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REPRESENTATION THEOREMS FOR OPERATORS OF TYPE $\ell_{p,q}^{v,\omega,\psi}$ AND $S_{\omega,\psi}$

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Abstract. Let $0 < p,\ q \leq \infty$, and $\psi : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing, subadditive function which satisfies $\psi(0) = 0$. For sequences $v = \{v_n\}$, $\omega = \{\omega_n\}$ of positive numbers we define sequence classes $\ell_{p,q}^{v,\omega,\psi}$ and $S_{\omega,\psi}$ and give representation theorems for an operator of type $\ell_{p,q}^{v,\omega,\psi}$ and $S_{\omega,\psi}$.

1. Introduction

Ever since the publication of a paper by E. Schmidt [8] in 1907, particular type of decreasing sequences of real numbers have been playing a significant role in the study of linear operators on Hilbert and Banach spaces. Indeed, the concept of $s-$numbers which arose in the study of integral operators has led to different possibilities of defining several equivalents of $s-$numbers on Banach spaces, e.g. Kolmogorov numbers, approximation numbers, Gelfand numbers etc. [1, 5]. In this paper we are concerned with approximation numbers which have been extensively studied in [5, 4] and [6]. Here one finds many applications of these numbers in Hilbert space theory, theory of operator ideals and nuclear spaces. Besides being responsible for introducing several types of operator ideals in the class of bounded linear operators, these numbers represent the linear operators of particular type as infinite series of finite rank operators [2, 7]. In this paper, certain sequence classes which, as particular cases, give rise to some well known sequence spaces, e.g. Lorentz sequence spaces and the space of rapidly decreasing sequences, have been defined and the representation theorems for those bounded linear operators

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whose sequences of approximation numbers belong to these classes, have been established.

2. Preliminaries and Notations

Throughout this paper $E,\ F$ will denote the Banach spaces over complex field and $\mathcal{L}(E,F)$ represents the space of all bounded linear maps between E and F. $\mathbb{N}_0$ stands for the set $\{0,1,2,\ldots\}$ and $\mathbb{N}$ represents the set of all natural numbers. For any scalar sequence $x=\{x_i\}\in c_0$, we denote by $\{s_i(x)\}$ its non-negative and decreasing rearrangement. For $T\in\mathcal{L}(E,F)$ and $n\in\mathbb{N}$, the n^{th} approximation number of T is defined as

$$a_n(T)=\inf\{\|T-A\| : A\in\mathcal{L}(E,F),\ \mathrm{rank}(A)<n\}.$$

For a detailed study on these numbers one is referred to [1] and [6]. For a class λ of scalar sequences, T is said to be of type λ if $\{a_n(T)\}\in\lambda$ and it is said to be pseudo-λ nuclear [3] if there exist sequences $\{\lambda_n\}\in\lambda,\ \{f_n\}\subset E^*$ and $\{y_n\}\subset F$ such that $\|f_n\|\le 1,\ \|y_n\|\le 1,\ n\in\mathbb{N}$ and T has following representation

$$T(x)=\sum_{n=1}^{\infty}\ \lambda_n\ f_n(x)\ y_n,\ \forall\ x\in E.$$

If the class λ contains the space ϕ of finitely non-zero sequences, finite rank operators are of type λ. For non-negative real-valued sequences $\{\alpha_n\}$ and $\{\beta_n\},\ \alpha_n\prec\beta_n$ means that there is a constant $c>0$ depending on various parameters but not on the index n with $\alpha_n\le c\beta_n,\ n\in\mathbb{N}$. We write $\alpha_n\sim\beta_n$ if $\alpha_n\prec\beta_n$ and $\beta_n\prec\alpha_n,\ n\in\mathbb{N}$.

3. Operators of the type $\ell_{p,q}^{v,\omega,\psi}$

Let $v=\{v_n\}$ and $\omega=\{\omega_n\}$ be any two sequences of positive numbers and $\psi:[0,\infty)\to[0,\infty)$ be a continuous function which is strictly increasing, subadditive and satisfies $\psi(0)=0$. Let us define

$$\ell_{p,q}^{v,\omega,\psi}=\{\{x_i\}\in c_0 : \{\sum_{k=1}^{\infty}[k^{1/p-1/q}\ v_k\ \psi(\omega_k\ s_k(x))]^q\}^{1/q}<\infty\},$$

for $0<p\le\infty$ and $0<q<\infty$;

$$\ell_{p,\infty}^{v,\omega,\psi}=\{\{x_i\}\in c_0 : \sup_{k=1}^{\infty}\ k^{1/p}\ v_k\ \psi(\omega_k\ s_k(x))<\infty\},$$

for $0<p<\infty$ and $q=\infty$; and

$$\ell_{\infty,\infty}^{v,\omega,\psi}=\{\{x_i\}\in c_0 : \sup_{k=1}^{\infty}\ v_k\ \psi(\omega_k\ s_k(x))<\infty\},$$

for $p=\infty$ and $q=\infty$.

Clearly ϕ is a subset of the above classes. Further, when ψ is the identity mapping and $v_n=\omega_n=1,\ \forall\ n\in\mathbb{N}$ the classes defined above coincide with the Lorentz sequence spaces.

PROPOSITION 3.1. *Let* $0 < p < \infty$ *and* $0 < q \le \infty$. *For decreasing sequences* $v = \{v_n\}$, $\omega = \{\omega_n\}$ *of positive numbers and a fixed* $r \in \{2, 3, 4, \ldots\}$,

$$x = \{x_i\} \in \ell_{p,q}^{v,\omega,\psi} \;\Leftrightarrow\; \{r^{k/p}\; v_{r^k}\; \psi(\omega_{r^k}\; s_{r^k}(x))\} \in \ell^q.$$

PROOF. For a fixed $r \in \{2, 3, 4, \ldots\}$ and $k \in \mathbb{N}_0$, let

$$U_k = \{n : r^k \le n < r^{k+1}\}.$$

Since

$$\sum_{U_k} n^{q/p-1} \;\sim\; r^{kq/p}$$

we get

$$\begin{aligned}
&\{\sum_{k=0}^{\infty}\sum_{U_k}[n^{1/p-1/q}\; v_n\; \psi(\omega_n\; s_n(x))]^q\}^{1/q} \\
&\qquad\le\; \{\sum_{k=0}^{\infty}[\sum_{U_k} n^{q/p-1}]\; v_{r^k}\; \psi(\omega_{r^k}\; s_{r^k}(x))^q\}^{1/q} \\
&\qquad\le\; c\; \{\sum_{k=0}^{\infty}[r^{k/p}\; v_{r^k}\; \psi(\omega_{r^k}\; s_{r^k}(x))]^q\}^{1/q},
\end{aligned}$$

for some $c > 0$. Therefore $\{r^{k/p}\; v_{r^k}\; \psi(\omega_{r^k}\; s_{r^k}(x))\} \in \ell^q \;\Rightarrow\; \{x_i\} \in \ell_{p,q}^{v,\omega,\psi}$.

For the converse, let

$$V_k = \{n : r^k < n \le r^{k+1}\},\; k \in \mathbb{N}_0.$$

Since

$$\sum_{V_k} n^{q/p-1} \;\sim\; r^{(k+1)q/p}$$

we get

$$\begin{aligned}
&\{\sum_{k=0}^{\infty}[r^{k/p}\; v_{r^k}\; \psi(\omega_{r^k}\; s_{r^k}(x))]^q\}^{1/q} \\
&\quad= \{v_1^q\; \psi(\omega_1\; s_1(x))^q \;+\; \sum_{k=0}^{\infty}[r^{(k+1)/p}\; v_{r^{(k+1)}}\; \psi(\omega_{r^{(k+1)}}\; s_{r^{(k+1)}}(x))]^q\}^{1/q} \\
&\quad\le c\; \{v_1^q\; \psi(\omega_1\; s_1(x))^q \;+\; \sum_{k=0}^{\infty}[\sum_{V_k} n^{q/p-1}]\; v_{r^{(k+1)}}^q\; [\psi(\omega_{r^{(k+1)}} s_{r^{(k+1)}}(x))]^q\}^{1/q} \\
&\quad\le c\; \{v_1^q\; \psi(\omega_1\; s_1(x))^q \;+\; \sum_{k=0}^{\infty}\sum_{V_k}[n^{1/p-1/q}\; v_n\; \psi(\omega_n\; s_n(x))]^q\}^{1/q},
\end{aligned}$$

for some $c > 0$. Hence we get the result. ∎

THEOREM 3.2. *Let $v = \{v_n\}$ and $\omega = \{\omega_n\}$ be two decreasing sequences of positive numbers. For $0 < p < \infty$, $0 < q \leq \infty$ and a fixed $r \in \{2, 3, 4, \ldots\}$, if $T \in \mathcal{L}(E, F)$ is of the type $\ell_{p,q}^{v,\omega,\psi}$, then T can be expressed as $T = \sum_{k=0}^{\infty} T_k$, where $T_k \in \mathcal{L}(E, F)$, rank $(T_k) \leq r^k$ and $\{r^{k/p}\ v_{r^k}\ \psi(\omega_{r^k}\ \|T_k\|)\} \in \ell^q$. Conversely if $v_n = \omega_n = 1,\ \forall\ n \in \mathbb{N}_0$ and*

$$(*) \qquad T = \sum_{k=0}^{\infty} T_k,$$

where $T_k \in \mathcal{L}(E, F)$, rank $(T_k) \leq r^k$ and $\sum_{k=0}^{\infty} \|T_k\| < \infty$ then T is of the type $\ell_{p,q}^{v,\omega,\psi}$ provided $\{r^{k/p}\psi(\|T_k\|)\} \in \ell^q$.

PROOF. For proving the first part, suppose $T \in \mathcal{L}(E, F)$ is of the type $\ell_{p,q}^{v,\omega,\psi}$ i.e. $\{a_n(T)\} \in \ell_{p,q}^{v,\omega,\psi}$. Then for a fixed $r \in \{2, 3, 4, \ldots\}$, we can find $S_k \in \mathcal{L}(E, F)$ such that $\text{rank}(S_k) < r^k,\ k \in \mathbb{N}_0$ and

$$\|T - S_k\| \leq 2a_{r^k}(T).$$

The result clearly holds for finite rank operator T. So we assume that $T \in \mathcal{L}(E, F)$ is such that $a_n(T) \neq 0,\ \forall\ n \in \mathbb{N}$. Let us define

$$T_0 = 0,\ T_1 = 0 \text{ and } T_{k+2} = S_{k+1} - S_k,\ \forall\ k \in \mathbb{N}_0.$$

Then

$$\text{rank}(T_{k+2})\ <\ r^{k+2};$$

and also

$$\begin{aligned}
&\|T_{k+2}\| \leq \|S_{k+1} - T\| + \|T - S_k\|\ \leq\ 4\ a_{r^k}(T),\\
&v_{r^k}\ \psi(\omega_{r^k}\ \|T_{k+2}\|)\ \leq\ 4\ v_{r^k}\ \psi(\omega_{r^k}\ a_{r^k}(T)),\ k \in \mathbb{N}_0,\\
&[\sum_{k=0}^{\infty} r^{kq/p} v_{r^k}^q \psi(\omega_{r^k}\|T_k\|)^q]^{1/q} \leq 4\ [\sum_{k=0}^{\infty} r^{kq/p} v_{r^k}^q \psi(\omega_{r^k} a_{r^k}(T))^q]^{1/q} < \infty,
\end{aligned}$$

by Proposition 3.1. Further we have

$$T = \lim_{k\to\infty} S_k = \sum_{k=0}^{\infty} T_k,$$

since $\{a_n(T)\} \in c_0$.

Conversely, we assume that T has the representation given in $(*)$. Let

$$A_h = \sum_{k=0}^{h-1} T_k.$$

Then $\text{rank}(A_h)\ <\ r^h$. We now fix ρ and θ such that $0 < \rho < \min(1, q)$ and $0 < \theta < 1$. Define t as

$$1/t = 1/\rho - 1/q.$$

Note that for fixed h and n,

$$\sum_{k=h}^{n} \psi(\|T_k\|) \ \leq \ (\sum_{k=h}^{n} \psi(\|T_k\|)^{\rho})^{1/\rho}.$$

We now apply the Hölder's inequality to get

$$\begin{aligned}\sum_{k=h}^{n} \psi(\|T_k\|) \ &\leq \ (\sum_{k=h}^{n} r^{-\theta kt/p})^{1/t} \ (\sum_{k=h}^{n} r^{\theta kq/p} \psi(\|T_k\|)^q)^{1/q} \\ &= \ c\ r^{-\theta h/p} \ (\sum_{k=h}^{n} r^{\theta kq/p} \psi(\|T_k\|)^q)^{1/q},\end{aligned}$$

for a positive constant c. Since ψ is sub additive, continuous and

$$a_{r^h}(T) \leq \|T - A_h\| \leq \sum_{k=h}^{\infty} \|T_k\|,$$

we get

$$\begin{aligned}\{\sum_{h=0}^{\infty} r^{hq/p} \psi(a_{r^h}(T))^q\}^{1/q} \ &\leq \ c\ \{\sum_{h=0}^{\infty} r^{(1-\theta)hq/p} [\sum_{k=h}^{\infty} r^{\theta kq/p} \ \psi(\|T_k\|)^q]\}^{1/q} \\ &= \ c\ \{\sum_{k=0}^{\infty} [\sum_{h=0}^{k} r^{(1-\theta)hq/p}] r^{\theta kq/p} \ \psi(\|T_k\|)^q\}^{1/q} \\ &\leq \ c_1\ \{\sum_{k=0}^{\infty} \ r^{kq/p} \ \psi(\|T_k\|)^q\}^{1/q} \ < \ \infty,\end{aligned}$$

where c_1 is a positive constant. Now the result follows from the preceding proposition. ∎

PROPOSITION 3.3. *Let $0 < q < \infty$, $p = \infty$, $v_n = (1 + \log n)^{\rho}$, where $-1/q < \rho < \infty$ and $\omega = \{\omega_n\}$ be a decreasing sequence of positive terms. For any $x = \{x_i\} \in \ell_{\infty,q}^{v,\omega,\psi}$, we have*

$$\begin{aligned}&\sum_{n=1}^{\infty} \{(1 + \log n)^{\rho} \ \psi(\omega_n s_n(x))\}^q \ n^{-1} \\ &\qquad \sim \psi(\omega_1 s_1(x))^q \ + \ \sum_{n=0}^{\infty} 2^{n(1+\rho q)} \{\psi(\omega_{\mu_n} s_{\mu_n}(x))\}^q,\end{aligned}$$

where $\mu_k = 2^{2^n}, \ n \in \mathbb{N}_0$.

PROOF. For $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\{(1+\log n)^{\rho}\ \psi(\omega_n s_n(x))\}^q\ n^{-1} \\
&\qquad = \psi(\omega_1 s_1(x))^q + \sum_{k=0}^{\infty}\sum_{n=\mu_k}^{\mu_{k+1}-1}(1+\log n)^{\rho q} n^{-1}\psi(\omega_n s_n(x))^q \\
&\qquad \leq \psi(\omega_1 s_1(x))^q + \sum_{k=0}^{\infty}[\sum_{n=\mu_k}^{\mu_{k+1}-1}(1+\log n)^{\rho q} n^{-1}]\psi(\omega_{\mu_k} s_{\mu_k}(x))^q.
\end{aligned}$$

Since

$$\sum_{n=\mu_k}^{\mu_{k+1}-1}(1+\log n)^{\rho q} n^{-1} \sim 2^{k(1+\rho q)},$$

we get

$$\sum_{n=1}^{\infty}\{(1+\log n)^{\rho}\ \psi(\omega_n s_n(x))\}^q\ n^{-1} \sim \psi(\omega_1 s_1(x))^q + \sum_{k=0}^{\infty} 2^{k(1+\rho q)}\psi(\omega_{\mu_k} s_{\mu_k}(x))^q.$$

For the converse, we note that

$$\begin{aligned}
\psi(\omega_1 s_1(x))^q &+ \sum_{k=0}^{\infty} 2^{k(1+\rho q)}\psi(\omega_{\mu_k} s_{\mu_k}(x))^q \\
&\sim \psi(\omega_1 s_1(x))^q + (1+\log 2)^{\rho q}\psi(\omega_2 s_2(x))^q \\
&\quad + \sum_{k=0}^{\infty} 2^{k(1+\rho q)}\psi(\omega_{\mu_{(k+1)}} s_{\mu_{(k+1)}}(x))^q \\
&\sim\ \psi(\omega_1 s_1(x))^q + (1+\log 2)^{\rho q}\psi(\omega_2 s_2(x))^q \\
&\quad + \sum_{k=0}^{\infty}[\sum_{n=\mu_k+1}^{\mu_{k+1}}(1+\log n)^{\rho q} n^{-1}]\psi(\omega_{\mu_{(k+1)}} s_{\mu_{(k+1)}}(x))^q \\
&\leq\ \psi(\omega_1 s_1(x))^q + (1+\log 2)^{\rho q}\psi(\omega_2 s_2(x))^q \\
&\quad + \sum_{k=3}^{\infty}[(1+\log k)^{\rho}\psi(\omega_k s_k(x))]^q\ k^{-1} \\
&=\ \sum_{k=1}^{\infty}[(1+\log k)^{\rho}\ \psi(\omega_k s_k(x))]^q\ k^{-1}.
\end{aligned}$$

$\square$

THEOREM 3.4. *Let* $0 < q < \infty,\ -1/q < \rho < \infty$, $v_n = (1+\log n)^{\rho}$, $n \in \mathbb{N}$ *and* $\{\omega_n\}$ *be a decresing sequence of positive numbers. If* $T \in \mathcal{L}(E, F)$ *is of the type* $\ell_{\infty,q}^{v,\omega,\psi}$ *then there exist operators* $T_n \in \mathcal{L}(E, F)$ *with rank*$(T_n) \leq \mu_n$, *where* $\mu_n = 2^{2^n}$, $n \in \mathbb{N}_0$, *such that* T *has a representation of the form*

$$T = \sum_{n=0}^{\infty} T_n$$

and

$$\sum_{n=0}^{\infty} 2^{n(1+\rho q)} \psi(\omega_{\mu_n} \|T_n\|)^q < \infty.$$

Conversely, if $\omega_n = 1$, $n \in \mathbb{N}_0$ *and* T *has a representation of the form*

$$T = \sum_{n=0}^{\infty} T_n$$

where $T_n \in \mathcal{L}(E,F)$, *rank*$(T_n) \leq \mu_n$, $n \in \mathbb{N}_0$, *such that the series* $\sum_{k=0}^{\infty} \|T_k\|$ *is convergent then* T *is of the type* $\ell_{\infty,q}^{v,\omega,\psi}$ *provided*

$$\sum_{n=0}^{\infty} 2^{n(1+\rho q)} \psi(\|T_n\|)^q < \infty.$$

Proof. We first assume that $T \in \mathcal{L}(E,F)$ is of the type $\ell_{\infty,q}^{v,\omega,\psi}$. By definition of $a_{\mu_k}(T)$, $k \in \mathbb{N}_0$, there exist operators $L_k \in \mathcal{L}(E,F)$ with $\operatorname{rank}(L_k) < \mu_k$ such that

$$\|T - L_k\| < 2\ a_{\mu_k}(T).$$

Define the operators $T_k\ 's$ as $T_0 = 0$, $T_1 = L_0$ and $T_k = L_{k-1} - L_{k-2}$, $k = 2, 3, \ldots$.

Then $\|T_1\| \leq 3\ a_1(T)$ and $\|T_k\| \leq 4\ a_{\mu_{k-2}}(T)$, $k = 2, 3, \ldots$ yield

$$\psi(\omega_1 \|T_1\|) \leq 3\ \psi(\omega_1 a_1(T))$$

and

$$\psi(\omega_{\mu_k} \|T_k\|) \leq 4\ \psi(\omega_{\mu_{k-2}} a_{\mu_{k-2}}(T)),\ k = 2, 3, \ldots.$$

Further, we note that $\operatorname{rank}(T_k) < \mu_k$. Since $\{a_k(T)\} \in c_0$, we have

$$T = \sum_{n=0}^{\infty} T_n.$$

Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2^{n(1+\rho q)} \psi(\omega_{\mu_n} \|T_n\|)^q \\
&\qquad \le 2^{(1+\rho q)} 3^q \psi(\omega_1 a_1(T))^q + \sum_{n=0}^{\infty} 2^{(n+2)(1+\rho q)} 4^q \psi(\omega_{\mu_n} a_{\mu_n}(T))^q \\
&\qquad \sim \psi(\omega_1 a_1(T))^q + \sum_{n=0}^{\infty} 2^{n(1+\rho q)} \psi(\omega_{\mu_n} a_{\mu_n}(T))^q \\
&\qquad \sim \sum_{n=1}^{\infty} ((1+\log n)^\rho \psi(\omega_n a_n(T)))^q n^{-1},
\end{aligned}$$

by Proposition 3.3. Hence

$$\sum_{n=0}^{\infty} 2^{n(1+\rho q)} \psi(\omega_{\mu_n} \|T_n\|)^q < \infty.$$

Conversely, let us assume that $T = \sum_{n=0}^{\infty} T_n$ where $\mathrm{rank}(T_n) \le 2^{2^n}, n \in \mathbb{N}_0$. For $n \in \mathbb{N}$ define

$$A_n = \sum_{k=0}^{n-1} T_k.$$

Then $\mathrm{rank}(A_n) < \mu_n,\ n \in \mathbb{N}$. Therefore

$$a_{\mu_n(T)} \le \|T - A_n\| \le \sum_{k=n}^{\infty} \|T_k\|,\ n \in \mathbb{N}$$

and

$$a_{\mu_0}(T) \le \|T\| \le \sum_{k=0}^{\infty} \|T_k\|.$$

Fix $n,\ m \in \mathbb{N}$ and let $0 < p < \min(1,q),\ 0 < \theta < (1+\rho q)/q,\ 1/s = 1/p - 1/q$. Applying the Holder's inequality, we get

$$\begin{aligned}
(\sum_{k=n}^{m} \psi(\|T_k\|)^p)^{1/p} &\le (\sum_{k=n}^{m} 2^{-\theta k s})^{1/s} (\sum_{k=n}^{m} 2^{\theta k q} \psi(\|T_k\|)^q)^{1/q} \\
&\le C\ 2^{-\theta n} (\sum_{k=n}^{m} 2^{\theta k q} \psi(\|T_k\|)^q)^{1/q},
\end{aligned}$$

where C is a positive constant.

Using the sub additivity and continuity of ψ and by the previous proposition

$$\begin{aligned}
\sum_{n=1}^{\infty}[(1+\log n)^{\rho}\ \psi(a_n(T))]^q\ n^{-1} &\sim \psi(a_1(T))^q\ +\ \sum_{n=0}^{\infty} 2^{n(1+\rho q)}\psi(a_{\mu_n}(T))^q \\
&\leq \psi(a_1(T))^q + C^q\ \sum_{n=0}^{\infty} 2^{n(1+\rho q-\theta q)}(\sum_{k=n}^{\infty} 2^{\theta k q}\psi(\|T_k\|)^q) \\
&\sim \psi(a_1(T))^q + \sum_{k=0}^{\infty}(\sum_{n=0}^{k} 2^{n(1+\rho q-\theta q)})2^{\theta k q}\psi(\|T_k\|)^q \\
&\leq K\ [\psi(a_1(T))^q + \sum_{k=0}^{\infty} 2^{k(1+\rho q)}\psi(\|T_k\|)^q] < \infty.
\end{aligned}$$

Here the positive constant K depends only on $q,\ \rho$ or θ. Hence we get the result. ∎

THEOREM 3.5. *Let $\mu_n = 2^{2^n},\ n \in \mathbb{N}_0$ and $T \in \mathcal{L}(E,F)$. For $0 < \rho < \infty$, define $v_n = (1+\log n)^{\rho},\ n \in \mathbb{N}$ and let $\{\omega_n\}$ be a decreasing sequence of positive numbers. If $\{a_n(T)\} \in \ell_{\infty,\infty}^{v,\omega,\psi}$ then for each $n \in \mathbb{N}_0$, there exist operators $T_n \in \mathcal{L}(E,F)$ with rank$(T_n) < \mu_n$ such that*

$$T = \sum_{n=0}^{\infty} T_n$$

and

$$\sup_{n=0}^{\infty}\{2^{n\rho}\ \psi(\omega_{\mu_n}\|T_n\|)\} < \infty.$$

Conversely, if $\omega_n = 1,\ \forall\ n \in \mathbb{N}$, $T \in \mathcal{L}(E,F)$ has following representation

$$T = \sum_{n=0}^{\infty} T_n,$$

where $T_n \in \mathcal{L}(E,F)$ with rank$(T_n) < \mu_n$ and the series $\sum_{n=0}^{\infty}\|T_n\|$ is convergent, then $\{a_n(T)\} \in \ell_{\infty,\infty}^{v,\omega,\psi}$ provided

$$\sup_{n=0}^{\infty}\{2^{n\rho}\ \psi(\|T_n\|)\} < \infty.$$

PROOF. Let $\{a_n(T)\} \in \ell_{\infty,\infty}^{v,\omega,\psi}$. By the definition of approximation numbers, there exists operator $L_n \in \mathcal{L}(E,F)$ with rank$(L_n) < \mu_n$ and

$$\|T - L_n\| < 2a_{\mu_n}(T).$$

Define $T_0 = 0,\ T_1 = L_0$ and $T_n = L_{n-1} - L_{n-2},\ n = 2,3,4,\ldots.$

Then

$$\|T - A_{n+1}\| = \|T - L_n\| \leq 2a_{\mu_n}(T)\ \to\ 0 \text{ as } n\ \to\ \infty.$$

Also, $\operatorname{rank}(T_n) < \mu_n,\ \|T_1\| \leq 3a_1(T)$ and $\|T_n\| \leq 4a_{\mu_{n-2}}(T),\ n = 2, 3, 4, \ldots$. Therefore

$$\psi(\omega_1\|T_1\|) \leq 3\psi(\omega_1 a_1(T)) \text{ and } \psi(\omega_{\mu_n}\|T_n\|) \leq 4\psi(\omega_{\mu_{n-2}} a_{\mu_{n-2}}(T)),$$

for $n = 2, 3, 4, \ldots$ and

$$\begin{aligned}
&\sup\{2^{n\rho}\psi(\omega_{\mu_n}\|T_n\|) : n \in \mathbb{N}_0\}\\
&\quad \leq \sup\{2^{\rho}\psi(\omega_1\|T_1\|),\ 2^{(n+2)\rho}\psi(\omega_{\mu_{n+2}}\|T_{n+2}\|) : n \in \mathbb{N}_0\}\\
&\quad \leq 2^{3\rho+2}\sup\{\psi(\omega_1\|T_1\|),\ \psi(\omega_2 a_2(T)), 2^{(n-1)\rho}\psi(\omega_{\mu_n} a_{\mu_n}(T)) : n \in \mathbb{N}\}\\
&\quad \leq c\ \sup\{\psi(\omega_1\|T_1\|),\ (1+\log 2)^{\rho}\ \psi(\omega_2 a_2(T)),\\
&\qquad\qquad \max\{(1+\log k)^{\rho}\psi(\omega_k a_k(T)) : \mu_{n-1} < k \leq \mu_n\}, n \in \mathbb{N}\}\\
&\quad = c\ \sup\{(1+\log n)^{\rho}\psi(\omega_n a_n(T)) : n \in \mathbb{N}\} < \infty,
\end{aligned}$$

where c is a positive constant depending only on ρ.

For the converse, for $n \in \mathbb{N}$ let us write

$$A_n = \sum_{k=0}^{n-1} T_k.$$

Then $\operatorname{rank}(A_n) < \mu_n,\ n \in \mathbb{N}$. Also, for $n < m$

$$\begin{aligned}
\sum_{k=n}^{m}\psi(\|T_k\|) &= \sum_{k=n}^{m} 2^{-k\rho}\ 2^{k\rho}\psi(\|T_k\|)\\
&\leq 2^{-n\rho}(1-2^{-\rho})^{-1}\sup\{2^{k\rho}\ \psi(\|T_k\|) : k = n, n+1, \ldots, m\}.
\end{aligned}$$

Since

$$a_{\mu_0}(T) \leq a_1(T) \leq \sum_{k=0}^{\infty}\|T_k\|$$

and

$$a_{\mu_n}(T) \leq \sum_{k=n}^{\infty}\|T_k\|,\ n \in \mathbb{N},$$

by the subadditivity and continuity of ψ, we get

$$\psi(a_{\mu_0}(T)) \leq \sum_{k=0}^{\infty}\psi(\|T_k\|) \leq (1-2^{-\rho})^{-1}\sup\{2^{k\rho}\ \psi(\|T_k\|) : k \in \mathbb{N}_0\}$$

and

$$\begin{aligned}
\psi(a_{\mu_n}(T)) &\leq \sum_{k=n}^{\infty}\psi(\|T_k\|)\\
&\leq 2^{-n\rho}(1-2^{-\rho})^{-1}\sup\{2^{k\rho}\ \psi(\|T_k\|) : k = n, n+1, \ldots\}.
\end{aligned}$$

We now have

$$\begin{aligned}\sup_{n=1}^{\infty}\{(1+\log n)^{\rho}\psi(a_n(T))\} &= \sup\{\psi(a_1(T)),\max\{(1+\log k)^{\rho}\psi(a_k(T)) : \\ &\qquad \mu_n \le k < \mu_{n+1}\} : n\in\mathbb{N}_0\} \\ &\le k_1 \ \sup\{\psi(a_1(T)), 2^{n\rho}\psi(a_{\mu_n}(T)) : n\in\mathbb{N}_0\} \\ &\le k_2 \ \sup_{n=1}^{\infty}\{2^{n\rho}\psi(\|T_n\|)\} < \infty;\end{aligned}$$

where the constants $k_1 > 0$ and $k_2 > 0$ depend only on ρ. ∎

4. Operators of the type $S_{\psi,\omega}$

Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous function which is strictly increasing, subadditive and satisfies $\psi(0) = 0$. Further let $\omega = \{\omega_n\}$ be an increasing sequence of positive numbers. We define

$$\begin{aligned}S_{\psi,\omega} &= \{\{x_n\} \subseteq \mathbb{K} : \forall\ k\in\mathbb{N}_0,\ \exists\ M_k > 0 \\ &\qquad \text{with } \sup_{n=0}^{\infty}(n+1)^k\psi(\omega_n|x_n|) \le M_k\}.\end{aligned}$$

If $\omega_n = 1$, $\forall\ n\in\mathbb{N}_0$ and ψ is the identity mapping, we get the class of all rapidly decreasing sequences.

Proposition 4.1. *For* $\{x_n\}\in S_{\psi,\omega}$,

$$\sum_{n=0}^{\infty}(n+1)^k\ (\psi(\omega_n|x_n|))^p < \infty$$

for each $p > 0$ *and* $k\in\mathbb{N}_0$.

Proof. Let $p > 0$ and $k\in\mathbb{N}_0$. We now choose a $r\in\mathbb{N}_0$ such that $rp \ge k+2$. Since $\{x_n\}\in S_{\psi,\omega}$, we can find a constant $M_r > 0$ such that

$$(n+1)^r\ \psi(\omega_n|x_n|) \le M_r$$

for each $n\in\mathbb{N}_0$. Then

$$(n+1)^k\ (\psi(\omega_n|x_n|))^p \ \le\ (n+1)^{rp-2}(\psi(\omega_n|x_n|))^p \ \le\ \frac{M_r^p}{(n+1)^2}$$

and

$$\sum_{n=0}^{\infty}(n+1)^k\ (\psi(\omega_n|x_n|))^p \ \le\ M_r^p\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\ <\ \infty.$$

∎

Proposition 4.2. $S_{\psi,\omega}\subseteq\ell^1$.

PROOF. Let $\{x_n\} \in S_{\psi,\omega}$ be such that $\{x_n\}$ is not in ℓ^1. So for each $k \in \mathbb{N}_0$ we can find an increasing sequence $\{n_k\}$ of natural numbers such that $n_k > k,\ \forall\ k$ and

$$\sum_{n=n_k+1}^{n_{k+1}} |x_n| \ \geq\ 2^k\ \omega_k^{-1}.$$

Since $\{\omega_k\}$ is an increasing sequence, we get

$$2^k \ \leq\ \omega_k \sum_{n=n_k+1}^{n_{k+1}} |x_n| \ \leq\ \sum_{n=n_k+1}^{n_{k+1}} \omega_n\ |x_n|.$$

Therefore

$$\psi(2^k) \ \leq\ \psi(\sum_{n=n_k+1}^{n_{k+1}} \omega_n\ |x_n|) \ \leq\ \sum_{n=n_k+1}^{n_{k+1}} \psi(\omega_n\ |x_n|)$$

using the increasing nature of ψ. Now $\psi(2^k) \to \infty$ as $k \to \infty$. This contradicts the conclusion in the preceding proposition. Hence $S_{\psi,\omega} \subseteq \ell^1$.

ALITER: by the assumption on ψ and ω, for any $\{x_n\} \in S_{\psi,\omega}$ we have

$$\psi(\omega_0 \sum_{n=0}^{\infty} |x_n|) \ \leq\ \psi(\sum_{n=0}^{\infty} \omega_n |x_n|) \ \leq\ \sum_{n=0}^{\infty} \psi(\omega_n |x_n|) \ \leq\ M_2 \sum_{n=0}^{\infty} (n+1)^{-2} \ <\ \infty,$$

where M_2 is the constant from Proposition 4.1. This shows $\{x_n\} \in \ell^1$. ∎

PROPOSITION 4.3. *If $\{x_n\}$ is a sequence such that the sequence $\{\omega_n |x_n|\}$ is decreasing and*

$$\sum_{n=0}^{\infty} (\psi(\omega_n |x_n|))^p \ <\ \infty$$

for each $p > 0$, then $\{x_n\} \in S_{\psi,\omega}$.

PROOF. By the hypothesis, for any $n \in \mathbb{N}_0$ and $p > 0$ we have

$$(n+1)\ (\psi(\omega_n |x_n|))^p \ \leq\ \sum_{m=0}^{n} (\psi(\omega_m |x_m|))^p \ \leq\ \sum_{n=0}^{\infty} (\psi(\omega_n |x_n|))^p \ \leq\ M$$

for some $M > 0$. Then

$$(n+1)^k\ \psi(\omega_n |x_n|) \ =\ [(n+1)\ \psi(\omega_n |x_n|)^{1/k}]^k \ \leq\ M^k \ <\ \infty,$$

for each $k \in \mathbb{N}_0$. Thus $\{x_n\} \in S_{\psi,\omega}$. ∎

THEOREM 4.4. *Let $T \in \mathcal{L}(E, F)$. If T is of the type $S_{\psi,\omega}$ then there exists an increasing sequence $v = \{v_n\}$ such that T is pseudo $S_{\psi,v}$- nuclear and*

$$S_{\psi,\omega} \subseteq S_{\psi,v}.$$

Conversely, if T is pseudo $S_{\psi,\omega}$- nuclear, then it is of the type $S_{\psi,\omega}$.

PROOF. Suppose T is of the type $S_{\psi,\omega}$ i.e. $\{a_n(T)\} \in S_{\psi,\omega}$. Now for each $n \in \mathbb{N}$, we can find operators $T_n \in \mathcal{L}(E,F)$ of rank $(T_n) < n$ such that

$$\|T - T_n\| \leq 2\, a_n(T).$$

Let $B_n = T_{n+1} - T_n$ and $b_n = \dim(\text{range}(B_n))$. Then

$$b_n \leq 2(n+1)$$

and

$$\|B_n\| \leq \|T_{n+1} - T\| + \|T - T_n\| \leq 4\, a_n(T).$$

Define $d_0 = 0,\ d_n = \sum_{i=1}^{n} b_i$ and let $v_j = \omega_n,\ d_{n-1} < j \leq d_n,\ n \in \mathbb{N}$. Then

$$\psi(v_{d_n}\, \|B_n\|) \leq 4\, \psi(v_{d_n}\, a_n(T)),$$

using the subadditive nature of ψ. We now have, for each $p > 0$

$$\sum_{n=0}^{\infty} b_n\, (\psi(v_{d_n}\, \|B_n\|))^p \leq 2.4^p \sum_{n=0}^{\infty} (n+1)(\psi(\omega_n\, a_n(T)))^p < \infty$$

by Proposition 4.1. Now each B_n has a representation of the form

$$B_n(x) = \sum_{i=1}^{b_n} \lambda_i^n\, f_i^n(x)\, y_i^n$$

for each $x \in E$, where $\|f_i^n\| \leq 1$, $\|y_i^n\| \leq 1$ and $\|\lambda_i^n\| \leq \|B_n\|$, for each $i = 1, 2, \ldots, b_n$. Then

$$\sum_{n=0}^{\infty} \sum_{i=1}^{b_n} (\psi(v_{d_{n-1}+i}\, |\lambda_i^n|))^p \leq \sum_{n=0}^{\infty} b_n\, (\psi(\omega_n\, \|B_n\|))^p < \infty$$

for each $p > 0$. Since $\{a_n(T)\} \in \ell^1$, $a_n(T) \to 0$ as $n \to \infty$. Therefore for each $x \in E$,

$$T(x) = \lim_{n\to\infty} T_{n+1}(x) = \sum_{n=0}^{\infty} B_n(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{b_n} \lambda_i^n\, f_i^n(x)\, y_i^n.$$

Thus T can be rewritten as

$$T(x) = \sum_{i=0}^{\infty} \mu_i\, g_i(x)\, z_i \tag{4.1}$$

where $g_n \in X^*,\ z_n \in Y,\ \|g_n\| \leq 1,\ \|z_n\| \leq 1$, for each $n \in \mathbb{N}_0$ and μ_i's are given by

$$\mu_{d_{n-1}+j} = \lambda_j^n,\ 1 \leq j \leq b_n.$$

Sequences $\{g_n\}$ and $\{z_n\}$ are defined in the similar fashion corresponding to $\{f_n\}$ and $\{y_n\}$ respectively. Note that

$$\sum_{n=0}^{\infty} \psi(v_n\, |\mu_n|))^p = \sum_{n=1}^{\infty} \sum_{j=1}^{b_n} \psi(v_{d_{n-1}+j} |\lambda_j^n|)^p \leq \sum_{n=1}^{\infty} b_n \psi(\omega_n \|B_n\|)^p < \infty.$$

for each $p > 0$. In particular when $p = 1$, we get $\{v_n|\mu_n|\} \in \ell_1 \subset c_0$, by the continuity and subadditivity of ψ. Thus the terms of the sequence $\{v_n|\mu_n|\}$ can be rearranged in a decreasing order. Let σ be the permutation which is being used in making such a rearrangement. Since the above series is convergent even if we replace n by $\sigma(n)$, by Proposition 4.3, we get $\{\mu_{\sigma(n)}\} \in S_{\psi,v}$. Further we note that the convergence of series in (4.1) is unconditional. Indeed for any permutation σ, we have

$$\|\sum_{i=n}^{m} \mu_{\sigma(i)}\ g_{\sigma(i)}(x)\ z_{\sigma(i)}\| \ \leq\ \|x\| \sum_{i=n}^{m} |\mu_{\sigma(i)}| \ \to\ 0$$

as $n,\ m\ \to \infty$ because $S_{\psi,v} \subseteq \ell^1$. Thus T is pseudo $S_{\psi,\omega}$ -nuclear.

Conversely, if T is pseudo $S_{\psi,\omega}$ -nuclear, then it can written as

$$T(x)\ =\ \sum_{i=0}^{\infty} \lambda_i\ f_i(x)\ y_i$$

where $f_n \in X^*,\ y_n \in Y,\ \|f_n\| \leq 1,\ \ \|y_n\| \leq 1$, for each $n \in \mathbb{N}_0$ and $\{\lambda_i\} \in S_{\psi,\omega}$. For $n \in \mathbb{N}_0$, we define

$$T_n(x)\ =\ \sum_{i=0}^{n-1} \lambda_i\ f_i(x)\ y_i.$$

It follows

$$a_n(T)\ \leq\ \|T - T_n\| \leq \sup_{\|x\|\leq 1} \|\sum_{i=n}^{\infty} \lambda_i\ f_i(x)\ y_i\|\ \leq\ \sum_{i=n}^{\infty} |\lambda_i|.$$

Using the increasing nature of $\{\omega_n\}$, we get

$$\omega_n\ a_n(T)\ \leq\ \sum_{i=n}^{\infty} \omega_i\ |\lambda_i|,$$

$$\psi(\omega_n\ a_n(T))\ \leq\ \sum_{i=n}^{\infty} \psi(\omega_i\ |\lambda_i|).$$

Therefore, for any $p \in [0, 1)$,

$$(\psi(\omega_n\ a_n(T)))^p\ \leq\ (\sum_{i=n}^{\infty} \psi(\omega_i\ |\lambda_i|))^p\ \leq\ \sum_{i=n}^{\infty} (\psi(\omega_i\ |\lambda_i|))^p,$$

so

$$\sum_{n=0}^{\infty} (\psi(\omega_n\ a_n(T)))^p\ \leq\ \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} (\psi(\omega_i\ |\lambda_i|))^p\ =\ \sum_{n=0}^{\infty} (n+1)\ (\psi(\omega_n\ |\lambda_n|))^p\ <\ \infty.$$

Since $\ell^p \subseteq \ell^q$, if $p \leq q$, we get the convergence of the above series for all $p > 0$. Now the result follows from Proposition 4.3. ∎

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