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ON THE n-FOLD PSEUDO-HYPERSPACE SUSPENSIONS OF CONTINUA

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Abstract. Let X be a (metric) continuum. Let n be a positive integer, let $C_n(X)$ denote the space of all nonempty closed subsets of X with at most n components and let $F_1(X)$ denote the space of singletons. The *n-fold pseudo-hyperspace suspension of* X is the quotient space $C_n(X)/F_1(X)$. We present properties of this hyperspace.

1. Introduction

In 1979 Sam B. Nadler Jr., introduced the hyperspace suspension of a continuum, $C(X)/F_1(X)$ [18]. In 2004 Sergio Macías, gave a natural generalization of such hyperspace and he defined the n-fold hyperspace suspension of a continuum X as the quotient space $C_n(X)/F_n(X)$ [12].

Now, we introduce the n-fold pseudo-hyperspace suspension of a continuum X as the quotient space $C_n(X)/F_1(X)$ topologized with the quotient topology. The purpose of this paper is to present a study of these hyperspaces. The paper is divided into six sections. In section 2, we give the basic definitions necessary for understanding the paper. In section 3, we present some general properties of the n-fold pseudo-hyperspace suspensions of continua. In section 4, we present results about the n-fold pseudo-hyperspace suspensions of locally connected continua. In section 5, we show that the n-fold pseudo-hyperspace suspensions of continua are zero-dimensional aposyndetic. In section 6, we show that a continuum X is indecomposable if and only if by removing two specific points from the n-fold pseudo-hyperspace suspension of X it becomes arcwise disconnected.

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2. Definitions

If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball about A of radious ε is denoted by $\mathcal{V}_{\varepsilon}^{d}(A)$; that is:

$$\mathcal{V}_{\varepsilon}^{d}(A) = \{y \in Y : d(y, a) < \varepsilon \text{ for some } a \in A\}.$$

We write $\mathcal{V}_{\varepsilon}(A)$ when the metric is understood.

The interior of A is denoted by $Int_Y(A)$, and the closure of A is denoted by $Cl_Y(A)$. We omit the subindices when there is no confusion. A map means a continuous function.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* of a space Y is a continuum contained in Y. A continuum is said to be *decomposable* provided that it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum is said to have *property* (b) provided that any map from X into the unit circle S^1 is homotopic to a constant map. A continuum X is *unicoherent* provided that if $X = A \cup B$, where A and B are subcontinua of X, then $A \cap B$ is connected. A continuum X is *uniformly pathwise connected* provided that it is the continuous image of the cone over the Cantor set [7, 3.5].

An *arc* is any space homeomorphic to $[0, 1]$. The *end points* of an arc α are the image of 0 and 1 under any homeomorphism from $[0, 1]$ onto α. An n-*cell* is any space homeomorphic to $[0, 1]^n$.

Given a continuum X and a positive integer n, $C_n(X)$ denotes the n-*fold hyperspace of* X; that is:

$$C_n(X) = \{A \subset X : A \text{ is nonempty, closed and has at most } n \text{ components}\},$$

topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 : A \subset \mathcal{V}_{\varepsilon}(B) \text{ and } B \subset \mathcal{V}_{\varepsilon}(A)\},$$

$\mathcal{H}$ always denotes the Hausdorff metric.

The symbol $F_n(X)$ denotes the n-*fold symmetric product of* X; that is:

$$F_n(X) = \{A \subset X : A \text{ has at most } n \text{ points}\}.$$

We agree that $C(X) = C_1(X)$.

It is known that $C_n(X)$ is an arcwise continuum (for $n = 1$ see [17, (1.12)], for $n \geq 2$ see [10, (3.1)]).

Notation 2.1. *Let* $U_1, ..., U_m$ *be nonempty subsets of a continuum* X, *we define*

$$\langle U_1, ..., U_m \rangle =$$

$$\left\{ A \in C_n(X) : A \subset \bigcup_{k=1}^{m} U_k \textit{ and } A \cap U_k \neq \emptyset \textit{ for each } k \in \{1, ..., m\} \right\}.$$

It is known that $C_n(X)$ can be topologized with the *Vietoris Topology* (see [17, (0.11)]), a base of which is given by:

$$\mathcal{B} = \{\langle U_1, \ldots, U_m \rangle : U_1, \ldots, U_m \text{ are nonempty open subsets of } X\}.$$

Also, it is known that the Vietoris Topology and the topology induced by the Hausdorff metric coincide [17, (0.13)].

An *order arc* in $C_n(X)$ is a one-to-one map $\alpha : [0,1] \to C_n(X)$ such that if $0 \leq s < t \leq 1$ then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$. If we have an order arc α, we use the notation α to refer to the map α or to the image of the map $(\alpha([0,1]))$, when there is no confusion.

By the *n-fold hyperspace suspension* of a continuum X, denoted by $HS_n(X)$, we mean the quotient space:

$$HS_n(X) = C_n(X)/F_n(X),$$

with the quotient topology. Remark that $HS_1(X)$ corresponds to the hyperspace suspension $HS(X)$ defined by Nadler in [18]. Thus, for the rest of the paper we assume that n is an integer such that $n \geq 2$.

By the *n-fold pseudo-hyperspace suspension* of a continuum X, denoted by $PHS_n(X)$, we mean the quotient space:

$$PHS_n(X) = C_n(X)/F_1(X),$$

with the quotient topology. The fact that $PHS_n(X)$ and $HS_n(X)$ are continua follows from [19, 3.10].

NOTATION 2.2. *Given a continuum X, $q_X^n : C_n(X) \to PHS_n(X)$ denotes the quotient map. Also, $q_X^n(F_1(X)) = \{F_X^n\}$ and $q_X^n(X) = T_X^n$.*

REMARK 2.3. Note that $PHS_n(X) \setminus \{F_X^n\}$ and $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ are homeomorphic to $C_n(X) \setminus F_1(X)$ and $C_n(X) \setminus (\{X\} \cup F_1(X))$, respectively.

A *free arc* in a continuum X is an arc β in X such that $\beta \setminus \{\text{end points}\}$ is open in X.

A *Hilbert cube* is any space homeomorphic to $\mathcal{Q} = \prod_{n=1}^{\infty} [0,1]_n$ with the product topology, where $[0,1]_n = [0,1]$ for each positive integer n.

Other definitions are given as required.

3. General properties

Let Y be a metric, compact space with the metric d. A closed subset A of Y is said to be a *Z-set in Y* provided that for each $\varepsilon > 0$, there is a continuous function f_ε from Y into $Y \setminus A$ such that f_ε is within ε of the identity map on Y (i.e., $d(f_\varepsilon(y), y) < \varepsilon$ for all $y \in Y$) [5, p. 78].

LEMMA 3.1. *$F_1([0,1])$ is a Z-set in $C_2([0,1])$.*

Proof. Let $\varepsilon > 0$, $I = [0,1]$ and let $f_\varepsilon : C_2(I) \to C_2(I) \backslash F_1(I)$ given by $f_\varepsilon(A) = Cl(\mathcal{V}^d_{\frac{\varepsilon}{2}}(A))$, it is easy to see that f_ε satisfies the requires properties. $\square$

Note that by [4, Lemma 2.2], $C_2(I)$ is a 4-cell. Hence, we have the following:

Corollary 3.2. *$F_1([0,1])$ is a subset of the manifold boundary of $C_2([0,1])$.*

Proof. Let β be the manifold boundary of $C_2([0,1])$. Since $F_1([0,1])$ is a Z-set in $C_2([0,1])$ (Lemma 3.1), we have that $F_1([0,1]) \subset \beta$ [5, p. 78]. $\square$

The following result is analogous to [15, 4.6] and its proof is very similar if we use Corollary 3.2.

Theorem 3.3. *$PHS_2([0,1])$ is homeomorphic to $[0,1]^4$.*

Theorem 3.4. *$PHS_n(S^1)$ does not have the fixed point property.*

Proof. Since $C(S^1)$ is a 2-cell [17, Example 0.55], we have that $C(S^1)$ is an absolute retract [20, Théorème II_m]. Then since $C(S^1) \subset C_n(S^1)$, there exists a retraction $G : C_n(S^1) \to C(S^1)$. Hence, $G(F_1(S^1)) = F_1(S^1)$. Then there exists a map $G_* : C_n(S^1)/F_1(S^1) \to C(S^1)/F_1(S^1)$ such that $G_* \circ q^n_{S^1} = q_{S^1} \circ G$, where $q^n_{S^1} : C_n(S^1) \to C_n(S^1)/F_1(S^1)$ and $q_{S^1} : C(S^1) \to C(S^1)/F_1(S^1)$ are the quotient maps [2, Theorem 7.7, p. 17]. Observe that $PHS_n(S^1) = C_n(S^1)/F_1(S^1)$ and $HS(S^1) = C(S^1)/F_1(S^1)$. Since G is a retraction, it is not difficult to see that $G_* : PHS_n(S^1) \to HS(S^1)$ is a retraction. Thus, since $HS(S^1)$ is a 2-sphere [18, p. 131], $PHS_n(S^1)$ does not have the fixed point property. $\square$

Remark 3.5. Let us observe that it is not known if $HS_n(S^1)$ has the fixed point property for $n \geq 2$.

Theorem 3.6. *If X is a contractible continuum and n is a positive integer, then $PHS_n(X)$ is contractible.*

Proof. Let $F : X \times [0,1] \to X$ be a map such that $F(x,0) = x$ and $F(x,1) = p$ for all $x \in X$ and some $p \in X$. Define $G : C_n(X) \times [0,1] \to C_n(X)$ by $G(A,t) = F(A \times \{t\})$. Then G is continuous, $G(A,0) = A$, $G(A,1) = \{p\}$ for each $A \in C_n(X)$ and if $A \in F_1(X)$, then $G(A,t) \in F_1(X)$ for all $t \in [0,1]$. Let $R : PHS_n(X) \times [0,1] \to PHS_n(X)$ given by

$$R(\chi,t) = \begin{cases} F^n_X & \text{if } \chi = F^n_X; \\ q^n_X\left(G\left((q^n_X)^{-1}(\chi), t\right)\right) & \text{if } \chi \neq F^n_X. \end{cases}$$

Then R is continuous [2, 4.3, p. 126]. Note that $R(\chi,0) = \chi$ and $R(\chi,1) = F^n_X$ for every $\chi \in PHS_n(X)$. Therefore, $PHS_n(X)$ is contractible. $\square$

COROLLARY 3.7. *$PHS_n([0,1])$ is not homeomorphic to $PHS_n(S^1)$, for any $n \geq 2$.*

PROOF. Since $[0,1]$ is contractible, by Theorem 3.6, $PHS_n([0,1])$ is contractible. Since $PHS_n(S^1)$ has a retract homeomorphic to S^2 (see proof of Theorem 3.4), $PHS_n(S^1)$ is not contractible. Therefore, $PHS_n([0,1])$ is not homeomorphic to $PHS_n(S^1)$. ☐

THEOREM 3.8. *If X is a continuum, then $PHS_n(X)$ is uniformly pathwise connected for any integer $n \geq 2$.*

PROOF. It is known that $C_n(X)$ is a continuous image of the Cantor fan [6, Remark, p. 29]. Hence, $PHS_n(X)$ is a continuous image of the Cantor fan. Therefore, $PHS_n(X)$ is uniformly pathwise connected [7, 3.5]. ☐

The proof of the following result is similar to that in [12, Theorem 3.6].

THEOREM 3.9. *If X is a finite-dimensional continuum and $n \geq 2$, then* $\dim(C_n(X)) = \dim(PHS_n(X))$.

COROLLARY 3.10. $\dim(PHS_n([0,1])) = \dim(PHS_n(S^1)) = 2n$ *for each integer $n \geq 2$.*

PROOF. Since $\dim(C_n([0,1])) = \dim(C_n(S^1)) = 2n$, for any positive integer n [13, Theorem 6.8.10], the result follows from Theorem 3.9. ☐

The techniques used to prove the following results are similar to the ones used in [12, 3.7 and 3.8].

Let us recall that a *simple m-od* in a continuum X is the union of m arcs emanating from a single point, v, and otherwise disjoint from each other.

THEOREM 3.11. *Let X be a continuum and let $n \geq 2$. Then:*

(a) $PHS_n(X)$ always contains n-cells.
(b) If X contains n pairwise disjoint decomposable subcontinua, then $PHS_n(X)$ contains $2n$-cells.
(c) If X contains a simple m-od, then $PHS_n(X)$ contains $(2n+m-2)$-cells.

The proof of the following theorem is similar to the one given for [12, 3.9].

A finite-dimensional continuum X is a *Cantor manifold* if for any subset A of X such that $\dim(A) \leq \dim(X) - 2$, then $X \backslash A$ is connected.

THEOREM 3.12. *Let X be a continuum, and let $n \geq 2$. If $C_n(X)$ is a finite-dimensional Cantor manifold such that* $\dim(C_n(X)) \geq n+2$, *then $PHS_n(X)$ is a finite-dimensional Cantor manifold.*

QUESTION 3.1. *Is it true that, for each continuum X and each $n \geq 2$,* $\dim(C_n(X)) \geq n+2$*?*

Corollary 3.13. *$PHS_n([0,1])$ and $PHS_n(S^1)$ are $2n$-dimensional Cantor manifolds, for each integer $n \geq 2$.*

Proof. It is known that $C_n([0,1])$ and $C_n(S^1)$ are $2n$-dimensional Cantor manifolds [14, Theorem 4.6]. Since $n \geq 2$, $2n \geq n+2$. Hence, the result follows from Theorem 3.12. $\square$

4. Local connectedness

In this section we present results about the n-fold pseudo-hyperspace suspensions of locally connected continua.

The techniques used to prove the following results are similar to the ones used in [12, 5.2 and 5.3].

Theorem 4.1. *Let X be a continuum and let $n \geq 2$. Then:*

(a) X is locally connected if and only if $PHS_n(X)$ is locally connected.
(b) If X is a contractible locally connected continuum without free arcs, then $PHS_n(X)$ is homeomorphic to the Hilbert cube. In particular, $C_n(X)$, $HS_n(X)$ and $PHS_n(X)$ are homeomorphic.

Corollary 4.2. *The n-fold pseudo-hyperspace suspensions of the Hilbert cube are homeomorphic to the Hilbert cube for each integer $n \geq 2$.*

Example 4.3. There exists a locally connected continuum X without free arcs such that $PHS_n(X)$ is not homeomorphic to the Hilbert cube (see [3, Example 5.3]).

The following theorem shows that $[0,1]$ and S^1 have unique n-fold pseudo-hyperspace suspensions.

Theorem 4.4. *Let X be a continuum, and let $n \geq 2$. If $PHS_n(X)$ is homeomorphic to $PHS_n([0,1])$, then X is homeomorphic to $[0,1]$. Also, if $PHS_n(X)$ is homeomorphic to $PHS_n(S^1)$, then X is homeomorphic to S^1.*

Proof. Suppose $PHS_n(X)$ is homeomorphic to $PHS_n([0,1])$. By an argument similar to the one given in [12, 5.7], we conclude that X is homeomorphic to either $[0,1]$ or S^1. Since $PHS_n([0,1])$ is not homeomorphic to $PHS_n(S^1)$ (Corollary 3.7), we have that X is homeomorphic to $[0,1]$.

In a similar way, we can show that if $PHS_n(X)$ is homeomorphic to $PHS_n(S^1)$, then X is homeomorphic to S^1. $\square$

A *finite graph* is a continuum that can be written as the union of finitely many arcs, any two of which can intersect in at most one or both of their end points.

Lemma 4.5. *Let X be a continuum, and let $n \geq 2$. Then $PHS_n(X)$ is locally connected and $\dim(PHS_n(X)) < \infty$ if and only if X is a finite graph.*

Proof. The result follows from Theorem 4.1 (a), Theorem 3.9 and [11, Theorem 5.1]. □

Corollary 4.6. *Let X be a continuum, let $n \geq 2$ and let G be a finite graph. If $PHS_n(X)$ is homeomorphic to $PHS_n(G)$, then X is a finite graph.*

Question 4.1. *Let G_1 and G_2 be a finite graphs, and let $n \geq 2$. If $PHS_n(G_1)$ is homeomorphic to $PHS_n(G_2)$, then is G_1 homeomorphic to G_2?*

5. Aposyndesis

In this section we show that $PHS_n(X)$ is zero-dimensional aposyndetic.

Theorem 5.1. *Let X be a continuum, and let $n \geq 2$. Then $PHS_n(X)$ has property (b).*

Proof. It is known that $C_n(X)$ has property (b) [10, 4.8]. Since the map q_X^n is monotone, and a monotone image of a continuum satisfying property (b) has property (b) [8, Theorem 2, p. 434], we have that $q_X^n(C_n(X)) = PHS_n(X)$ has property (b). □

Corollary 5.2. *$PHS_n(X)$ is unicoherent.*

Proof. Since $PHS_n(X)$ has property (b) (Theorem 5.1), the result follows from [5, Lemma 19.7]. □

A continuum is said to be *colocally connected at* $p \in X$ provided for each open set V of X such that $p \in V$, there exist an open set W of X such that $p \in W \subset V$ and $X \backslash W$ is connected. The continuum X is *colocally connected* provided is collocaly connected at each one of its points.

Theorem 5.3. *Let X be a continuum, and let $n \geq 2$. Then $PHS_n(X)$ is colocally connected.*

Proof. Let $\chi \in PHS_n(X)$ and let $\mathcal{U}$ be an open subset of $PHS_n(X)$ such that $\chi \in \mathcal{U}$. We consider two cases.

Case 1. $\chi = F_X$. Then $(q_X^n)^{-1}(\chi) = F_1(X)$ and $(q_X^n)^{-1}(\mathcal{U})$ is open in $C_n(X)$ such that $F_1(X) \subset (q_X^n)^{-1}(\mathcal{U})$. We have that, for each $x \in X$, there exists an open set V_X in X such that $\{x\} \subset \langle V_X \rangle \subset (q_X^n)^{-1}(\mathcal{U})$. Let $\mathcal{V} = \bigcup_{x \in X} q_X^n(\langle V_X \rangle)$. Hence, $\mathcal{V}$ is an open set in $PHS_n(X)$ such that $F_X^n \in \mathcal{V} \subset \mathcal{U}$. To show that $PHS_n(X) \backslash \mathcal{V}$ is connected, we need to prove that any element of $PHS_n(X) \backslash \mathcal{V}$ can be joined to T_X^n with an arc in $PHS_n(X) \backslash \mathcal{V}$. For this, let $\xi \in PHS_n(X) \backslash \mathcal{V}$. We observe that $(q_X^n)^{-1}(\xi) \in C_n(X) \backslash \left(\bigcup_{x \in X} \langle V_X \rangle\right)$. Let γ be an order arc in $C_n(X)$ such that $\gamma(0) = (q_X^n)^{-1}(\xi)$ and $\gamma(1) = X$. Then, $\gamma \cap \left(\bigcup_{x \in X} \langle V_X \rangle\right) = \emptyset$. Hence, $q_X^n(\gamma)$ is an arc in $PHS_n(X)$ such that

$(q_X^n(\gamma))(0) = \xi$ and $(q_X^n(\gamma))(1) = T_X^n$. We note that $q_X^n(\gamma) \cap \mathcal{V} = \emptyset$. Hence, $q_X^n(\gamma) \subset PHS_n(X)\backslash\mathcal{V}$. Since ξ is an arbitrary element of $PHS_n(X)\backslash\mathcal{V}$, we have that $PHS_n(X)\backslash\mathcal{V}$ is connected.

Case 2. $\chi \neq F_X^n$. We note that $(q_X^n)^{-1}(\mathcal{U})$ is open in $C_n(X)$ such that $(q_X^n)^{-1}(\chi) \in (q_X^n)^{-1}(\mathcal{U})$. Let $\varepsilon > 0$ be such that $\mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right) \subset (q_X^n)^{-1}(\mathcal{U})$ and $\mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right) \cap F_1(X) = \emptyset$. Since $C_n(X)$ is colocally connected [10, Theorem 5.1], there exists an open set W in $C_n(X)$ such that $(q_X^n)^{-1}(\chi) \in W \subset \mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right)$ and $C_n(X)\backslash W$ is connected. Hence, $q_X^n(W)$ is open in $PHS_n(X)$ such that $\chi \in q_X^n(W) \subset \mathcal{U}$ and, moreover, $PHS_n(X)\backslash q_X^n(W) = q_X^n(C_n(X)\backslash W)$. By the continuity of q_X^n, we have that $PHS_n(X)\backslash q_X^n(W)$ is connected. Therefore, $PHS_n(X)$ is colocally connected. $\square$

Clearly, colocal connectedness implies aposyndesis. Hence, we have the following result as a consequence of Theorem 5.3.

Corollary 5.4. *If X is a continuum and $n \geq 2$, then $PHS_n(X)$ is aposyndetic.*

A continuum X is said to be *zero-dimensional aposyndetic* provided that for each zero-dimensional closed subset A of X and for each $p \in X\backslash A$, there exists a subcontinuum M of X such that $p \in Int_X(M)$ and $M \cap A = \emptyset$.

Theorem 5.5. *Let X be a continuum, and let $n \geq 2$. Then $PHS_n(X)$ is zero-dimensional aposyndetic.*

Proof. Let $\mathcal{Z}$ be a nonempty zero-dimensional closed subset of $PHS_n(X)$ and let $\chi \in PHS_n(X)\backslash\mathcal{Z}$. We consider three cases.

Case 1. $\chi \in PHS_n(X)\backslash\{F_X^n, T_X^n\}$. Note that $(q_X^n)^{-1}(\mathcal{Z})$ is closed in $C_n(X)$ and $(q_X^n)^{-1}(\chi) \notin (q_X^n)^{-1}(\mathcal{Z})$. Let $\varepsilon > 0$ be such that

$$Cl_{C_n(X)}\left(\mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right)\right) \cap F_1(X) = \emptyset$$

and

$$Cl_{PHS_n(X)}(q_X^n\left(\mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right)\right) \cap \mathcal{Z} = \emptyset.$$

Now, let $\mathcal{Z}' = \mathcal{Z}\backslash\{F_X^n\}$. Hence, $\dim(\mathcal{Z}') \leq 0$ and

$$\mathcal{V}_\varepsilon^{\mathcal{H}}\left((q_X^n)^{-1}(\chi)\right) \cap (q_X^n)^{-1}(\mathcal{Z}') = \emptyset.$$

Then there exists a subcontinuum W of $C_n(X)$ such that $(q_X^n)^{-1}(\chi) \in Int_{C_n(X)}(W) \subset W \subset C_n(X)\backslash(q_X^n)^{-1}(\mathcal{Z}')$ [16, Theorem 2.6]. Note that in the proof of [16, Theorem 2.6], the fact that the zero-dimensional set $\mathcal{Z}$ is closed is used only to construct an open set about a point whose closure missed $\mathcal{Z}$. To construct W, only the fact that $\dim(\mathcal{Z}') \leq 0$ is used (see [16, Lemma 2.5]). Observe that W can be constructed in such a way that

$W \cap F_1(X) = \emptyset$ (in the proof of [16, Theorem 2.6], the continuum constructed is contained in a set of the form $\left\{A \in C_n(X) : A \in \mu^{-1}([s,t])\right\}$, where μ is a Whitney map and $0 \leq s < t \leq 1$. Since $(q_X^n)^{-1}(\chi) \notin F_1(X)$, we can take $s \neq 0$). Hence, $q_X^n(W)$ is a subcontinuum of $PHS_n(X)$ such that $\chi \in Int_{PHS_n(X)}(q_X^n(W)) \subset q_X^n(W) \subset PHS_n(X) \setminus \mathcal{Z}'$. Since $W \cap F_1(X) = \emptyset$, we have that $F_X^n \notin q_X^n(W)$. Therefore, $q_X^n(W) \subset PHS_n(X) \setminus \mathcal{Z}$.

Case 2. $\chi = T_X^n$. Since $PHS_n(X)$ is normal, there exists an open subset $\mathcal{U}$ in $PHS_n(X)$ such that $Cl_{PHS_n(X)}(\mathcal{U}) \cap \mathcal{Z} = \emptyset$ and $T_X^n \in \mathcal{U}$. From the existence of order arcs in $C_n(X)$, it is easy to see that $C_n(X)$ is locally connected at X. Thus, $PHS_n(X)$ is locally connected at T_X^n. Hence, there exists an open and connected subset $\mathcal{V}$ of $PHS_n(X)$ such that $T_X^n \in \mathcal{V} \subset \mathcal{U}$. We obtain that $Cl_{PHS_n(X)}(\mathcal{V}) \cap \mathcal{Z} = \emptyset$. Therefore, $Cl_{PHS_n(X)}(\mathcal{V})$ is a subcontinuum of $PHS_n(X)$ such that $T_X^n \in Int\left(Cl_{PHS_n(X)}(\mathcal{V})\right) \subset Cl_{PHS_n(X)}(\mathcal{V}) \subset PHS_n(X) \setminus \mathcal{Z}$.

Case 3. $\chi = F_X^n$. Let $\mathcal{Z}' = (q_X^n)^{-1}(\mathcal{Z})$. Hence, $\mathcal{Z}'$ is a closed subset of $C_n(X)$ such that $\dim(\mathcal{Z}') = 0$ and $\mathcal{Z}' \cap F_1(X) = \emptyset$. Thus, for each $x \in X$, there exists a subcontinuum W_x of $C_n(X)$ such that $\{x\} \in Int_{C_n(X)}(W_x) \subset W_x \subset C_n(X) \setminus \mathcal{Z}'$ [16, Theorem 2.6]. Since $F_1(X)$ is compact, there exist a finite cover $\left\{W_x^i\right\}_{i=1}^k$ such that $F_1(X) \subset Int_{C_n(X)}\left(W_x^i\right)$. Let $W = \bigcup\left\{W_x^i : x \in X\right\}$. We have that, W is a subcontinuum of $C_n(X)$ such that $F_1(X) \subset Int_{C_n(X)}(W) \subset W \subset C_n(X) \setminus \mathcal{Z}'$. Therefore, $q_X^n(W)$ is a subcontinuum of $PHS_n(X)$ such that $F_X^n \in Int_{PHS_n(X)}(q_X^n(W)) \subset q_X^n(W) \subset PHS_n(X) \setminus \mathcal{Z}$. ∎

6. Points that arcwise disconnect

In this section we show that a continuum X is indecomposable if and only if by removing the set $\{T_X^n, F_X^n\}$ from $PHS_n(X)$ it becomes arcwise disconnected.

Given a continuum X and any point $p \in X$, the composant of p in X is the set

$$\{x \in X : \text{ there is a proper subcontinuum } A \text{ of } X \text{ such that } p, x \in A\}.$$

By a composant of X we mean a composant of some point in X [19, 5.20].

Lemma 6.1. *Let $n \geq 2$ be an integer. If X is a decomposable continuum, then $C_n(X) \setminus (\{X\} \cup F_1(X))$ is arcwise connected.*

Proof. Since $C_n(X) \setminus (\{X\} \cup F_n(X))$ is arcwise connected [12, Lemma 6.1], note that it suffices to show that any point $\boldsymbol{x}$ in $F_n(X) \setminus F_1(X)$ can be joined to some point A of $C_n(X) \setminus (\{X\} \cup F_n(X))$ with an arc in $C_n(X) \setminus (\{X\} \cup F_1(X))$.

Let $\boldsymbol{x} = \{x_1, ..., x_k\}$ in $F_n(X) \setminus F_1(X)$. Since X is normal, there exist $U_1, ..., U_k$, open sets in X such that $x_i \in U_i$, for each $i \in \{1, ..., k\}$ and

$U_i \cap U_j = \emptyset$, $i \neq j$, where $i, j \in \{1, ..., k\}$. Then there exist subcontinua $B_1, ..., B_k$ of X such that $x_i \in B_i \subset U_i$, for each $i \in \{1, ..., k\}$ [19, Corollary 5.5]. Note that $\bigcup_{i=1}^{n} B_i \in C_n(X) \setminus (\{X\} \cup F_n(X)) \subset C_n(X) \setminus (\{X\} \cup F_1(X))$. Let $A = \bigcup_{i=1}^{n} B_i$. Since $\boldsymbol{x} \subset A$, by [17, Theorem 1.8], there exists an order arc $\alpha : [0,1] \to C_n(X) \setminus (\{X\} \cup F_1(X))$ such that $\alpha(0) = \boldsymbol{x}$, and $\alpha(1) = A$. Therefore, $C_n(X) \setminus (\{X\} \cup F_1(X))$ is arcwise connected. □

THEOREM 6.2. *Let n be a positive integer. A continuum X is indecomposable if and only if $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ is not arcwise connected.*

PROOF. Suppose X is an indecomposable continuum. Then, $C_n(X) \setminus \{X\}$ is not arcwise connected [10, Theorem 6.3]. Hence, $C_n(X) \setminus (\{X\} \cup F_1(X))$ is not arcwise connected either (any two nondegenerate subcontinua A and B of X, contained in two different composants of X, are in two different arc components of $C_n(X) \setminus (\{X\} \cup F_1(X))$). Since $C_n(X) \setminus (\{X\} \cup F_1(X))$ is homeomorphic to $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ (Remark 2.3), $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ is not arcwise connected.

Suppose X is a decomposable continuum. By Lemma 6.1, the space $C_n(X) \setminus (\{X\} \cup F_1(X))$ is arcwise connected. Since $C_n(X) \setminus (\{X\} \cup F_1(X))$ is homeomorphic to $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ (Remark 2.3), $PHS_n(X) \setminus \{T_X^n, F_X^n\}$ is arcwise connected. □

QUESTION 6.1. *If X is an indecomposable continuum and $n \geq 2$, then is $PHS_n(X)$ homeomorphic to $HS_n(X)$?*

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