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ENERGY DECAY ESTIMATES FOR A WAVE EQUATION WITH NONLINEAR BOUNDARY FEEDBACK

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Abstract. We study a wave equation in one dimensional space with nonlinear dissipative boundary feedback at both ends. We prove existence and uniqueness of solution, strong and uniform exponential decay of energy under some conditions in the nonlinear feedback. Decay rate estimates of the energy are given under weak growth assumptions on the feedback functions.

1. Introduction and main results

We consider the wave equation with a variable coefficient, controlled at the boundary by the two feedbacks laws L_1 and L_2 to be determined in the sequel:

$$\left\{ \begin{array}{rcllr} y_{tt}(x,t) - (ay_x)_x(x,t) & = & 0, & 0 < x < 1, & t > 0, \\ (ay_x)(0,t) & = & L_1(t), & & t > 0, \\ -(ay_x)(1,t) & = & L_2(t), & & t > 0. \end{array} \right. \tag{1.1}$$

We assume that the function $a(.)$ belongs to $H^1(0,1)$ and that

$$0 < \underline{a} \leq a(x), \text{ for all } x \in [0,1]. \tag{1.2}$$

The aim of this paper is to show that the system (1.1) is well posed in the terms of the semigroups of contractions and is asymptotically stabilized by the nonlinear feedback laws L_1 and L_2 given as follows:

$$\begin{array}{rcl} L_1(t) & = & k_{p,0}y(0,t) + k_{v,0}f(y_t(0,t)), \\ L_2(t) & = & k_{p,1}y(1,t) + k_{v,1}g(y_t(1,t)), \end{array} \tag{1.3}$$

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where $k_{p,0}$, $k_{p,1}, k_{v,0}$ and $k_{v,1}$ are non negative constants such that

$$k_{v,0} + k_{v,1} > 0, \tag{1.4}$$

$$k_{p,0} + k_{p,1} > 0, \tag{1.5}$$

f and g are suitable nonlinear functions in $C^0(\mathbb{R})$.

The boundary stabilization of (1.1) has been studied by many authors. Conrad and Rao [4] have proved that the feedback law

$$\begin{aligned} L_1(t) &= k_{p,0}y(0,t) + f(y_t(0,t)), \\ L_2(t) &= k_{p,0}y(1,t) + f(y_t(1,t)), \quad k_{p,0} > 0 \end{aligned}$$

stabilizes asymptotically the system under a suitable growth condition of f. Indeed, in the more general cases for which f is a maximal monotone graph, the strong asymptotic stabilization has been proved by Chen and Wang [2] and Lasieska [8, 9].

Let us mention the work of B. Chentouf et al. [1], where a damping model is considered and the equation tolerates a term βy, $\beta = cte > 0$, the system is asymptotically stabilized by the nonlinear feedback law depending only on the boundary velocities:

$$\begin{aligned} L_1(t) &= k_{v,0}f(y_t(0,t)), \\ L_2(t) &= k_{v,1}f(y_t(1,t)), \\ k_{v,0} + k_{v,1} &> 0, \end{aligned}$$

under a suitable choice of f.

In the linear version of this paper Cherkaoui [3] have proved the strong and uniform exponential decay of the energy and also existence of a Riesz basis associated with a spectral formulation of the problem.

With the feedback laws L_1 and L_2 in (1.3), we introduce the energy associated with the system (1.1) as follows:

$$E(t) = \frac{1}{2}\int_0^1 [y_t^2(x,t) + a(x)y_x^2(x,t)]dx + \frac{1}{2}k_{p,0}y^2(0,t) + \frac{1}{2}k_{p,1}y^2(1,t). \tag{1.6}$$

We derive $E(t)$ with respect to t and integrate by parts, and we show formally that:

$$E'(t) = -k_{v,0}y_t(0,t)f(y_t(0,t)) - k_{v,1}y_t(1,t)g(y_t(1,t)). \tag{1.7}$$

Throughout this paper, both f and g are nondecreasing functions in $C^0(\mathbb{R})$ such that

$$f(0) = g(0) = 0, \quad f(s).s \text{ and } g(s).s > 0 \quad \forall s \neq 0. \tag{1.8}$$

Assumption (1.8) implies that the energy $E(t)$ is non-increasing and a Lyapounov function.

Let us define the Hilbert space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ equipped with the inner product:

$$(1.9)\quad < (u,v),(w,z) >_{\mathcal{H}} = \int_0^1 (au_x w_x + vz)dx + k_{p,1}u(1)w(1) + k_{p,0}u(0)w(0).$$

We consider the following nonlinear operator:

(1.10)

$$\mathcal{D}(A) = \left\{ (u,v) \in H^2(0,1) \times H^1(0,1) : \begin{array}{c} (au_x)(0) = k_{p,0}u(0) + k_{v,0}f(v(0)) \\ -(au_x)(1) = k_{p,1}u(1) + k_{v,1}g(v(1)) \end{array} \right\},$$

and for all $(u,v) \in \mathcal{D}(A)$

$$(1.11)\qquad A(u,v) = (v, (au_x)_x)$$

with the initial data $W_0 = (y_0, y_1)$, the closed-loop system (1.1) can be written as an evolution equation on $\mathcal{H}$ called problem (P)

$$(1.12)\qquad \left\{ \begin{array}{ccc} \dot{W}(t) & = & AW(t) \\ W(0) & = & W_0 \end{array} \right.$$

where $W(t) = (y(.,t), y_t(.,t))$.

Throughout this paper, $k_{p,0}$ and $k_{p,1}$ are assumed to satisfy the hypothesis (1.5).

Our main results are stated below.

THEOREM 1.1. *The operator $A : \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ defined by (1.10) and (1.11) generates a C_0-semigroup of contractions $S(t)$ on the energy space $\mathcal{H}$.*

If $k_{v,0}+ k_{v,1} > 0$, then for all initial data $(y_0, y_1) \in \mathcal{H}$, the energy $E(t)$ of the problem (P) converges to zero as $t \to +\infty$.

THEOREM 1.2. *Assume that $a \in H^1(0,1)$ satisfies (1.2) and the hypothesis (1.8) holds.*

If $k_{v,0}+ k_{v,1} > 0$ then we have:

(i) If there exist positive constants C_1, C_2, C_3 and C_4 such that for all $x \in \mathbb{R}$

$$(1.13)\qquad \begin{array}{ccccc} C_1 |x| & \leq & |f(x)| & \leq & C_2 |x|, \\ C_3 |x| & \leq & |g(x)| & \leq & C_4 |x|, \end{array}$$

then given any $M > 1$, there exists a constant $\omega > 0$ such that

$$E(t) \leq ME(0)e^{-\omega t}, \quad \forall t \geq 0.$$

(ii) If there exist positive constants C_1, C_2, C_3 and C_4 and $(p,q) \in [1, +\infty[^2$ with $\max(p,q) = p \vee q > 1$ such that for all $x \in \mathbb{R}$,

$$(1.14)\qquad \begin{array}{ccccc} C_1 \min(|x|, |x|^p) & \leq & |f(x)| & \leq & C_2 |x|, \\ C_3 \min(|x|, |x|^q) & \leq & |g(x)| & \leq & C_4 |x|, \end{array}$$

then given any $M > 1$, *there exists a constant* $\omega > 0$ *depending on* $E(0)$ *such that*

$$E(t) \leq ME(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}}, \quad \forall t \geq 0.$$

The next section is devoted to the proof of our main results.

2. Proofs of the main results

2.1. *Proof of Theorem 1.1.*

2.1.1. *Proof of the well-posedness.* By the Lumer-Phillips theorem, it is sufficient to prove that A is an m-dissipative operator.

First, for $w = (y, z) \in D(A)$, we have $Aw = (z, (ay_x)_x)$,

$$\begin{aligned} \langle Aw, w\rangle_{\mathcal{H}} &= \langle (z, (ay_x)_x), (y, z)\rangle_{\mathcal{H}} \\ &= \int_0^1 (az_x y_x + (ay_x)_x z)dx + k_{p,1} z(1)y(1) + k_{p,0} z(0)y(0). \end{aligned}$$

Using (1.10), we deduce that

$$\langle Aw, w\rangle_{\mathcal{H}} = -k_{v,0} z(0) f(z(0)) - k_{v,1} z(1) g(z(1)),$$

which implies that A is dissipative.

Next, we show the maximality of A, i. e. for any given $(f_1, f_2) \in \mathcal{H}$, there exists $(u, v) \in \mathcal{D}(A)$ such that $(I - A)(u, v) = (f_1, f_2)$. Equivalently, we seek u and v satisfying

$$\left\{ \begin{array}{rcl} u - v &=& f_1, \\ v - (au_x)_x &=& f_2, \\ (au_x)(0) &=& k_{p,0} u(0) + k_{v,0} f(v(0)), \\ -(au_x)(1) &=& k_{p,1} u(1) + k_{v,1} g(v(1)), \\ u &\in& H^2(0,1), v \in H^1(0,1). \end{array} \right. \tag{2.1}$$

Eliminating the unknown v in equation (2.1), we obtain the following reduced problem

$$\left\{ \begin{array}{rcl} u - (au_x)_x &=& f_1 + f_2 = F \in L^2(0,1), \\ (au_x)(0) &=& k_{p,0} u(0) + k_{v,0} f(u(0) - f_1(0)), \\ -(au_x)(1) &=& k_{p,1} u(1) + k_{v,1} g(u(1) - f_1(1)), \\ u &\in& H^2(0,1). \end{array} \right. \tag{2.2}$$

Now, let us define two functions F_0 and F_1 by

$$F_0(x) = k_{v,0} \int_0^x f(s)ds \quad \text{and} \quad F_1(x) = k_{v,1} \int_0^x g(s)ds, \quad \forall x \in \mathbb{R}. \tag{2.3}$$

From the hypothesis (1.8), we deduce that F_0 and F_1 are two convex functions such that

$$F_i \in C^1(\mathbb{R}); \quad F_i(s) \geq 0, \quad \forall s \in \mathbb{R}, \ i = 0, 1. \tag{2.4}$$

In turn, let us define the function $J(.)$ on $H^1(0,1)$ by:

$$
\begin{aligned}
J(w) &= \frac{1}{2}\int_0^1 \left(aw_x^2 + w^2\right) dx - \int_0^1 Fwdx + \frac{k_{p,0}}{2}w^2(0) + \frac{k_{p,1}}{2}w^2(1) \\
&\quad + F_0\left(w(0) - f_1(0)\right) + F_1\left(w(1) - f_1(1)\right). \qquad (2.5)
\end{aligned}
$$

From (2.4), we deduce that the functional $J(.)$ is convex, coercive and strongly continuous in the space $H^1(0,1)$. Hence there exists a unique function $u \in H^1(0,1)$ such that

$$J(u) = \inf_{w \in H^1(0,1)} J(w).$$

This implies that the function $\lambda \mapsto J(u + \lambda w)$ admits a minimum at $\lambda = 0$ and thus

$$\frac{d}{d\lambda}\left[J(u+\lambda w)\right]|_{\lambda=0} = 0 \quad \forall w \in H^1(0,1),$$

this means that for any $w \in H^1(0,1)$, we have

$$
\begin{aligned}
&\int_0^1 uwdx + \int_0^1 au_x w_x dx - \int_0^1 Fwdx + k_{p,0}w(0)u(0) + k_{p,1}w(1)u(1) \\
&+ k_{v,0}w(0)f(u(0) - f_1(0)) + k_{v,1}w(1)g(u(1) - f_1(1)) = 0. \qquad (2.6)
\end{aligned}
$$

In particular for any $w \in C_0^\infty(0,1)$,

$$-\int_0^1 au_x w_x dx = \int_0^1 \left[u - F\right] wdx,$$

this implies the Euler-Lagrange equations

$$u - (au_x)_x = F \in L^2(0,1). \qquad (2.7)$$

Then the $H^2(0,1)$ regularity follows. Integrating equation (2.6) by parts and using equation (2.7), one obtains

$$
\begin{aligned}
(au_x)(0) &= k_{p,0}u(0) + k_{v,0}f\left(u(0) - f_1(0)\right), \\
-(au_x)(1) &= k_{p,1}u(1) + k_{v,1}g\left(u(1) - f_1(1)\right).
\end{aligned} \qquad (2.8)
$$

Therefore, u is the unique solution of system (2.2). Now, we define an element (u,v) by u, solution of (2.2), $v = u - f_1$, which satisfies clearly system (2.1) and thus A is an m-dissipative operator on $\mathcal{H}$.

REMARK 2.1. (i) Given $(y_0, y_1) \in \mathcal{D}(A)$, we define

$$w(t) = \mathcal{S}(t)(y_0, y_1) = (y(.,t),\, y_t(.,t)).$$

Using the regularity result of Haraux [6], we obtain the following smoothness results:

$$y \in C(\mathbb{R}^+; H^2(0,1)) \cap C^1(\mathbb{R}^+; H^1(0,1)) \cap C^2(\mathbb{R}^+; L^2(0,1)).$$

(ii) If $(y_0, y_1) \in \mathcal{H}$, then the Problem (P) admits a unique weak solution

$$(y,\, y_t) = \mathcal{S}(t)(y_0, y_1) \in C(\mathbb{R}^+;\, \mathcal{H}),$$

and by using again a result of Haraux [6], one obtains

$$y \in C(\mathbb{R}^+; H^1(0,1)) \cap C^1(\mathbb{R}^+; L^2(0,1)).$$

2.1.2. *Proof of the asymptotic stability.* We can assume without loss of generality, that $k_{v,0} > 0$ and $k_{v,1} \geq 0$. According to the density of $\mathcal{D}(A)$ in $\mathcal{H}$ and the contraction of the semigroup $\mathcal{S}(t)$, it is enough to prove Theorem 1.1 for any initial data $(y_0, y_1) \in \mathcal{D}(A)$. Let $(y_0, y_1) \in \mathcal{D}(A)$, it is clear that $E(t) \geq 0$ for all $t \geq 0$, and if we set $w = (y,\, y_t)$ we get

$$\begin{aligned} \frac{dE(t)}{dt} &= \left\langle w, \frac{dw}{dt} \right\rangle_{\mathcal{H}} = \langle w, Aw \rangle_{\mathcal{H}} \\ &= -k_{v,0} y_t(0,t) f(y_t(0,t)) - k_{v,1} y_t(1,t) g(y_t(1,t)) \leq 0, \end{aligned} \tag{2.9}$$

so, $E(t)$ is a Lyapounov function.

The resolvent of A is compact, and according to Dafermos [5], it follows that the trajectory $O^+(y_0, y_1) = \{(y(t),\, y_t(t)), t \geq 0.\}$ is relatively compact in E for initial data in $\mathcal{D}(A)$. We apply the Lasalle's invariance principle (see [7] and [10]) to the ω-limit set

$$\begin{aligned} \omega(y_0, y_1) = \Big\{ &(z_0, z_1) \in \mathcal{H} : (z_0, z_1) = \lim_{n \to +\infty} \mathcal{S}(t_n)(y_0, y_1) \\ &\text{where } t_n \to +\infty \text{ as } n \to +\infty \Big\} \end{aligned}$$

of the trajectory $O^+(y_0, y_1)$. Note that

$$\mathcal{S}(t)(y_0, y_1) \to \omega(y_0, y_1) \text{ as } t \to +\infty.$$

In order to show the asymptotic stability, it is sufficient to prove that the ω-limit set reduces to $\{(0,0)\}$. For this, first (2.9) implies

$$E(t) - E(s) + \int_s^t \left[k_{v,0} y_t(0,\sigma) f(y_t(0,\sigma)) + k_{v,1} y_t(1,\sigma) g(y_t(1,\sigma))\right] d\sigma = 0. \tag{2.10}$$

Second, let $(z_0, z_1) \in \omega(y_0, y_1) \subset D(A)$ and let $(z(t),\, z_t(t))$ be the trajectory associated with (z_0, z_1), so according to (2.9) we obtain:

$$\int_s^t \left[k_{v,0} z_t(0,\sigma) f(z_t(0,\sigma)) + k_{v,1} z_t(1,\sigma) g(z_t(1,\sigma))\right] d\sigma = 0,$$

therefore, we deduce from (1.8) that

$$\left\{ \begin{array}{rcl} k_{v,0} z_t(0,t) f(z_t(0,t)) &=& 0, \\ k_{v,1} z_t(1,t) g(z_t(1,t)) &=& 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} z(0,t) &=& c = const \\ k_{v,1} z_t(1,t) &=& 0 \end{array} \right. \qquad \forall t \geq 0.$$

Thus $\omega(y_0, y_1)$ is included in the set of all initial data whose associated solution has constant energy, so z verifies the following system:

$$\text{(2.11)} \quad \begin{cases} z_{tt}(x,t) - (az_x)_x(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ a(0)z_x(0,t) = k_{p,0}c, & t > 0, \\ -a(1)z_x(1,t) = k_{p,1}z(1,t), & t > 0, \\ z_t(0,t) = 0, & t > 0, \\ (z(0), z_t(0)) = (z_0, z_1) \in \omega(y_0, y_1). \end{cases}$$

First, we prove that c $= 0$. For this, we consider the function $h(x) = 1 + k_{p,1}\int_x^1 \frac{ds}{a(s)}$ and we multiply the first equation of (2.11) by h and integrate in x and t. A straightforward computation shows that

$$\text{(2.12)} \qquad \left[\int_0^1 h(x)z_t(x,t)dx\right]_0^T = -\left[h(0)k_{p,0} + k_{p,1}\right]cT.$$

The left term of (2.12) is bounded uniformly with respect to T. So, we divide (2.12) by T and let it goes to infinity, we get $c = 0$, then (2.11) yields

$$\text{(2.13)} \quad \begin{cases} z_{tt}(x,t) - (az_x)_x(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ z_x(0,t) = 0, & t > 0, \\ -(az_x)(1,t) = k_{p,1}z(1,t), & t > 0, \\ z(0,t) = 0, & t > 0. \end{cases}$$

To achieve the proof, we distinguish two cases.

Case $k_{p,1} = 0$. (2.13) can be written as follows

$$\begin{cases} z_{tt}(x,t) - (az_x)_x(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ z_x(0,t) = z_x(1,t) = 0, & t > 0, \\ z(0,t) = 0, & t > 0. \end{cases}$$

This system has a unique solution $z = 0$ (see [11]), so, the asymptotic stability follows for this case.

Case $k_{p,1} > 0$. We multiply the first equation of (2.13) by φz_x and integrate in x and t. Then one obtains

$$\left[\int_0^1 z_t\varphi z_x dx\right]_0^T - \frac{1}{2}\varphi(1)\int_0^T z_t^2(1,t)dt + \frac{1}{2}\int_0^T\int_0^1 \varphi_x z_t^2 dxdt$$
$$= \frac{1}{2}\int_0^T \frac{\varphi(1)}{a(1)}k_{p,1}^2 z^2(1,t)dt - \frac{1}{2}\int_0^T\int_0^1 a\left(\frac{\varphi}{a}\right)_x az_x^2 dxdt$$

$$\begin{aligned}\left[\int_0^1 z_t\varphi z_x dx\right]_0^T &= \frac{1}{2}\int_0^T \frac{\varphi(1)}{a(1)}k_{p,1}^2 z^2(1,t)dt - \frac{1}{2}\int_0^T\int_0^1 a\varphi_x\frac{1}{a}z_t^2 dxdt \\ &\quad -\frac{1}{2}\int_0^T\int_0^1 a\left(\frac{\varphi}{a}\right)_x az_x^2 dxdt + \frac{1}{2}\varphi(1)\int_0^T z_t^2(1,t)dt.\end{aligned}$$

The function φ is chosen so that

$$\varphi \in H^1(0,1), \quad -a(x)\varphi_x \geq k_{p,1}, \quad \varphi(1) \geq 1 \text{ and } -a\left(\frac{\varphi}{a}\right)_x \geq k_{p,1}.$$

With this choice and (1.2), we obtain

$$\frac{k_{p,1}}{2\underline{a}}\left[\int_0^T \left[\int_0^1 (z_t^2 + a(x)z_x^2)dx + k_{p,1}z^2(1,t)\right]dt\right] + \frac{1}{2}\int_0^T z_t^2(1,t)dt \leq \left[\int_0^1 z_t\varphi z_x dx\right]_0^T,$$

which implies that

$$\int_0^T E(t)dt \quad \leq \quad C(E(T)+E(0)) = 2CE(0),$$

where C is a positive constant. The energy is constant, so

$$E(T) \leq \frac{2}{T}CE(0) \quad \forall T > 0,$$

and then

$$E(t) = 0 \quad \forall t \geq 0,$$

i.e. $z \equiv 0$ and so $z_t \equiv 0$. Thus $\omega(y_0, y_1) = \{(0,0)\}$. Therefore, the proof of Theorem 1.1. is complete.

2.2. *Proof of Theorem 1.2.* We adopt the method used by Conrad and Rao [4] but in our case we have two nonlinear feedback functions f and g satisfying more general growth conditions. We introduce the same functional defined in [4]

$$\rho(t) = 2\int_0^1 y_t\varphi y_x dx + C_0\int_0^1 y_t\psi dx$$

where C_0 is a positive constant and ψ is the solution of the problem

$$\begin{cases} (a\psi_x)_x(x,t) = 0, & 0 < x < 1, \\ \psi(0,t) = y(0,t), & \\ \psi(1,t) = y(1,t). & \end{cases}$$

We verify that the following inequalities hold

$$\int_0^1 a\psi_x y_x dx = \left[\int_0^1 a^{-1}(s)ds\right]^{-1}[y(1,t)-y(0,t)]^2 \geq 0,$$

$$\int_0^1 \psi^2 dx \leq 5\left[y^2(0,t) + y^2(1,t)\right],$$

similarly,

$$\int_0^1 \psi_t^2 dx \leq 5\left[y_t^2(0,t) + y_t^2(1,t)\right].$$

The function φ is chosen so that

$$\varphi \in H^1(0,1), \quad \varphi_x \geq 1, \quad a\left(\frac{\varphi}{a}\right)_x \geq 1, \quad \varphi(0) < 0, \quad \varphi(1) > 0$$

and

$$k_{p,0}\varphi(0)a(1) + k_{p,1}\varphi(1)a(0) = 0.$$

We can show that there exist positive constants K_0, K_1 and K_2 such that for any $t \geq 0$,

$$|\rho(t)| \leq K_0 E(t), \tag{2.14}$$

$$\rho'(t) \leq -E(t) + K_1\left[y_t^2(0,t) + y_t^2(1,t)\right] + K_2\left[f^2(y_t(0,t)) + g^2(y_t(1,t))\right]. \tag{2.15}$$

Given $\varepsilon > 0$, we introduce (see [4]) the perturbed energy by

$$E_\varepsilon(t) = E(t) + \varepsilon\rho(t)\left[E(t)\right]^{\frac{(p\vee q)-1}{2}}. \tag{2.16}$$

In the formula (2.16), the exponent $p \vee q$ seems to be new. This together with the non-increasing of the energy $E(t)$ implies that for any $M > 1$

$$M^{-1/2}\left[E_\varepsilon(t)\right]^{\frac{(p\vee q)+1}{2}} \leq \left[E(t)\right]^{\frac{(p\vee q)+1}{2}} \leq M^{1/2}\left[E_\varepsilon(t)\right]^{\frac{(p\vee q)+1}{2}} \tag{2.17}$$

with

$$\varepsilon \leq K_0^{-1}\left[E(0)\right]^{\frac{1-(p\vee q)}{2}}\left(1 - M^{-\frac{1}{(p\vee q)+1}}\right).$$

Now, we calculate the derivative of the perturbed energy $E_\varepsilon(t)$.

$$E'_\varepsilon(t) = E'(t) + \varepsilon\frac{(p\vee q)-1}{2}\rho(t)E'(t)\left[E(t)\right]^{\frac{(p\vee q)-3}{2}} + \varepsilon\rho'(t)\left[E(t)\right]^{\frac{(p\vee q)-1}{2}} \tag{2.18}$$

on the other hand, from (1.13), (1.14) and (2.15), one obtains

$$\rho'(t) \leq -E(t) + K_3 y_t^2(0,t) + K_4 y_t^2(1,t) \tag{2.19}$$

where

$$K_3 = K_1 + K_2 C_2^2 \quad \text{and} \quad K_4 = K_1 + K_2 C_4^2.$$

Plugging (1.7), (2.14) and (2.19) into equation (2.18), one obtains

$$\begin{aligned} E'_\varepsilon(t) \quad \leq \quad & \left[-1 + \varepsilon\frac{(p\vee q)-1}{2}K_0\left[E(0)\right]^{\frac{(p\vee q)-1}{2}}\right] \\ & \left[k_{v,0}y_t(0,t)f(y_t(0,t)) + k_{v,1}y_t(1,t)g(y_t(1,t))\right] \\ & +\varepsilon\left[E(t)\right]^{\frac{(p\vee q)-1}{2}}\left[K_3 y_t^2(0,t) + K_4 y_t^2(1,t)\right] - \varepsilon\left[E(t)\right]^{\frac{(p\vee q)+1}{2}}. \end{aligned} \tag{2.20}$$

Now we distinguish the case $p \vee q = 1$ and $p \vee q > 1$.

(i) Case $p \vee q = 1$. In this case (2.20) yields

$$\begin{aligned} E'_\varepsilon(t) \quad \leq \quad & \left(\varepsilon\frac{K_4}{C_3} - k_{v,1}\right)y_t(1,t)g(y_t(1,t)) \\ & + \left(\varepsilon\frac{K_3}{C_1} - k_{v,0}\right)y_t(0,t)f(y_t(0,t)) - \varepsilon E(t). \end{aligned}$$

If we choose $\varepsilon \le \varepsilon_0 = \min\left(\frac{C_3 k_{v,1}}{K_4}, \frac{C_1 k_{v,0}}{K_3}\right)$, then from (1.8) and (2.17), the previous inequality becomes

$$E'_\varepsilon(t) \le -\varepsilon E(t) \le -\varepsilon M^{-1/2} E_\varepsilon(t).$$

Now, integrating $\frac{E'_\varepsilon(t)}{E_\varepsilon(t)}$ on $[0,t]$, and using (2.17), we get

$$E(t) \le M E(0) e^{-\omega t}, \quad \forall t \ge 0,$$

where $\omega = \varepsilon M^{-1/2}$, with $\varepsilon \le \min\left(\varepsilon_0, K_0^{-1}(1 - M^{-\frac{1}{2}})\right)$.

(ii) Case $p \vee q > 1$.

If $y_t^2(1,t) > 1$, it follows from hypothesis (1.8) and (1.14) that

$$\varepsilon K_4 [E(t)]^{\frac{(p\vee q)-1}{2}} y_t^2(1,t) \le \frac{\varepsilon K_4}{C_3} [E(0)]^{\frac{(p\vee q)-1}{2}} y_t(1,t) g(y_t(1,t)). \tag{2.21}$$

However, while $y_t^2(1,t) \le 1$, by Young's inequality, we have for any parameter $\delta > 0$,

$$\begin{aligned} \varepsilon K_4 [E(t)]^{\frac{(p\vee q)-1}{2}} y_t^2(1,t) \quad &\le \quad \frac{2\varepsilon}{(p\vee q)+1} (K_4\delta)^{\frac{(p\vee q)+1}{2}} |y_t(1,t)|^{(p\vee q)+1} \\ &\quad + \varepsilon \frac{(p\vee q)-1}{(p\vee q)+1} \delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} [E(t)]^{\frac{(p\vee q)+1}{2}}. \end{aligned}$$

Since $\min(|y_t(1,t)|, |y_t(1,t)|^q) = |y_t(1,t)|^q$, and $p\vee q \ge q$, we have

$$|y_t(1,t)|^{1+(p\vee q)} \le |y_t(1,t)|^q \le \frac{1}{C_3} y_t(1,t) g(y_t(1,t)).$$

This implies that

$$\begin{aligned} \varepsilon K_4 [E(t)]^{\frac{(p\vee q)-1}{2}} y_t^2(1,t) \quad &\le \quad \frac{2\varepsilon}{p\vee q+1} \frac{(K_4\delta)^{\frac{(p\vee q)+1}{2}}}{C_3} y_t(1,t) g(y_t(1,t)) \\ &\quad + \varepsilon \frac{(p\vee q)-1}{(p\vee q)+1} \delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} [E(t)]^{\frac{(p\vee q)+1}{2}}. \end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), one has

$$\begin{aligned} \varepsilon K_4 [E(t)]^{\frac{(p\vee q)-1}{2}} y_t^2(1,t) \quad &\le \quad \varepsilon \frac{(p\vee q)-1}{(p\vee q)+1} \delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} [E(t)]^{\frac{(p\vee q)+1}{2}} \\ &\quad + \varepsilon K_5 y_t(1,t) g(y_t(1,t)) \end{aligned} \tag{2.23}$$

where $K_5 = \frac{K_4}{C_3} [E(0)]^{\frac{(p\vee q)-1}{2}} + \frac{2}{(p\vee q)+1} \frac{(K_4\delta)^{\frac{(p\vee q)+1}{2}}}{C_3}$. Similarly, we can show that

$$\begin{aligned} \varepsilon K_3 [E(t)]^{\frac{(p\vee q)-1}{2}} y_t^2(0,t) \quad &\le \quad \varepsilon \frac{(p\vee q)-1}{(p\vee q)+1} \delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} [E(t)]^{\frac{(p\vee q)+1}{2}} \\ &\quad + \varepsilon K_6 y_t(0,t) f(y_t(0,t)), \end{aligned} \tag{2.24}$$

with $K_6 = \frac{K_3}{C_1}\left[E(0)\right]^{\frac{(p\vee q)-1}{2}} + \frac{2}{(p\vee q)+1}\frac{(K_3\delta)^{\frac{(p\vee q)+1}{2}}}{C_1}$. Inserting (2.23) and (2.24) into (2.20), we obtain

$$\begin{aligned} E_\varepsilon'(t) \le \varepsilon &\left[2\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} - 1\right]\left[E(t)\right]^{\frac{(p\vee q)+1}{2}} \\ &+ \left[-k_{v,0} + \varepsilon k_{v,0}\frac{(p\vee q)-1}{2}K_0\left[E(0)\right]^{\frac{(p\vee q)-1}{2}} + \varepsilon K_6\right] y_t(0,t)f(y_t(0,t)) \\ &+ \left[-k_{v,1} + \varepsilon k_{v,1}\frac{(p\vee q)-1}{2}K_0\left[E(0)\right]^{\frac{(p\vee q)-1}{2}} + \varepsilon K_5\right] y_t(1,t)g(y_t(1,t)). \end{aligned}$$

This implies that

$$E_\varepsilon'(t) \le -\mu\varepsilon\left[E(t)\right]^{\frac{(p\vee q)+1}{2}} \tag{2.25}$$

provided that δ is chosen such that for some $\mu > 0$, $2\frac{(p\vee q)-1}{(p\vee q)+1}\delta^{-\frac{(p\vee q)+1}{(p\vee q)-1}} - 1 \le -\mu < 0$, and ε is chosen as follows

$$-k_{v,i} + \varepsilon\left[K_{6-i} + k_{v,i}\frac{(p\vee q)-1}{2}K_0\left[E(0)\right]^{\frac{(p\vee q)-1}{2}}\right] \le 0,$$

where $i = 0,\ 1$. Combining (2.17) and (2.25), we get

$$E_\varepsilon'(t) \le -\mu\varepsilon M^{-1/2}\left[E_\varepsilon(t)\right]^{\frac{(p\vee q)+1}{2}}. \tag{2.26}$$

Finally, solving the differential inequality (2.26) and using (2.17), we obtain

$$E(t) \le ME(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}}$$

with $\omega = \frac{(p\vee q)-1}{2}\mu\varepsilon M^{-\frac{p\vee q}{(p\vee q)+1}}\left[E(0)\right]^{\frac{(p\vee q)-1}{2}}$.

This completes the proof of Theorem 1.2.

REMARK 2.2. The result of Theorem 1.2 (ii) remains true if we replace (1.14) with

$$\begin{aligned} C_1|x| &\le |f(x)| \le C_2\max\left(|x|,\ |x|^{\frac{1}{p}}\right), \\ C_3|x| &\le |g(x)| \le C_4\max\left(|x|,\ |x|^{\frac{1}{q}}\right), \end{aligned} \tag{2.27}$$

or with

$$\begin{aligned} C_1|x| &\le |f(x)| \le C_2\max\left(|x|,\ |x|^{\frac{1}{p}}\right), \\ C_3\min\left(|x|,\ |x|^{q}\right) &\le |g(x)| \le C_4|x|, \end{aligned} \tag{2.28}$$

where $(p,q) \in [1,+\infty[^2$ with $\max(p,q) = p\vee q > 1$.

For the cases (2.27) and (2.28) the energy decay estimates is: given any $M > 1$, there exists a constant $\omega > 0$ depending on $E(0)$ such that

$$E(t) \le ME(0)(1+\omega t)^{-\frac{2}{(p\vee q)-1}}, \quad \forall t \ge 0.$$

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