E)v_{2-m-i}^{(l)}) \otimes w_i + \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot v_{2-m-i}^{(l)} \otimes (\tau_4(E)w_i) \end{aligned}$$

$$= - \sum_{i=0}^4 ((2 - m - i)A_{[l,2m;0,i]} + (i + 1)A_{[l,2m;0,i+1]}) \cdot v_{1-m-i}^{(l)} \otimes w_i.$$

Here we put $A_{[l,2m;0,5]} = 0$. By direct computation, we confirm

$$(2 - m - i)A_{[l,2m;0,i]} + (i + 1)A_{[l,2m;0,i+1]} = 0$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Hence

$$(\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)}) = 0.$$

Moreover, we have

$$(\tau_l \otimes \tau_4)(H) \circ I_{2m}^l(v_0^{(l+2m)}) = (l + 2m)I_{2m}^l(v_0^{(l+2m)}),$$

since

$$\begin{aligned} (\tau_l \otimes \tau_4)(H)(v_i^{(l)} \otimes w_j) &= (\tau_l(H)v_i^{(l)}) \otimes w_j + v_i^{(l)} \otimes (\tau_4(H)w_j) \\ &= (l + 4 - 2i - 2j)v_i^{(l)} \otimes w_j. \end{aligned}$$

This means $I_{2m}^l(v_0^{(l+2m)})$ is the highest weight vector of the V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ with respect to a Borel subalgebra $(\mathbf{C} \cdot H) \oplus (\mathbf{C} \cdot E)$ of $\mathfrak{sl}(2, \mathbf{C})$.

Therefore, in order to complete the proof, it suffices to confirm

$$(\tau_l \otimes \tau_4)(F) \circ I_{2m}^l(v_k^{(l+2m)}) = I_{2m}^l \circ \tau_{l+2m}(F)(v_k^{(l+2m)})$$

for each $0 \leq k \leq l + 2m$.

We confirm these equations by direct computation. □

The coefficients $A_{[l,2m;k,i]}$ in the above proposition satisfy the following relations.

LEMMA 3.3. *The coefficients $A_{[l,2m;k,i]}$ in Proposition 3.2 satisfy following relations:*

$$\begin{aligned} A_{[l,2m;l+2m-k,0]} &= (-1)^m A_{[l,2m;k,4]}, \quad A_{[l,2m;l+2m-k,2]} = (-1)^m A_{[l,2m;k,2]}, \\ 3\{(k - m + 1)A_{[l,2m;k,1]} + (l - k + m + 1)A_{[l,2m;k,3]}\} \\ &= (ml + m^2 + m - 6)A_{[l,2m;k,2]}. \end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq k \leq l + 2m$.

PROOF. These are obtained by direct computation. □

3.5. *The contragredient representation of (τ_l, V_l) .* We denote by (τ^*, V^*) the contragredient representation of (τ, V) . Here we note that V_l^* is equivalent to V_l as $SU(2)$ -modules, since the irreducible $l + 1$ -dimensional representation of $SU(2)$ is unique up to isomorphism.

LEMMA 3.4. Let $\{v_k^{(l)*} \mid 0 \leq k \leq l\}$ be the dual basis of the standard basis $\{v_k^{(l)} \mid 0 \leq k \leq l\}$. Via the isomorphism between V_l and V_l^* as K -modules we have the identification

$$v_k^{(l)} = (-1)^k \frac{(l-k)!k!}{l!} v_{l-k}^{(l)*}$$

for $0 \leq k \leq l$.

PROOF. We denote by \langle, \rangle the canonical pairing on $V_l^* \otimes_{\mathbf{C}} V_l$.

Since

$$\langle \tau_l^*(H)v_k^{(l)*}, v_m^{(l)} \rangle = -\langle v_k^{(l)*}, \tau_l(H)v_m^{(l)} \rangle = (2m-l)\delta_{km} = (2k-l)\delta_{km},$$

we have $\tau_l^*(H)v_k^{(l)*} = (2k-l)v_k^{(l)*}$. Similarly, we obtain

$$\tau_l^*(E)v_k^{(l)*} = (k+1)v_{k+1}^{(l)*}, \quad \tau_l^*(F)v_k^{(l)*} = (l-k+1)v_{k-1}^{(l)*}.$$

From these equations, the identification $v_0^{(l)} = v_l^{(l)*}$ determines the isomorphism in the statement. \square

4. THE (\mathfrak{g}, K) -MODULE STRUCTURES OF THE P_0 -PRINCIPAL SERIES REPRESENTATIONS

4.1. *The irreducible decomposition of $\pi_{(\nu_0, \sigma_0)}|_K$ as a K -module.* We set

$$L^2_{(M_0, \sigma_0)}(K) = \{f \in L^2(K) \mid f(mx) = \sigma_0(m)f(x) \text{ for a.e. } m \in M, x \in K\}$$

and give a K -module structure by the right regular action of K . Then the restriction map $r_K: H_{(\nu_0, \sigma_0)} \ni f \mapsto f|_K \in L^2_{(M_0, \sigma_0)}(K)$ is an isomorphism of K -modules.

$L^2(K)$ has a $K \times K$ -bimodule structure by the two sided regular action:

$$((k_1, k_2)f)(x) = f(k_1^{-1}xk_2), \quad x \in K, f \in L^2(K), (k_1, k_2) \in K \times K.$$

Then we define a homomorphism $\Phi_l: V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \rightarrow L^2(K)$ of $K \times K$ -bimodules by

$$w \otimes v \mapsto (x \mapsto \langle w, \tau_{2l}(x)v \rangle).$$

Then Peter-Weyl's theorem tells that

$$\widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} \Phi_l}: \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} V_{2l}^* \otimes_{\mathbf{C}} V_{2l}} \rightarrow L^2(K)$$

is an isomorphism as $K \times K$ -bimodules. Here $\widehat{\bigoplus}$ means a Hilbert space direct sum.

Since $L^2_{(M_0, \sigma_0)}(K) \subset L^2(K)$, we have an irreducible decomposition of $L^2_{(M_0, \sigma_0)}(K)$:

$$L^2_{(M_0, \sigma_0)}(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0])} \otimes_{\mathbf{C}} V_{2l}.$$

Here $V[\sigma_0]$ means the σ_0 -isotypic component in $(\tau|_{M_0}, V)$ for a K -module (τ, V) . Therefore we obtain an isomorphism

$$r_K^{-1} \circ \bigoplus_{l \in \mathbf{Z}_{\geq 0}} \widehat{\Phi}_l : \bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l} \rightarrow H_{(\nu_0, \sigma_0)}.$$

Since M_0 is generated by the two elements

$$m_{0,1} = \text{diag}(-1, 1, -1), \quad m_{0,2} = \text{diag}(1, -1, -1) \in M_0,$$

we note that $v \in V_{2l}[\sigma_0]$ if and only if

$$\tau_{2l}(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_{0,i}}v \quad (i = 1, 2)$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi_1^{-1}(m_{0,1}) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \varphi_1^{-1}(m_{0,2}) = \left\{ \pm \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\},$$

we have $\tau_{2l}(m_{0,1})v_k^{(2l)} = (-1)^k v_{2l-k}^{(2l)}$ and $\tau_{2l}(m_{0,2})v_k^{(2l)} = (-1)^{l-k} v_k^{(2l)}$. Hence we have

$$V_{2l}[\sigma_0] = \bigoplus_{k \in Z(\sigma_0; l)} \mathbf{C} \cdot (v_{2l-k}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)} v_k^{(2l)}),$$

where $\varepsilon(\sigma_0; l) \in \{0, 1\}$ such that $\varepsilon(\sigma_0; l) \equiv l - \sigma_{0,1} - \sigma_{0,2} \pmod{2}$ and

$$Z(\sigma_0; l) = \begin{cases} \{k \in \mathbf{Z} \mid 0 \leq k \leq l, k \equiv l - \sigma_{0,2} \pmod{2}\} & \text{if } \varepsilon(\sigma_0; l) = 0, \\ \{k \in \mathbf{Z} \mid 0 \leq k \leq l - 1, k \equiv l - \sigma_{0,2} \pmod{2}\} & \text{if } \varepsilon(\sigma_0; l) = 1. \end{cases}$$

We see that $\{v_{2l-k}^{(2l)*} + (-1)^{\varepsilon(\sigma_0; l)} v_k^{(2l)*} \mid k \in Z(\sigma_0; l)\}$ is the basis of $V_{2l}^*[\sigma_0]$, by using the identification $V_{2l}^* = V_{2l}$ in Lemma 3.4.

Now we define the elementary function $s(l; p, q) \in H_{(\nu_0, \sigma_0)}$ by

$$s(l; p, q) = r_K^{-1} \circ \Phi_l((v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0; l)} v_p^{(2l)*}) \otimes v_q^{(2l)})$$

for $l \in \mathbf{Z}_{\geq 0}$, $p \in Z(\sigma_0; l)$ and $0 \leq q \leq 2l$.

For each $p \in Z(\sigma_0; l)$, we put $S(l; p)$ a column vector of degree $2l+1$ whose $q+1$ -th component is $s(l; p, q)$, i.e. ${}^t(s(l; p, 0), s(l; p, 1), \dots, s(l; p, 2l))$.

Moreover we denote by $\langle S(l; p) \rangle$ the subspace of $H_{(\nu_0, \sigma_0)}$ generated by the functions in the entries of the vector $S(l; p)$, i.e. $\langle S(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot s(l; p, q) \simeq V_{2l}$. Via the isomorphism between $\langle S(l; p) \rangle$ and V_{2l} , we identify $\{s(l; p, q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

PROPOSITION 4.1. *As an unitary representation of K , it has an irreducible decomposition:*

$$H_{(\nu_0, \sigma_0)} \simeq \bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}.$$

Then the τ_{2l} -isotypic component of $\pi_{(\nu_0, \sigma_0)}$ is given by

$$\bigoplus_{p \in Z(\sigma_0; l)} \langle S(l; p) \rangle.$$

COROLLARY 4.2. The multiplicity $d(\sigma_0; l)$ of τ_{2l} in $\pi_{(\nu_0, \sigma_0), K}$ is given by

$$d(\sigma_0; l) = \begin{cases} (l+2)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is even,} \\ (l-1)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is odd,} \\ l/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is even,} \\ (l+1)/2 & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is odd.} \end{cases}$$

4.2. *General setting.* Let $H_{(\nu_i, \sigma_i), K}$ be the K -finite part of $H_{(\nu_i, \sigma_i)}$. In order to describe the action of \mathfrak{g} or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, it suffices to investigate the action of \mathfrak{p} or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

For a K -type (τ_{2l}, V_{2l}) of $\pi_{(\nu_i, \sigma_i)}$ and a K -homomorphism $\eta: V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$, we define a linear map

$$\tilde{\eta}: \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$$

by $X \otimes v \mapsto \pi_{(\nu_i, \sigma_i)}(X)\eta(v)$. Here we denote differential of $\pi_{(\nu_i, \sigma_i)}$ again by $\pi_{(\nu_i, \sigma_i)}$. Then $\tilde{\eta}$ is K -homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K .

Since

$$V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}} \simeq V_{2l} \otimes_{\mathbf{C}} V_4 \simeq \bigoplus_{-2 \leq m \leq 2} V_{2(l+m)},$$

there are five injective K -homomorphisms

$$I_{2m}^{2l}: V_{2(l+m)} \rightarrow V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}, \quad -2 \leq m \leq 2$$

for general $l \in \mathbf{Z}_{\geq 0}$. Then we define \mathbf{C} -linear maps

$$\Gamma_{l,m}^i: \text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K}) \rightarrow \text{Hom}_K(V_{2(l+m)}, H_{(\nu_i, \sigma_i), K}), \quad -2 \leq m \leq 2$$

by $\eta \mapsto \tilde{\eta} \circ I_{2m}^{2l}$.

Now we settle the goal of this paper:

- PROBLEM 4.1. (i) Describe the injective K -homomorphism I_{2m}^{2l} in terms of the standard basis.
 (ii) Determine the matrix representations of the linear homomorphisms $\Gamma_{l,m}^i$ with respect to the induced basis defined in the next subsection.

We have already accomplished (i) in Proposition 3.2. We accomplish (ii) in Theorem 4.5 and 5.5. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_{(\nu_i, \sigma_i)}[\tau_{2l}]$ and $H_{(\nu_i, \sigma_i)}[\tau_{2(l+m)}]$. Here $H_{(\nu_i, \sigma_i)}[\tau]$ is τ -isotypic component of $H_{(\nu_i, \sigma_i)}$.

4.3. *The canonical blocks of elementary functions.* Let $\eta: V_{2l} \rightarrow H_{(\nu_i, \sigma_i), K}$ be a non-zero K -homomorphism. Then we identify η with the column vector of degree $2l + 1$ whose $q + 1$ -th component is $\eta(v_q^{(2l)})$ for $0 \leq q \leq 2l$, i.e. ${}^t(\eta(v_0^{(2l)}), \eta(v_1^{(2l)}), \dots, \eta(v_{2l}^{(2l)}))$.

By this identification, we identify $S(l; p)$ with the K -homomorphism

$$V_{2l} \ni v_q^{(2l)} \mapsto s(l; p, q) \in H_{(\nu_0, \sigma_0), K}, \quad 0 \leq q \leq 2l$$

for $p \in Z(\sigma_0; l)$. We note that $\{S(l; p) \mid p \in Z(\sigma_0; l)\}$ is a basis of the space $\text{Hom}_K(V_{2l}, H_{(\nu_0, \sigma_0), K})$ and we call it *the induced basis from the standard basis*.

We define a certain matrix of elementary functions corresponding to the induced basis $\{S(l; p) \mid p \in Z(\sigma_0; l)\}$ of $\text{Hom}_K(V_{2l}, H_{(\nu_0, \sigma_0), K})$ for each K -type τ_{2l} of our principal series representation $\pi_{(\nu_0, \sigma_0)}$.

DEFINITION 4.3. *The following $(2l + 1) \times d(\sigma_0; l)$ matrix $\mathbf{S}(\sigma_0; l)$ is called the canonical block of elementary functions for τ_{2l} -isotypic component of $\pi_{(\nu_0, \sigma_0)}$: When $(\sigma_{0,1}, \sigma_{0,2}) = (0, 0)$, we consider the matrix*

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l)) & \text{if } l \text{ is even,} \\ (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l-2)) & \text{if } l \text{ is odd.} \end{cases}$$

When $(\sigma_{0,1}, \sigma_{0,2}) = (1, 0)$, we consider the matrix

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l-2)) & \text{if } l \text{ is even,} \\ (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l)) & \text{if } l \text{ is odd.} \end{cases}$$

When $\sigma_{0,2} = 1$, we consider the matrix

$$\mathbf{S}(\sigma_0; l) = \begin{cases} (S(l; 1), S(l; 3), S(l; 5), \dots, S(l; l-1)) & \text{if } l \text{ is even,} \\ (S(l; 0), S(l; 2), S(l; 4), \dots, S(l; l-1)) & \text{if } l \text{ is odd.} \end{cases}$$

4.4. *The $\mathfrak{p}_{\mathbf{C}}$ -matrix corresponding to I_{2m}^{2l} .* For two integers c_0, c_1 such that $c_0 \leq c_1$ and a rational function $f(x)$ in the variable x , we denote by

$$\text{Diag}_{c_0 \leq n \leq c_1} (f(n))$$

the diagonal matrix of size $c_1 - c_0 + 1$ with an entry $f(n)$ at the $(n - c_0 + 1, n - c_0 + 1)$ -th component. Let $\mathbf{e}_i^{(l)}$ ($0 \leq i \leq l$) be the column unit vector of degree $l + 1$ with its $i + 1$ -th component 1 and the remaining components 0. Moreover, let $\mathbf{e}_i^{(l)}$ be the column zero vector of degree $l + 1$ when $i < 0$ or $l < i$.

In this subsection, we define $\mathfrak{p}_{\mathbf{C}}$ -matrix $\mathfrak{C}_{l,m}$ of size $(2(l+m)+1) \times (2l+1)$ corresponding to I_{2m}^{2l} with respect to the standard basis.

Let $\sum_{i=0}^4 \iota_i^{(l,m)} \otimes X_i$ be the image of I_{2m}^{2l} under the composite of natural linear maps

$$\begin{aligned} \text{Hom}_K(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) &\rightarrow \text{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) \\ &\simeq \text{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l}) \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}. \end{aligned}$$

Then we define $\mathfrak{p}_{\mathbf{C}}$ -matrix $\mathfrak{C}_{l,m} = \sum_{i=0}^4 R(t_i^{(l,m)}) \otimes X_i$ where $R(t_i^{(l,m)})$ is the matrix representation of $t_i^{(l,m)}$ with respect to the standard basis. Explicit expression of the matrix $R(t_i^{(l,m)})$ of size $(2(l+m)+1) \times (2l+1)$ is given by

$$\begin{aligned} & \left(O_{2(l+m)+1, m+2}, R(t_i^{(l,m)}), O_{2(l+m)+1, m+2} \right) \\ & = \left(O_{2(l+m)+1, 4-i}, \text{Diag}_{0 \leq k \leq 2(l+m)} (A_{[2l, 2m; k, i]}), O_{2(l+m)+1, i} \right) \end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Here we omit the symbol $O_{m,n}$ when $m = 0$ or $n = 0$.

For a column vector $\mathbf{v} = {}^t(v_0, v_1, \dots, v_{2l}) \in (H_{(\nu_i, \sigma_i), K})^{\oplus 2l+1}$ which is identified with an element of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$, we define $\mathfrak{C}_{l,m} \mathbf{v} \in (H_{(\nu_i, \sigma_i), K})^{\oplus 2(l+m)+1} \simeq \mathbf{C}^{2(l+m)+1} \otimes_{\mathbf{C}} H_{(\nu_i, \sigma_i), K}$ by

$$\mathfrak{C}_{l,m} \mathbf{v} = \sum_{\substack{0 \leq j \leq 4 \\ 0 \leq q \leq 2l}} (R(t_j^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes (\pi_{(\nu_i, \sigma_i)}(X_j)v_q).$$

Here $R(t_j^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}$ is the ordinal product of matrices $R(t_j^{(l,m)})$ and $\mathbf{e}_q^{(2l)}$.

From the definition of $\mathfrak{C}_{l,m}$, we note that the vector $\mathfrak{C}_{l,m} \mathbf{v}$ is identified with the image of \mathbf{v} under $\Gamma_{l,m}^i$.

4.5. *The contiguous relations.*

LEMMA 4.4. *The standard basis X_i ($0 \leq i \leq 4$) in $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned} X_0 &= -2\sqrt{-1}E_{e_2-e_3} + H_2 + \sqrt{-1}K_{23}, \\ X_1 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\ X_2 &= -\frac{1}{3}(2H_1 - H_2), \\ X_3 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \\ X_4 &= 2\sqrt{-1}E_{e_2-e_3} + H_2 - \sqrt{-1}K_{23}. \end{aligned}$$

PROOF. We obtain the assertion immediately from Lemma 3.1. □

We give the matrix representation of $\Gamma_{l,m}^0$ with respect to the induced basis as follows.

THEOREM 4.5. *For $l \in \mathbf{Z}_{\geq 0}$, $-2 \leq m \leq 2$ such that $d(\sigma_0; l) > 0$ and $d(\sigma_0; l+m) > 0$, we have*

$$(4.1) \quad \mathfrak{C}_{l,m} \mathbf{S}(\sigma_0; l) = \mathbf{S}(\sigma_0; l+m) \cdot R(\Gamma_{l,m}^0)$$

with the matrix representation $R(\Gamma_{l,m}^0) \in M_{d(\sigma_0;l+m),d(\sigma_0;l)}(\mathbf{C})$ of $\Gamma_{l,m}^0$ with respect to the induced basis $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$:

We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_0;l,m),d(\sigma_0;l)} \\ R(\Gamma_{l,m}^0) \end{pmatrix}$$

as follows:

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (2\mathbf{Z})$,

$$\begin{aligned} & \begin{pmatrix} \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),-1]}^{(0)} \right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),0]}^{(0)} \right) \end{pmatrix} \\ & + \begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-2} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),1]}^{(0)} \right) & \gamma_{[l,m;l,1]}^{(0)} \cdot e_{d(\sigma_0;l)-3}^{(d(\sigma_0;l)-2)} \end{pmatrix}. \end{aligned}$$

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (1 + 2\mathbf{Z})$,

$$\begin{aligned} & \begin{pmatrix} \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),-1]}^{(0)} \right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),0]}^{(0)} \right) \end{pmatrix} \\ & + \begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-2} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),1]}^{(0)} \right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix}. \end{aligned}$$

When $\sigma_{0,2} = 0$, $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (2\mathbf{Z})$,

$$\begin{aligned} & \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),-1]}^{(0)} \right) \\ & + \begin{pmatrix} O_{1,d(\sigma_0;l)-1} & 0 \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-2} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),0]}^{(0)} \right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix} \\ & + \begin{pmatrix} O_{2,d(\sigma_0;l)-2} & O_{2,1} & O_{2,1} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-3} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),1]}^{(0)} \right) & O_{d(\sigma_0;l)-2,1} & -\gamma_{[l,m;l,1]}^{(0)} \cdot e_{d(\sigma_0;l)-3}^{(d(\sigma_0;l)-3)} \end{pmatrix}. \end{aligned}$$

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (1 + 2\mathbf{Z})$,

$$\begin{pmatrix} \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),-1]}^{(0)} \right) \\ O_{2,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),0]}^{(0)} \right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix}$$

$$+ \left(\begin{array}{c} O_{2,d(\sigma_0;l)} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),1]}^{(0)} \right) \end{array} \right).$$

When $\sigma_{0,2} = 1$,

$$\begin{aligned} & \left(\begin{array}{c} \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),-1]}^{(0)} \right) \\ O_{1,d(\sigma_0;l)} \end{array} \right) + \left(\begin{array}{c} O_{1,d(\sigma_0;l)} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-1} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),0]}^{(0)} \right) \end{array} \right) \\ & + \left(\begin{array}{c} O_{2,d(\sigma_0;l)-1} \\ \text{Diag}_{0 \leq k \leq d(\sigma_0;l)-2} \left(\gamma_{[l,m;2k+\delta(\sigma_0;l),1]}^{(0)} \right) \end{array} \right) (-1)^{\varepsilon(\sigma_0;l+m)} \gamma_{[l,m;l-1,1]}^{(0)} \cdot e_{d(\sigma_0;l)-2}^{(d(\sigma_0;l)-2)}. \end{aligned}$$

Here

$$\gamma_{[l,m;p,1]}^{(0)} = (\nu_{0,2} + \rho_{0,2} - l + p) A_{[2l,2m;2l-p+m-2,0]},$$

$$\begin{aligned} \gamma_{[l,m;p,0]}^{(0)} &= -\frac{1}{3} \left(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2} + lm - 3 + \frac{m(m+1)}{2} \right) A_{[2l,2m;2l-p+m,2]}, \\ \gamma_{[l,m;p,-1]}^{(0)} &= (\nu_{0,2} + \rho_{0,2} + l - p) A_{[2l,2m;2l-p+m+2,4]}, \end{aligned}$$

$$n(\sigma_0; l, m) = \begin{cases} (2-m)/2 & \text{if } m \in \{0, \pm 2\}, \\ (3-m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (2\mathbf{Z}), \\ (1-m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (1 + 2\mathbf{Z}), \end{cases}$$

and $\delta(\sigma_0; l) \in \{0, 1\}$ such that $\delta(\sigma_0; l) \equiv l - \sigma_{0,2} \pmod{2}$.

In the above equations, we put $A_{[2l,2m;k,i]} = 0$ for $k < 0$ or $k > 2(l+m)$, and omit the symbols $\text{Diag}_{c \leq n \leq c-1}(f(n))$, $O_{0,n}$, $O_{m,0}$ and $\mathbf{e}_i^{(-1)}$.

PROOF. Since

$$s(l; p, q)(1_3) = \langle (v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0;l)} v_p^{(2l)*}), v_q^{(2l)} \rangle = \delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)} \delta_{pq},$$

we have

$$(4.2) \quad S(l; p)(1_3) = \mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)} \mathbf{e}_p^{(2l)}.$$

Hence $S(l; p)(1_3)$ ($p \in Z(\sigma_0; l)$) are linearly independent over \mathbf{C} . Thus we note that it suffices to evaluate the both side of the equation (4.1) at $1_3 \in G$.

First, we compute $\{\pi_{(\nu_0, \sigma_0)}(X_i) s(l; p, q)\}(1_3)$ for $0 \leq i \leq 4$, $p \in Z(\sigma_0; l)$ and $0 \leq q \leq 2l$. Since $\{s(l; p, q) \mid 0 \leq q \leq 2l\}$ is the standard basis of $\langle S(l; p) \rangle$, we have

$$\begin{aligned} \{\pi_{(\nu_0, \sigma_0)}(\sqrt{-1}K_{23})s(l; p, q)\}(1_3) &= (l-q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)} \delta_{pq}), \\ \{\pi_{(\nu_0, \sigma_0)}(K_{13} + \sqrt{-1}K_{12})s(l; p, q)\}(1_3) &= -q(\delta_{2l-p+1q} + (-1)^{\varepsilon(\sigma_0;l)} \delta_{p+1q}), \\ \{\pi_{(\nu_0, \sigma_0)}(K_{13} - \sqrt{-1}K_{12})s(l; p, q)\}(1_3) &= \end{aligned}$$

$$= (2l - q)(\delta_{2l-p-1q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1q}).$$

Moreover, we obtain

$$\begin{aligned} \{\pi_{(\nu_0,\sigma_0)}(E_\alpha)s(l;p,q)\}(1_3) &= 0 \quad (\alpha \in \Sigma^+), \\ \{\pi_{(\nu_0,\sigma_0)}(H_i)s(l;p,q)\}(1_3) &= (\nu_{0,i} + \rho_{0,i})s(l;p,q)(1_3) \\ &= (\nu_{0,i} + \rho_{0,i})(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}) \quad (i = 1, 2), \end{aligned}$$

from the definition of principal series representation. From these computations and Iwasawa decomposition in Lemma 4.4, we obtain

$$\begin{aligned} \{\pi_{(\nu_0,\sigma_0)}(X_0)s(l;p,q)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} + l - q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}), \\ \{\pi_{(\nu_0,\sigma_0)}(X_1)s(l;p,q)\}(1_3) &= -\frac{q}{2}(\delta_{2l-p+1q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p+1q}), \\ \{\pi_{(\nu_0,\sigma_0)}(X_2)s(l;p,q)\}(1_3) \\ &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}), \\ \{\pi_{(\nu_0,\sigma_0)}(X_3)s(l;p,q)\}(1_3) &= -\frac{2l-q}{2}(\delta_{2l-p-1q} - (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1q}), \\ \{\pi_{(\nu_0,\sigma_0)}(X_4)s(l;p,q)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} - l + q)(\delta_{2l-pq} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}). \end{aligned}$$

We set

$$\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p) = \sum_{0 \leq q \leq 2l} \mathbf{e}_q^{(2l)} \otimes (\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q)).$$

Then we obtain

$$\begin{aligned} \{\pi_{(\nu_0,\sigma_0)}(X_0)S(l;p)\}(1_3) &= (\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_p^{(2l)}, \\ \{\pi_{(\nu_0,\sigma_0)}(X_1)S(l;p)\}(1_3) &= -\frac{2l-p+1}{2}\mathbf{e}_{2l-p+1}^{(2l)} - (-1)^{\varepsilon(\sigma_0;l)}\frac{p+1}{2}\mathbf{e}_{p+1}^{(2l)}, \\ \{\pi_{(\nu_0,\sigma_0)}(X_2)S(l;p)\}(1_3) \\ &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}_p^{(2l)}), \\ \{\pi_{(\nu_0,\sigma_0)}(X_3)S(l;p)\}(1_3) &= -\frac{p+1}{2}\mathbf{e}_{2l-p-1}^{(2l)} - (-1)^{\varepsilon(\sigma_0;l)}\frac{2l-p+1}{2}\mathbf{e}_{p-1}^{(2l)}, \\ \{\pi_{(\nu_0,\sigma_0)}(X_4)S(l;p)\}(1_3) \\ &= (\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_p^{(2l)}. \end{aligned}$$

Let us compute $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$. By the above equations, we have

$$\begin{aligned} \{\mathfrak{C}_{l,m}S(l;p)\}(1_3) \\ &= \sum_{\substack{0 \leq i \leq 4 \\ 0 \leq q \leq 2l}} (R(\iota_i^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes \{(\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q))\}(1_3) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{0 \leq i \leq 4} R(l_i^{(l,m)}) \cdot \{(\pi_{(\nu_0, \sigma_0)}(X_i)S(l; p))\}(1_3) \\
 &= R(l_0^{(l,m)}) \cdot \{(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_p^{(2l)}\} \\
 &+ R(l_1^{(l,m)}) \cdot \left\{ -\frac{2l-p+1}{2}\mathbf{e}_{2l-p+1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{p+1}{2}\mathbf{e}_{p+1}^{(2l)} \right\} \\
 &+ R(l_2^{(l,m)}) \cdot \left\{ -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}\mathbf{e}_p^{(2l)}) \right\} \\
 &+ R(l_3^{(l,m)}) \cdot \left\{ -\frac{p+1}{2}\mathbf{e}_{2l-p-1}^{(2l)} - (-1)^{\varepsilon(\sigma_0; l)}\frac{2l-p+1}{2}\mathbf{e}_{p-1}^{(2l)} \right\} \\
 &+ R(l_4^{(l,m)}) \cdot \{(\nu_{0,2} + \rho_{0,2} + l - p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0; l)}(\nu_{0,2} + \rho_{0,2} - l + p)\mathbf{e}_p^{(2l)}\}.
 \end{aligned}$$

Since

$$R(l_i^{(l,m)})\mathbf{e}_q^{(2l)} = A_{[2l, 2m; i+q+m-2, i]}\mathbf{e}_{i+q+m-2}^{(2(l+m))}, \quad -2 \leq m \leq 2,$$

we obtain

$$\begin{aligned}
 (4.3) \quad &\{\mathfrak{C}_{l,m}S(l; p)\}(1_3) \\
 &= \sum_{-1 \leq i \leq 1} \{ \alpha_{[l, m; p, i]}\mathbf{e}_{2(l+m)-(p+m+2i)}^{(2(l+m))} + (-1)^{\varepsilon(\sigma_0; l)}\beta_{[l, m; p, i]}\mathbf{e}_{p+m+2i}^{(2(l+m))} \},
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha_{[l, m; p, 1]} &= (\nu_{0,2} + \rho_{0,2} - l + p)A_{[2l, 2m; 2l-p+m-2, 0]}, \\
 \alpha_{[l, m; p, 0]} &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{[2l, 2m; 2l-p+m, 2]} \\
 &\quad - \frac{2l-p+1}{2}A_{[2l, 2m; 2l-p+m, 1]} - \frac{p+1}{2}A_{[2l, 2m; 2l-p+m, 3]}, \\
 \alpha_{[l, m; p, -1]} &= (\nu_{0,2} + \rho_{0,2} + l - p)A_{[2l, 2m; 2l-p+m+2, 4]}, \\
 \beta_{[l, m; p, 1]} &= (\nu_{0,2} + \rho_{0,2} - l + p)A_{[2l, 2m; p+m+2, 4]}, \\
 \beta_{[l, m; p, 0]} &= -\frac{1}{3}(2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{[2l, 2m; p+m, 2]} \\
 &\quad - \frac{p+1}{2}A_{[2l, 2m; p+m, 1]} - \frac{2l-p+1}{2}A_{[2l, 2m; p+m, 3]}, \\
 \beta_{[l, m; p, -1]} &= (\nu_{0,2} + \rho_{0,2} + l - p)A_{[2l, 2m; p+m-2, 0]}.
 \end{aligned}$$

By the relations of the coefficients $A_{[2l, 2m; k, i]}$ in Lemma 3.3, we see that

$$\alpha_{[l, m; p, i]} = (-1)^m \beta_{[l, m; p, i]} = \gamma_{[l, m; p, i]}^{(0)}, \quad -1 \leq i \leq 1.$$

Therefore, (4.3) become

$$\begin{aligned}
 (4.4) \quad &\{\mathfrak{C}_{l,m}S(l; p)\}(1_3) \\
 &= \sum_{-1 \leq i \leq 1} \gamma_{[l, m; p, i]}^{(0)} \{ \mathbf{e}_{2(l+m)-(p+m+2i)}^{(2(l+m))} + (-1)^{\varepsilon(\sigma_0; l)+m} \mathbf{e}_{p+m+2i}^{(2(l+m))} \}.
 \end{aligned}$$

From the equations (4.2), (4.4) and $\varepsilon(\sigma_0; l) + m \equiv \varepsilon(\sigma_0; l + m) \pmod 2$, we obtain the assertion. \square

5. THE (\mathfrak{g}, K) -MODULE STRUCTURES OF THE P_i -PRINCIPAL SERIES REPRESENTATIONS FOR $i = 1, 2$

In this section, we set $i = 1$ or 2 .

5.1. *The discrete series representations of $SL^\pm(2, \mathbf{R})$.* The set of equivalence classes of discrete series representations of $SL^\pm(2, \mathbf{R})$ is exhausted by the induced representation $D_k = \text{Ind}_{SL(2, \mathbf{R})}^{SL^\pm(2, \mathbf{R})}(D_k^+)$. Here D_k^+ is the discrete series representation of $SL(2, \mathbf{R})$ with Blattner parameter k , i.e. the one whose minimal $SO(2)$ -type is given by the character

$$SO(2) \ni \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mapsto e^{\sqrt{-1}kt} \in \mathbf{C}^\times.$$

We denote by D_k^- the contragredient representation of D_k^+ and set $y_0 = \text{diag}(1, -1) \in O(2)$. Then a discrete series representation D_k is uniquely determined by specifying the $SL(2, \mathbf{R})$ -module structure together with the action of y_0 . Since $D_k|_{SL(2, \mathbf{R})} = D_k^+ \oplus D_k^-$ and $D_k^+ \oplus D_k^-$ is infinitesimally equivalent with a subrepresentation of some principal series representation of $SL(2, \mathbf{R})$, we obtain the following realization of associated $(\mathfrak{sl}(2, \mathbf{C}), O(2))$ -module of D_k :

$$V_{D_k, O(2)} = \bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} W_{k+2\alpha} \quad (W_p = \mathbf{C} \cdot \chi_p + \mathbf{C} \cdot \chi_{-p})$$

and

$$\begin{aligned} D_k(\kappa_t)\chi_p &= e^{\sqrt{-1}pt}\chi_p & D_k(y_0)\chi_p &= \chi_{-p}, & D_k(w)\chi_p &= \sqrt{-1}p\chi_p, \\ D_k(x_+)\chi_p &= (k+p)\chi_{p+2}, & D_k(x_-)\chi_p &= (k-p)\chi_{p-2}, \end{aligned}$$

where

$$\begin{aligned} w &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_\pm = \begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbf{C}), \\ \kappa_t &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2) \quad (t \in \mathbf{R}). \end{aligned}$$

Here we denote differential of D_k again by D_k and the $O(2)$ -finite part of V_{D_k} by $V_{D_k, O(2)}$. See [1, §2.5] for details.

5.2. *The irreducible decompositions of $\pi_{(\nu_1, \sigma_1)}|_K$ and $\pi_{(\nu_2, \sigma_2)}|_K$ as K -modules.* We identify M_i with $SL^\pm(2, \mathbf{R})$ by natural isomorphisms $m_i: SL^\pm(2, \mathbf{R}) \rightarrow M_i$ defined by

$$m_1(h) = \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix}, \quad m_2(h) = \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix}$$

for $h \in SL^\pm(2, \mathbf{R})$. Then we may put $\sigma_i = D_k \circ m_i^{-1}$ for some $k \geq 2$.

We analyze the K -type of the representation space $H_{(\nu_i, \sigma_i)}$ of the P_i -principal series representation. The target V_{σ_i} of functions \mathbf{f} in $H_{(\nu_i, \sigma_i)}$ has a decomposition:

$$V_{\sigma_i} = V_{D_k} = \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} W_{k+2\alpha}}.$$

Denote the corresponding decomposition of \mathbf{f} by

$$\mathbf{f}(x) = \sum_{\alpha=0}^{\infty} (f_{k+2\alpha}(x) \otimes \chi_{k+2\alpha} + f_{-(k+2\alpha)}(x) \otimes \chi_{-(k+2\alpha)}).$$

From the definition of the space $H_{(\nu_i, \sigma_i)}$, we have

$$\mathbf{f}|_K(mx) = \sigma_i(m)\mathbf{f}|_K(x) \quad (\text{a.e. } x \in K, m \in K_i = M_i \cap K \simeq O(2)).$$

For $m = m_i(\kappa_t)$, $m_i(y_0)$, comparing the coefficients of χ_p in the left hand side with those in the right hand side, we have the equations

$$f_p|_K(m_i(\kappa_t)x) = e^{\sqrt{-1}pt} f_p|_K(x), \quad f_p|_K(m_i(y_0)x) = f_{-p}|_K(x).$$

Moreover, from the equality of inner products

$$\int_K \|\mathbf{f}|_K(x)\|_{\sigma_i}^2 dx = \sum_{\varepsilon \in \{\pm 1\}, \alpha \in \mathbf{Z}_{\geq 0}} \left\{ \int_K |f_{\varepsilon(k+2\alpha)}|_K(x)|^2 dx \right\} \|\chi_{\varepsilon(k+2\alpha)}\|_{\sigma_i}^2,$$

we have $f_p|_K \in L^2(K)$. Therefore $\mathbf{f}|_K$ belongs to

$$\widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} L_i^2(K; W_{k+2\alpha})}$$

where

$$L_i^2(K; W_p) = \left\{ \mathbf{f}: K \rightarrow W_p \mid \begin{array}{l} \mathbf{f}(x) = f(x) \otimes \chi_p + f(m_i(y_0)x) \otimes \chi_{-p}, \\ f \in L^2_{(K_i^\circ, \chi_p)}(K), \quad x \in K \end{array} \right\},$$

$$L^2_{(K_i^\circ, \chi_p)}(K) = \left\{ f \in L^2(K) \mid \begin{array}{l} f(m_i(\kappa_t)x) = e^{\sqrt{-1}pt} f(x), \\ m_i(\kappa_t) \in K_i^\circ, \quad x \in K \end{array} \right\}.$$

Here K_i° means the connected component of K_i , which is isomorphic to $SO(2)$. We easily see that the restriction map

$$r_K^{(i)}: H_{(\nu_i, \sigma_i)} \ni \mathbf{f} \mapsto \mathbf{f}|_K \in \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} L_i^2(K; W_{k+2\alpha})}$$

is a K -isomorphism.

By Peter-Weyl's theorem, we have the following irreducible decomposition of $L^2_{(K_i^\circ, \chi_p)}(K)$:

$$L^2_{(K_i^\circ, \chi_p)}(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}} (V_{2l}^*[\xi_{(i; -p)}]) \otimes_{\mathbf{C}} V_{2l}}.$$

Here

$$\xi_{(i;p)} : K_i^\circ \ni m_i(\kappa_t) \mapsto e^{\sqrt{-1}pt} \in \mathbf{C}^\times$$

and $V[\xi_{(i;p)}]$ means the $\xi_{(i;p)}$ -isotypic component in $(\tau|_{K_i^\circ}, V)$ for a K -module (τ, V) .

In this section, we denote by $\{v_{1,q}^{(2l)} \mid 0 \leq q \leq 2l\}$ the standard basis of V_{2l} . We define another basis $\{v_{2,q}^{(2l)} \mid 0 \leq q \leq 2l\}$ of V_{2l} by

$$v_{2,q}^{(2l)} = \tau_{2l}(u_c)v_{1,q}^{(2l)} = \frac{1}{2^l}(x+y)^q(-x+y)^{2l-q} \quad (0 \leq q \leq 2l)$$

where

$$u_c = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3).$$

We note that $v \in V_{2l}[\xi_{(i;-p)}]$ if and only if

$$\tau_{2l}(m_i(\kappa_t))v = \xi_{(i;-p)}(m_i(\kappa_t))v = e^{-\sqrt{-1}pt}v \quad (t \in \mathbf{R})$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi^{-1}(m_1(\kappa_t)) = \varphi^{-1}(u_c^{-1}m_2(\kappa_t)u_c) = \left\{ \pm \operatorname{diag}(e^{-\sqrt{-1}t/2}, e^{\sqrt{-1}t/2}) \right\},$$

we have $\tau_{2l}(m_i(\kappa_t))v_{i,q}^{(2l)} = e^{\sqrt{-1}(q-l)t}v_{i,q}^{(2l)}$. Hence we have

$$V_{2l}[\xi_{(i;-p)}] = \begin{cases} \mathbf{C} \cdot v_{i,l-p}^{(2l)} & \text{if } -l \leq p \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

By the identification $V_{2l}^* = V_{2l}$ in Lemma 3.4, we obtain

$$L^2_{(K_i^\circ, \chi_p)}(K) \simeq \bigoplus_{l \in \mathbf{Z}_{\geq 0}} (\mathbf{C} \cdot v_{i,l+p}^{(2l)*}) \otimes_{\mathbf{C}} V_{2l}.$$

Here we put $v_{i,l+p}^{(2l)*} = 0$ if $p < -l$ or $l < p$. Moreover, since

$$\begin{aligned} \varphi^{-1}(m_1(y_0)) &= \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \\ \varphi^{-1}(u_c^{-1}m_2(y_0)u_c) &= \left\{ \pm \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right\}, \end{aligned}$$

we have

$$\tau_{2l}^*(m_1(y_0)^{-1})v_{1,l+p}^{(2l)*} = (-1)^{l+p}v_{1,l-p}^{(2l)*}, \quad \tau_{2l}^*(m_2(y_0)^{-1})v_{2,l+p}^{(2l)*} = (-1)^l v_{2,l-p}^{(2l)*}.$$

For $0 \leq p \leq l-k$ such that $p \equiv l-k \pmod{2}$, we define the elementary function $t_i(l; p, q) \in H_{(\nu_i, \sigma_i)}$ by

$$t_i(l; p, q) = r_K^{(i)-1}(\tilde{t}_i(l; p, q))$$

where

$$\begin{aligned} \tilde{t}_1(l; p, q)(x) &= \langle v_{1, 2l-p}^{(2l)*}, \tau_{2l}(x)v_{1,q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^p \langle v_{1,p}^{(2l)*}, \tau_{2l}(x)v_{1,q}^{(2l)} \rangle \otimes \chi_{p-l}, \\ \tilde{t}_2(l; p, q)(x) &= \langle v_{2, 2l-p}^{(2l)*}, \tau_{2l}(x)v_{1,q}^{(2l)} \rangle \otimes \chi_{l-p} + (-1)^l \langle v_{2,p}^{(2l)*}, \tau_{2l}(x)v_{1,q}^{(2l)} \rangle \otimes \chi_{p-l}. \end{aligned}$$

Let $T_i(l; p)$ be a column vector of degree $2l+1$ with its $q+1$ -th component $t_i(l; p, q)$, i.e. ${}^t(t_i(l; p, 0), t_i(l; p, 1), \dots, t_i(l; p, 2l))$.

Moreover we denote by $\langle T_i(l; p) \rangle$ the subspace of $H_{(\nu_i, \sigma_i)}$ generated by the functions in the entries of the vector $T_i(l; p)$, i.e.

$$\langle T_i(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot t_i(l; p, q) \simeq V_{2l}.$$

Via the isomorphism between $\langle T_i(l; p) \rangle$ and V_{2l} , we identify $\{t_i(l; p, q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

PROPOSITION 5.1. *As an unitary representation of K , it has an irreducible decomposition:*

$$H_{(\nu_i, \sigma_i)} = \widehat{\bigoplus_{\substack{l \in \mathbf{Z}_{\geq 0}, 0 \leq p \leq l-k \\ p \equiv l-k \pmod{2}}} \langle T_i(l; p) \rangle}$$

for $i = 1, 2$. Then the τ_{2l} -isotypic component of $\pi_{(\nu_i, \sigma_i)}$ is given by

$$\bigoplus_{\substack{0 \leq p \leq l-k \\ p \equiv l-k \pmod{2}}} \langle T_i(l; p) \rangle.$$

COROLLARY 5.2. *The multiplicity $d(\sigma_i; l)$ of τ_{2l} in $\pi_{(\nu_i, \sigma_i), K}$ is given by*

$$d(\sigma_i; l) = \begin{cases} (l-k+2)/2 & \text{if } k \leq l \text{ and } l-k \text{ is even,} \\ (l-k+1)/2 & \text{if } k \leq l \text{ and } l-k \text{ is odd,} \\ 0 & \text{if } k > l. \end{cases}$$

5.3. *The canonical blocks of elementary functions.* By the identification introduced in Subsection 4.3, we identify $T_i(l; p)$ with the K -homomorphism

$$V_{2l} \ni v_{1,q}^{(2l)} \mapsto t_i(l; p, q) \in H_{(\nu_i, \sigma_i), K}, \quad 0 \leq q \leq 2l$$

for $0 \leq p \leq l-k$ such that $p \equiv l-k \pmod{2}$. We note that $\{T_i(l; p) \mid 0 \leq p \leq l-k, p \equiv l-k \pmod{2}\}$ is a basis of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ and we call it *the induced basis from the standard basis*.

For each K -type τ_{2l} of our P_i -principal series representation $\pi_{(\nu_i, \sigma_i)}$, we define a certain matrix of elementary functions corresponding to the induced basis $\{T_i(l; p) \mid 0 \leq p \leq l-k, p \equiv l-k \pmod{2}\}$ of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$.

DEFINITION 5.3. For $l \in \mathbf{Z}_{\geq 0}$ such that $d(\sigma_i; l) > 0$, the following $(2l + 1) \times d(\sigma_i; l)$ matrix $\mathbf{T}_i(\sigma_i; l)$ is called the canonical block of elementary functions for τ_{2l} -isotypic component of $\pi_{(\nu_i, \sigma_i)}$: When $l - k$ is even, we consider the matrix

$$\mathbf{T}_i(\sigma_i; l) = (T_i(l; 0), T_i(l; 2), T_i(l; 4), \dots, T_i(l; l - k)).$$

When $l - k$ is odd, we consider the matrix

$$\mathbf{T}_i(\sigma_i; l) = (T_i(l; 1), T_i(l; 3), T_i(l; 5), \dots, T_i(l; l - k)).$$

5.4. The contiguous relations.

LEMMA 5.4. (i) The standard basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = (\mathfrak{n}_{1, \mathbf{C}} \oplus \mathfrak{a}_{1, \mathbf{C}} \oplus \mathfrak{m}_{1, \mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X_0 &= m_1(x_-), & X_1 &= - (E_{e_1 - e_3} + \sqrt{-1}E_{e_1 - e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\ X_2 &= -\frac{1}{3}H^{(1)}, & X_3 &= (E_{e_1 - e_3} - \sqrt{-1}E_{e_1 - e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \\ X_4 &= m_1(x_+). \end{aligned}$$

(ii) The standard basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \text{Ad}(u_c^{-1})(\mathfrak{n}_{2, \mathbf{C}} \oplus \mathfrak{a}_{2, \mathbf{C}} \oplus \mathfrak{m}_{2, \mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X_0 &= -\text{Ad}(u_c^{-1})m(x_-), \\ X_1 &= \text{Ad}(u_c^{-1})(E_{e_1 - e_3} - \sqrt{-1}E_{e_2 - e_3}) - \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\ X_2 &= \frac{1}{3}\text{Ad}(u_c^{-1})H^{(2)}, \\ X_3 &= -\text{Ad}(u_c^{-1})(E_{e_1 - e_3} + \sqrt{-1}E_{e_2 - e_3}) + \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \\ X_4 &= -\text{Ad}(u_c^{-1})m(x_+), \end{aligned}$$

PROOF. We obtain the assertion immediately from Lemma 3.1. □

We give the matrix representation of $\Gamma_{l, m}^i$ with respect to the induced basis as follows.

THEOREM 5.5. For $i = 1, 2$ and $-2 \leq m \leq 2$, we have the following equation with the matrix representation $R(\Gamma_{l, m}^i) \in M_{d(\sigma_i; l+m), d(\sigma_i; l)}(\mathbf{C})$ of $\Gamma_{l, m}^i$ with respect to the induced basis $\{T_i(l; p) \mid 0 \leq p \leq l - k, p \equiv l - k \pmod{2}\}$:

$$(5.1) \quad \mathfrak{C}_{l, m} \mathbf{T}_i(\sigma_i; l) = \mathbf{T}_i(\sigma_i; l + m) \cdot R(\Gamma_{l, m}^i).$$

We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_i; l, m), d(\sigma_i; l)} \\ R(\Gamma_{l, m}^i) \end{pmatrix}$$

by

$$\begin{aligned} & \left(\begin{array}{c} \text{Diag} \\ 0 \leq j \leq d(\sigma_i; l) - 1 \end{array} \left(\gamma_{[l, m; 2j + \delta(\sigma_i; l), -1]}^{(i)} \right) \right) + \left(\begin{array}{c} O_{1, d(\sigma_i; l)} \\ \text{Diag} \\ 0 \leq j \leq d(\sigma_i; l) - 1 \end{array} \left(\gamma_{[l, m; 2j + \delta(\sigma_i; l), 0]}^{(i)} \right) \right) \\ & + \left(\begin{array}{c} O_{2, d(\sigma_i; l) - 1} \\ \text{Diag} \\ 0 \leq j \leq d(\sigma_i; l) - 2 \end{array} \left(\gamma_{[l, m; 2j + \delta(\sigma_i; l), 1]}^{(i)} \right) \right) O_{d(\sigma_i; l) - 1, 1}. \end{aligned}$$

Here

$$\begin{aligned} \gamma_{[l, m; p, 1]}^{(i)} &= (-1)^{i+1} (k - l + p) A_{[2l, 2m; 2l - p + m - 2, 0]}, \\ \gamma_{[l, m; p, 0]}^{(i)} &= \frac{(-1)^i}{3} \left(\nu_i + \rho_i + lm - 3 + \frac{m(m+1)}{2} \right) A_{[2l, 2m; 2l - p + m, 2]}, \\ \gamma_{[l, m; p, -1]}^{(i)} &= (-1)^{i+1} (k + l - p) A_{[2l, 2m; 2l - p + m + 2, 4]}, \\ n(\sigma_i; l, m) &= \begin{cases} (2 - m)/2 & \text{if } m \in \{0, \pm 2\}, \\ (3 - m)/2 & \text{if } (m, l - k) \in \{\pm 1\} \times (2\mathbf{Z}), \\ (1 - m)/2 & \text{if } (m, l - k) \in \{\pm 1\} \times (1 + 2\mathbf{Z}), \end{cases} \end{aligned}$$

and $\delta(\sigma_i; l) \in \{0, 1\}$ such that $\delta(\sigma_i; l) \equiv l - k \pmod{2}$.

In the above equations, we put $A_{[2l, 2m; p, j]} = 0$ for $p < 0$ or $p > 2(l + m)$, and omit the symbols $\text{Diag } (f(n))$ ($c_0 > c_1$), $O_{m, n}$ ($m \leq 0$ or $n \leq 0$).

PROOF. By the similar computation in the proof of Theorem 4.5 using Lemma 5.4 (i), we obtain the assertion in the case of $i = 1$. In the case of $i = 2$, the value of $T_2(l; p)$ at $u_c \in G$ is given by

$$T_2(l; p)(u_c) = \mathbf{e}_{2l-p}^{(2l)} \otimes \chi_{l-p} + (-1)^l \mathbf{e}_p^{(2l)} \otimes \chi_{p-l}.$$

Thus, by the similar computation using Lemma 5.4 (ii), we also obtain the assertion in the case of $i = 2$ evaluating the both side of the equation (5.1) at $u_c \in G$. \square

6. THE ACTION OF $\mathfrak{p}_{\mathbf{C}}$

The linear map $\Gamma_{l, m}^i$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. In this section, we give the explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.

6.1. *The projectors for $V_l \otimes_{\mathbf{C}} V_4$.* For $-2 \leq m \leq 2$, we describe a surjective $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism P_{2m}^l from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m} in terms of the standard basis as follows.

LEMMA 6.1. *Let $\{v_q^{(l)} \mid 0 \leq q \leq l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\geq 0}$. We put $v_q^{(l)} = 0$ when $q < 0$ or $q > l$.*

We define linear maps $P_{2m}^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+2m}$ ($-2 \leq m \leq 2$) by

$$P_{2m}^l(v_q^{(l)} \otimes w_r) = B_{[l, 2m; q, r]} \cdot v_{q+r+m-2}^{(l+2m)},$$

when V_{l+2m} -component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish.

Here the coefficients $B_{[l,2m;q,r]} = b(l, 2m; q, r)/d'(l, 2m)$ are defined by following formulae.

FORMULA 1: The coefficients of $P_4^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+4}$ are given as follows:

$$b(l, 4; q, r) = 1 \quad (0 \leq r \leq 4), \quad d'(l, 4) = 1.$$

FORMULA 2: The coefficients of $P_2^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l+2}$ are given as follows:

$$\begin{aligned} b(l, 2; q, 0) &= 4q, & b(l, 2; q, 1) &= -(l - 4q), & b(l, 2; q, 2) &= -2(l - 2q), \\ b(l, 2; q, 3) &= -(3l - 4q), & b(l, 2; q, 4) &= -4(l - q), & d'(l, 2) &= l + 4. \end{aligned}$$

FORMULA 3: The coefficients of $P_0^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_l$ are given as follows:

$$\begin{aligned} b(l, 0; q, 0) &= 6q(q - 1), & b(l, 0; q, 1) &= -3q(l - 2q + 1), \\ b(l, 0; q, 2) &= l^2 - 6lq + 6q^2 - l, & b(l, 0; q, 3) &= 3(l - 2q - 1)(l - q), \\ b(l, 0; q, 4) &= 6(l - q)(l - q - 1), & d'(l, 0) &= (l + 3)(l + 2). \end{aligned}$$

FORMULA 4: The coefficients of $P_{-2}^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l-2}$ are given as follows:

$$\begin{aligned} b(l, -2; q, 0) &= 4q(q - 1)(q - 2), & b(l, -2; q, 1) &= -q(q - 1)(3l - 4q + 2), \\ b(l, -2; q, 2) &= 2q(l - 2q)(l - q), \\ b(l, -2; q, 3) &= -(l - 4q - 2)(l - q)(l - q - 1), \\ b(l, -2; q, 4) &= -4(l - q)(l - q - 1)(l - q - 2), & d'(l, -2) &= (l + 2)(l + 1)l. \end{aligned}$$

FORMULA 5: The coefficients of $P_{-4}^l: V_l \otimes_{\mathbf{C}} V_4 \rightarrow V_{l-4}$ are given as follows:

$$\begin{aligned} b(l, -4; q, 0) &= q(q - 1)(q - 2)(q - 3), \\ b(l, -4; q, 1) &= -q(q - 1)(q - 2)(l - q), \\ b(l, -4; q, 2) &= q(q - 1)(l - q)(l - q - 1), \\ b(l, -4; q, 3) &= -q(l - q)(l - q - 1)(l - q - 2), \\ b(l, -4; q, 4) &= (l - q)(l - q - 1)(l - q - 2)(l - q - 3), \\ d'(l, -4) &= (l + 1)l(l - 1)(l - 2). \end{aligned}$$

Then P_{2m}^l is the generator of $\text{Hom}_{\mathfrak{sl}(2, \mathbf{C})}(V_l \otimes_{\mathbf{C}} V_4, V_{l+2m})$ such that $P_{2m}^l \circ I_{2m}^l = \text{id}_{V_{l+2m}}$.

PROOF. The composite

$$V_l \otimes_{\mathbf{C}} V_4 \simeq V_l^* \otimes_{\mathbf{C}} V_4^* \simeq (V_l \otimes_{\mathbf{C}} V_4)^* \ni f \mapsto f \circ I_{2m}^l \in V_{l+2m}^* \simeq V_{l+2m}$$

is a surjective $\mathfrak{sl}(2, \mathbf{C})$ -homomorphism from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m} , which is unique up to scalar multiple. Therefore we obtain the assertion from Proposition 3.2 and Lemma 3.4. \square

6.2. *The action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.*

PROPOSITION 6.2. (i) *An explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{s(l; p, q) \mid l \geq 0, p \in Z(\sigma_0; l), 0 \leq q \leq 2l\}$ of $H_{(\nu_0, \sigma_0), K}$ is given by following equation:*

$$\begin{aligned} & \pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) \\ &= \sum_{\substack{-1 \leq j \leq 1 \\ -2 \leq m \leq 2}} \gamma_{[l, m; p, j]}^{(0)} B_{[2l, 2m; q, r]} s(l+m; p+m+2j, q+m+r-2). \end{aligned}$$

Here we put

$$\begin{aligned} \gamma_{[0, m; 0, j]}^{(0)} &= B_{[0, 2m; 0, r]} = 0 \text{ for } m < 2, \quad \gamma_{[1, m; p, j]}^{(0)} = B_{[2, 2m; q, r]} = 0 \text{ for } m < 0, \\ s(l; p, q) &= 0 \text{ whenever } p \leq l \text{ such that } p \notin Z(\sigma_0; l) \text{ or } q < 0 \text{ or } q > 2l, \\ s(l; p, q) &= (-1)^{\varepsilon(\sigma_0; l)} s(l; 2l-p, q) \text{ for } p > l. \end{aligned}$$

(ii) *For $i = 1, 2$, the explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{t_i(l; p, q) \mid l \geq k, 0 \leq p \leq l-k, p \equiv l-k \pmod{2}, 0 \leq q \leq 2l\}$ of $H_{(\nu_i, \sigma_i), K}$ is given by following equation:*

$$\begin{aligned} & \pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) \\ &= \sum_{\substack{-1 \leq j \leq 1 \\ -2 \leq m \leq 2}} \gamma_{[l, m; p, j]}^{(i)} B_{[2l, 2m; q, r]} t_i(l+m; p+m+2j, q+m+r-2) \end{aligned}$$

Here we put $t_i(l; p, q) = 0$ unless $0 \leq p \leq l-k, p \equiv l-k \pmod{2}$ and $0 \leq q \leq 2l$.

PROOF. Since

$$\begin{aligned} \pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) &= \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^0(S(l; p)) \circ P_{2m}^l(v_q^{(2l)} \otimes X_r), \\ \pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) &= \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^i(T_i(l; p)) \circ P_{2m}^l(v_q^{(2l)} \otimes X_r) \quad (i = 1, 2), \end{aligned}$$

we obtain the assertion from Theorem 4.5, 5.5 and Lemma 6.1. □

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REFERENCES

[1] D. Bump, Automorphic forms and representations, Cambridge Studies in Advanced Mathematics **55**, Cambridge University Press, Cambridge, 1997.
 [2] T. Fujimura, *On some degenerate principal series representations of $O(p, 2)$* , J. Lie Theory **11** (2001), 23-55.

- [3] R. Howe, *K-type structure in the principal series of GL_3 . I*, in: Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997), volume 26 of *Adv. Stud. Pure Math.*, pages 77–98. Math. Soc. Japan, Tokyo, 2000.
- [4] R. E. Howe and E.-C. Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, *Bull. Amer. Math. Soc. (N.S.)* **28** (1993), 1-74.
- [5] H. Kraljević, *Representations of the universal covering group of the group $SU(n, 1)$* , *Glas. Mat. Ser. III* **8(28)** (1973), 23-72.
- [6] S. T. Lee, *Degenerate principal series representations of $Sp(2n, \mathbf{R})$* , *Compositio Math.* **103** (1996), 123-151.
- [7] S. T. Lee and H. Y. Loke, *Degenerate principal series representations of $Sp(p, q)$* , *Israel J. Math.* **137** (2003), 355-379.
- [8] H. Manabe, T. Ishii, and T. Oda, *Principal series Whittaker functions on $SL(3, \mathbf{R})$* , *Japan. J. Math. (N.S.)* **30** (2004), 183-226.
- [9] T. Miyazaki, *Whittaker functions for generalized principal series representations of $SL(3, \mathbf{R})$* , to appear in *Manuscripta Math.*
- [10] T. Miyazaki, *The (\mathfrak{g}, K) -module structures of principal series representations of $Sp(3, \mathbf{R})$* , preprint.
- [11] T. Oda, *The standard (\mathfrak{g}, K) -modules of $Sp(2, \mathbf{R})$ I*, submitted.
- [12] E. Thieleker, *On the integrable and square-integrable representations of $Spin(1, 2m)$* , *Trans. Amer. Math. Soc.* **230** (1977), 1-40.

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