MATHEMATICAL COMMUNICATIONS
Math. Commun. **29**(2024), 1–19

Computability of sets with attached arcs*

Zvonko Iljazović[1,†] and Matea Jelić[2]

[1] *Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, HR-10 000 Zagreb, Croatia*
[2] *Faculty of Civil Engineering, Architecture and Geodesy, University of Split, Matice hrvatske 15, HR-21 000, Split, Croatia*

Received June 17, 2023; accepted September 13, 2023

Abstract. We consider topological spaces A that have a computable type, which means that any semicomputable set in a computable topological space which is homeomorphic to A is computable. Moreover, we consider topological pairs (A, B), $B \subseteq A$, which have a computable type, which means the following: if S and T are semicomputable sets in a computable topological space such that S is homeomorphic to A by a homeomorphism which maps T to B, then S is computable. We prove the following: if B has a computable type and A is obtained by gluing finitely many arcs to B along their endpoints, then (A, B) has a computable type. We also examine spaces obtained in the same way by gluing chainable continua.

AMS subject classifications: 03D78, 03F60

Keywords: computable topological space, semicomputable set, computable set, computable type, chainable continuum

1. Introduction

A compact set $S \subseteq \mathbb{R}$ is semicomputable if its complement $\mathbb{R} \setminus S$ can be effectively exhausted by rational open intervals. A compact set $S \subseteq \mathbb{R}$ is computable if it is semicomputable and we can effectively enumerate all rational open intervals which intersect S.

A semicomputable set need not be computable. There exists $\gamma > 0$ such that $[0, \gamma]$ is a semicomputable set which is not computable [20]. In fact, while each nonempty computable set contains computable numbers (moreover, they are dense in it), there exists a nonempty semicomputable set $S \subseteq \mathbb{R}$ which does not contain any computable number [23].

The notions of a semicomputable set and a computable set can be naturally defined in Euclidean space $\mathbb{R}^n$ as well as in more general ambient spaces – computable metric spaces and computable topological spaces. While for a set S in a computable topological space X the implication

$$S \text{ semicomputable } \Rightarrow \ S \text{ computable} \tag{1}$$

*This work was fully supported by the Croatian Science Foundation under the project 7459 CompStruct.
†Corresponding author. *Email addresses:* `zilj@math.hr` (Z. Iljazović), `mkalinic@gradst.hr` (M. Jelić)

https://www.mathos.unios.hr/mc

does not hold in general, there are certain additional assumptions under which (1) holds. It turns out that topology of S plays an important role in view of (1). More specifically, there are topological spaces A such that (1) holds in any computable topological space X whenever S is homeomorphic to A. We say that such an A has a computable type. It is known that each sphere in Euclidean space has a computable type; moreover, each compact manifold has a computable type [20, 13, 14, 18]. In particular, each circle has a computable type. However, not only manifolds have a computable type – the Warsaw circle also has a computable type. In fact, any circularly chainable continuum which is not chainable has a computable type [12, 10, 16].

On the other hand, $[0,1]$ does not have a computable type. But if S is a set in a computable topological space X and $f : [0,1] \to S$ is a homeomorphism such that $f(0)$ and $f(1)$ are computable points (which is equivalent to saying that $f(\{0,1\})$ is a semicomputable set), then implication (1) holds. The following definition arises. We say that a topological pair (A, B) (i.e., a pair of topological spaces such that $B \subseteq A$) has a computable type if (1) holds whenever there exists a homeomorphism $f : A \to S$ such that $f(B)$ is a semicomputable set in X.

So $([0,1], \{0,1\})$ has a computable type. Moreover, (B^n, S^{n-1}) has a computable type, where B^n is the unit closed ball and S^{n-1} is the unit sphere in $\mathbb{R}^n$ [20, 13]. In fact, $(M, \partial M)$ has a computable type if M is a compact manifold with boundary [14, 18]. Furthermore, if K is a continuum chainable from a to b, then $(K, \{a, b\})$ has a computable type [12, 10, 16].

A computable type of topological spaces called graphs has been investigated in [15]. Amir and Hoyrup examined conditions under which a finite polyhedra has a computable type (see [1]). Certain results regarding a computable type and (in)computability of semicomputable sets can be found in [2, 6, 19, 17, 11, 8, 24, 9, 7].

A general question is the following: if A is a topological space obtained from topological spaces which have computable types (using some standard topological construction), does A have a computable type? For example, if A_1 and A_2 have computable types, does $A_1 \times A_2$ have a computable type?

In this paper, we consider a topological space B and a space A obtained by gluing finitely many arcs to B along their endpoints. In general, if B has a computable type, A need not have a computable type. Take for example $A = [0,1]$ and $B = \{0,1\}$. But, we prove the following: if B has a computable type, then (A, B) has a computable type. Actually, we prove a more general result involving circularly chainable and chainable continua.

2. Preliminaries

In this section, we give some basic facts about computable metric and topological spaces. See [22, 27, 25, 26, 4, 3, 12].

Let $k \in \mathbb{N}$, $k \geq 1$. A function $f : \mathbb{N}^k \to \mathbb{Q}$ is said to be computable if there are computable (i.e. recursive) functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1},$$

for each $x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be computable if there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x,i)| < 2^{-i},$$

for each $x \in \mathbb{N}^k$, $i \in \mathbb{N}$.

For a set X, let $\mathcal{F}(X)$ denote the family of all finite subsets of X. A function $\Theta : \mathbb{N} \to \mathcal{F}(\mathbb{N})$ is called computable if the set

$$\{(x,y) \in \mathbb{N}^2 \mid y \in \Theta(x)\}$$

is computable and if there is a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$\Theta(x) \subseteq \{0, \dots, \varphi(x)\}$$

for each $x \in \mathbb{N}$.

From now on, let $\mathbb{N} \to \mathcal{F}(\mathbb{N})$, $j \to [j]$ be some fixed computable function whose range is the set of all nonempty finite subsets of $\mathbb{N}$.

2.1. Computable metric space

A triple (X, d, α) is said to be a computable metric space if (X, d) is a metric space, and $\alpha = (\alpha_i)$ is a sequence in X such that $\alpha(\mathbb{N}) \subseteq X$ is dense in (X, d) and such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

For example, if d is the Euclidean metric on $\mathbb{R}^n$, where $n \in \mathbb{N} \setminus \{0\}$, and $\alpha : \mathbb{N} \to \mathbb{Q}^n$ is some effective enumeration of $\mathbb{Q}^n$, then $(\mathbb{R}^n, d, \alpha)$ is a computable metric space.

Let (X, d, α) be a fixed computable metric space. For $x \in X$ and $r > 0$, let $B(x, r)$ denote the open ball in (X, d) with radius r centered at x.

Let $i \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$. We say that $B(\alpha_i, r)$ is an (open) rational ball in (X, d, α).

Let $q : \mathbb{N} \to \mathbb{Q}$ be some fixed computable function whose image is the set of all positive rational numbers and let $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$. For $i \in \mathbb{N}$ we define

$$I_i = B(\alpha_{\tau_1(i)}, q_{\tau_2(i)}). \tag{2}$$

Note that $(I_i)_{i \in \mathbb{N}}$ is an enumeration of all rational balls. Every finite union of rational balls will be called a rational open set. For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i \in [j]} I_i.$$

Clearly, $\{J_j \mid j \in \mathbb{N}\}$ is the family of all rational open sets in (X, d, α).

Let $S \subseteq X$ be a closed set in (X, d). We say that S is a computably enumerable (c.e.) set in (X, d, α) if the set

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

is a c.e. subset of $\mathbb{N}$.

Let $S \subseteq X$ be a compact set in (X, d). We say that S is a semicomputable set in (X, d, α) if the set

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

is a c.e. subset of $\mathbb{N}$.

Finally, we say that S is a computable set in (X, d, α) if S is both c.e. and semicomputable in (X, d, α).

These definitions do not depend on the choice of functions q, τ_1, τ_2 and $([j])_{j \in \mathbb{N}}$.

It can be shown that a nonempty subset S of X is computable in (X, d, α) if and only if S can be effectively approximated by a finite subset of $\{\alpha_i \mid i \in \mathbb{N}\}$ with any given precision. More precisely, S is computable in (X, d, α) if and only if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d_H(S, \{\alpha_i \mid i \in [f(k)]\}) < 2^{-k},$$

for each $k \in \mathbb{N}$, where d_H is the Hausdorff metric (see Proposition 2.6 in [14]).

2.2. Computable topological space

A more general ambient space is a computable topological space. The notion of a computable topological space is not new, see e.g. [28, 29]. We will use the notion of a computable topological space which corresponds to the notion of a SCT_2 space from [28] (which is an effective second countable Hausdorff space).

Let $(X, \mathcal{T})$ be a topological space and (I_i) a sequence in $\mathcal{T}$ such that the set $\{I_i \mid i \in \mathbb{N}\}$ is a basis for $\mathcal{T}$. A triple $(X, \mathcal{T}, (I_i))$ is called a computable topological space if there exist c.e. subsets $C, D \subseteq \mathbb{N}^2$ such that:

1. if $i, j \in \mathbb{N}$ are such that $(i, j) \in C$, then $I_i \subseteq I_j$;
2. if $i, j \in \mathbb{N}$ are such that $(i, j) \in D$, then $I_i \cap I_j = \emptyset$;
3. if $x \in X$ and $i, j \in \mathbb{N}$ are such that $x \in I_i \cap I_j$, then there is $k \in \mathbb{N}$ such that $x \in I_k$ and $(k, i), (k, j) \in C$,
4. if $x, y \in X$ are such that $x \neq y$, then there are $i, j \in \mathbb{N}$ such that $x \in I_i$, $y \in I_j$ and $(i, j) \in D$.

Let $(X, \mathcal{T}, (I_i))$ be a fixed computable topological space. We define $J_j := \bigcup_{i \in [j]} I_i$.

We say that a closed set S in $(X, \mathcal{T})$ is computably enumerable in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbb{N} \mid S \cap I_i \neq \emptyset\}$ is a c.e. subset od $\mathbb{N}$.

Furthermore, we say that S is semicomputable in $(X, \mathcal{T}, (I_i))$ if S is a compact set in $(X, \mathcal{T})$ and $\{j \in \mathbb{N} \mid S \subseteq J_j\}$ is a c.e. subset of $\mathbb{N}$.

We say that S is computable in $(X, \mathcal{T}, (I_i))$ if S is both c.e. and semicomputable in $(X, \mathcal{T}, (I_i))$.

The definition of a semicomputable set (and a computable set) does not depend on the choice of the sequence $([j])_{j \in \mathbb{N}}$.

If (X, d, α) is a computable metric space, then $(X, \mathcal{T}_d, (I_i))$ is a computable topological space, where $\mathcal{T}_d$ is a topology induced by the metric d and (I_i) are the sequences defined by (2) (see e.g. [18]). Clearly, S is c.e./semicomputable/computable in (X, d, α) if and only if S is c.e./semicomputable/computable in $(X, \mathcal{T}_d, (I_i))$.

We say that $x \in X$ is a computable point in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbb{N} \mid x \in I_i\}$ is a c.e. subset of $\mathbb{N}$.

The proofs of the following facts, which will be used frequently in this paper, can be found in [18].

Theorem 1. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. There exist c.e. subsets $\mathcal{C}, \mathcal{D} \subseteq \mathbb{N}^2$ such that:*

1. *if $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{C}$, then $J_i \subseteq J_j$;*
2. *if $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{D}$, then $J_i \cap J_j = \emptyset$;*
3. *if $\mathcal{F}$ is a finite family of nonempty compact sets in $(X, \mathcal{T})$ and $A \subseteq \mathbb{N}$ is a finite subset of $\mathbb{N}$, then for each $K \in \mathcal{F}$ there is $i_K \in \mathbb{N}$ such that*
   - *(i) if $K \in \mathcal{F}$, then $K \subseteq J_{i_K}$;*
   - *(ii) if $K, L \in \mathcal{F}$ are such that $K \cap L = \emptyset$, then $(i_K, i_L) \in \mathcal{D}$;*
   - *(iii) if $a \in A$ and $K \in \mathcal{F}$ are such that $K \subseteq J_a$, then $(i_K, a) \in \mathcal{C}$.*

Proposition 1. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $S \subseteq X$ be a semicomputable set in this space.*

- *(i) If $m \in \mathbb{N}$, then $S \setminus J_m$ is a semicomputable set in $(X, \mathcal{T}, (I_i))$.*
- *(ii) If $k \in \mathbb{N} \setminus \{0\}$, then the set $\{(j_1, \dots, j_k) \in \mathbb{N}^k \mid S \subseteq J_{j_1} \cup \dots \cup J_{j_k}\}$ is c.e.*

The proof of the following proposition can be found in [15].

Proposition 2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $x_0, \dots, x_n \in X$. Then the following holds:*

$$\begin{aligned} x_0, \dots, x_n \text{ are computable points } &\iff \{x_0, \dots, x_n\} \text{ is a semicomputable set} \\ &\iff \{x_0, \dots, x_n\} \text{ is a computable set.} \end{aligned}$$

If $(X, \mathcal{T}, (I_i))$ is a computable topological space, then the topological space $(X, \mathcal{T})$ need not be metrizable (see Example 3.2 in [18]). However, if S is a compact set in $(X, \mathcal{T})$, then S, as a subspace of $(X, \mathcal{T})$, is a compact Hausdorff second countable space, which implies that S is a normal second countable space and therefore it is metrizable. This fact will be very important to us later and we will use it often.

Let A be a topological space. We say that A has a computable type if the following holds: if $(X, \mathcal{T}, (I_i))$ is a computable topological space and S a semicomputable set in this space such that S and A are homeomorphic, then S is computable.

Moreover, let A be a topological space and let B be a subspace of A. We say that (A, B) has a computable type if the following holds: if $(X, \mathcal{T}, (I_i)$ is a computable topological space, S and T semicomputable sets in this space and $f : A \to S$ a homeomorphism such that $f(B) = T$, then S is computable.

2.3. Chainable and circulary chainable continua

Let X be a set and $\mathcal{C} = (C_0, \ldots, C_m)$ a finite sequence of subsets of X. We say that $\mathcal{C}$ is a chain in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j|,$$

for all $i, j \in \{0, \ldots, m\}$.

Figure 1: *Chain*

We say that $\mathcal{C}$ is a circular chain in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j| < m,$$

for all $i, j \in \{0, \ldots, m\}$.

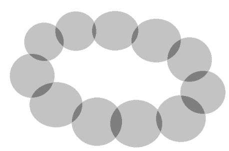

Figure 2: *Circular chain*

Let $A \subseteq X$ and $a, b \in A$. We say that $C_0, \ldots, C_m$ covers A if $A \subseteq C_0 \cup \cdots \cup C_m$, and we say it covers A from a to b if also $a \in C_0$ and $b \in C_m$.

Let (X, d) be a metric space. A (circular) chain $C_0, \ldots, C_m$ is said to be a ϵ-(circular) chain, for some $\epsilon > 0$, if $\operatorname{diam} C_i < \epsilon$, for each $i \in \{0, \ldots, m\}$ and it is said to be an open (circular) chain if every C_i is open in (X, d). In the same way we define the notion of a compact (circular) chain.

A connected and compact metric space is called a continuum.

Let (X, d) be a continuum. We say that (X, d) is a (circulary) chainable continuum if for every $\epsilon > 0$ there is an open ϵ-(circular) chain in (X, d) which covers X.

Suppose $a, b \in X$. We say that (X, d) is a continuum chainable from a to b if for every $\epsilon > 0$ there is an open ϵ-chain $C_0, \ldots, C_m$ which covers X from a to b.

We similarly define the notions of an open and a compact (circular) chain in a topological space.

A topological space which is Hausdorff, connected and compact is called a continuum.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. We say that $\mathcal{A}$ refines $\mathcal{B}$ if for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Let X be a topological space which is a continuum. We say that X is a (circulary) chainable continuum if for each open cover $\mathcal{U}$ of X there is an open (circular) chain

$C_0, \dots, C_m$ in X which covers X and such that $\{C_0, \dots, C_m\}$ refines $\mathcal{U}$. We similarly define that X is a continuum chainable from a to b.

It follows easily that a metric space (X, d) is a (circulary) chainable continuum if and only if topological space $(X, \mathcal{T}_d)$ is a (circulary) chainable continuum. Moreover, (X, d) is a continuum chainable from a to b if and only if $(X, \mathcal{T}_d)$ is a continuum chainable from a to b. See Section 3 in [10].

Remark 1. *Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then it is easy to see that X is a (circularly) chainable continuum if and only if Y is a (circularly) chainable continuum. Furthermore, if $a, b \in X$, then X is a continuum chainable from a to b if and only if Y is a continuum chainable from $f(a)$ to $f(b)$.*

The proofs of the following facts can be found in [16].

Proposition 3. *Let (X, d) be a continuum and $a, b \in X$. Then (X, d) is a chainable continuum from a to b if and only if for each $\epsilon > 0$ there is a compact ϵ-chain in (X, d) which covers X from a to b.*

Proposition 4. *Let (X, d) be a continuum. Then (X, d) is a (circulary) chainable continuum if and only if for each $\epsilon > 0$ there is a compact ϵ-(circular) chain in (X, d) which covers X.*

Example 1. *We have that $[0, 1]$ (with the Euclidean metric) is a continuum chainable from 0 to 1. This can be easily concluded from Proposition 3. (Thus $[0, 1]$ with the Euclidean topology is a continuum chainable from 0 to 1.)*

Similarly, the unit circle S^1 in $\mathbb{R}^2$ is a circularly chainable continuum. However, S^1 is not a chainable continuum (see [5]).

A topological space homeomorphic to $[0, 1]$ is called an arc. If A is an arc and $f : [0, 1] \to A$ a homeomorphism, then we say that $f(0)$ and $f(1)$ are endpoints of A (this definition does not depend on the choice of f).

If A is an arc with endpoints a and b, then by Example 1 and Remark 1 we have that A is a continuum chainable from a to b.

A topological space homeomorphic to S^1 is called a topological circle. By Example 1 and Remark 1 each topological circle is a circularly chainable continuum which is not chainable.

Example 2. *Let*

$$K = (\{0\} \times [-1, 1]) \cup \left\{ \left(x, \sin \frac{1}{x}\right) \mid 0 < x \leq 1 \right\}.$$

Let $a = (0, -1)$ and $b = (1, \sin 1)$. It is known that K is a continuum chainable from a to b. However, K is not an arc since K is not locally connected.

Furthermore, let

$$W = K \cup (\{0\} \times [-2, -1]) \cup ([0, 1] \times \{-2\}) \cup (\{1\} \times [-2, \sin 1]).$$

The space W is called the Warsaw circle. It is known that W is a circularly chainable continuum which is not chainable. Since W is not locally connected, W is not a topological circle.

3. Spaces with attached arcs

The following result was proved in [10] (Theorem 2): if $(X, \mathcal{T}, (I_i))$ is a computable topological space and K a semicomputable set in this space which, as a subspace of $(X, \mathcal{T})$, is a continuum chainable from a to b, where a and b are computable points, then K is a computable set in $(X, \mathcal{T}, (I_i))$. In other words, if K is a continuum chainable from a to b, then $(K, \{a, b\})$ has a computable type (note that by Proposition 2, the condition that a and b are computable points is equivalent to the fact that $\{a, b\}$ is a semicomputable set).

Now we prove a more general result (the result from [10] follows from the following result for $S = \{a, b\}$ and $L = \emptyset$).

Proposition 5. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Suppose K, as a subspace of $(X, \mathcal{T})$, is a continuum chainable from a to b, $a, b \in X$, $a \neq b$. Let $S \subseteq X$ be such that $S \cap K = \{a, b\}$ and let $L \subseteq X$ be a compact set in $(X, \mathcal{T})$ such that $L \cap K \subseteq \{a, b\}$ (see Figure 3). Suppose S and $S \cup L \cup K$ are semicomputable sets in $(X, \mathcal{T}, (I_i))$. Then K is a c.e. set in $(X, \mathcal{T}, (I_i))$.*

Proof. Since K is compact, it is metrizable. Let d be the metric on K which induces the topology on K, i.e., the relative topology on K in $(X, \mathcal{T})$.

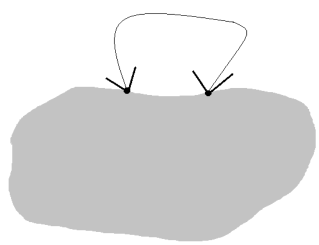

Figure 3: *$S \cup L \cup K$: the grey set is S, the union of the short straight lines is L and the arc whose endpoints lie is S is K.*

Since X is Hausdorff, there are $U_a, U_b \in \mathcal{T}$ such that $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \emptyset$.

Assume that $(S \cup L) \setminus (U_a \cup U_b) \neq \emptyset$. The sets K and $(S \cup L) \setminus (U_a \cup U_b)$ are disjoint and compact in $(X, \mathcal{T})$. Namely, since $K \cap (S \cup L) = \{a, b\}$ and $a, b \notin (S \cup L) \setminus (U_a \cup U_b)$, the sets K and $(S \cup L) \setminus (U_a \cup U_b)$ are disjoint. We have that $(S \cup L) \setminus (U_a \cup U_b)$ is compact in $(X, \mathcal{T})$ because it is closed and contained in $S \cup L$ (which is compact).

By Theorem 1 there exists $\mu \in \mathbb{N}$ such that

$$(S \cup L) \setminus (U_a \cup U_b) \subseteq J_\mu \text{ and } K \cap J_\mu = \emptyset \tag{3}$$

(this can also be easily concluded from the fact that $(X, \mathcal{T})$ is Hausdorff). Let us denote

$$S' = (S \cup L \cup K) \setminus J_\mu.$$

By (3) we have $(S \cup L) \setminus J_\mu \subseteq U_a \cup U_b$ and therefore

$$S' = (S \setminus J_\mu) \cup (L \setminus J_\mu) \cup K = A \cup B \cup L_1 \cup L_2 \cup K,$$

where $A = (S \setminus J_\mu) \cap U_a$, $B = (S \setminus J_\mu) \cap U_b$, $L_1 = (L \setminus J_\mu) \cap U_a$ and $L_2 = (L \setminus J_\mu) \cap U_b$. By Proposition 1 the set S' is semicomputable in $(X, \mathcal{T}, (I_i))$.

We claim that A and B are semicomputable sets in $(X, \mathcal{T}, (I_i))$. Namely, $S \setminus J_\mu = A \cup B$ and A and B are open in $S \setminus J_\mu$. Since these sets are disjoint, they are also closed in $S \setminus J_\mu$. The fact that $S \setminus J_\mu$ is compact now implies that A and B are compact in $(X, \mathcal{T})$. It follows that there exist $\alpha, \beta \in \mathbb{N}$ such that

$$A \subseteq J_\alpha,\ B \subseteq J_\beta \text{ and } J_\alpha \cap J_\beta = \emptyset.$$

Then $A = (S \setminus J_\mu) \setminus J_\beta$ and $B = (S \setminus J_\mu) \setminus J_\alpha$, i.e., A and B are semicomputable sets in $(X, \mathcal{T}, (I_i))$. In a similar way we conclude that L_1 and L_2 are compact in $(X, \mathcal{T})$.

So S' is a semicomputable set and

$$S' = A \cup B \cup L_1 \cup L_2 \cup K,$$

where A and B are semicomputable, L_1 and L_2 are compact, $(A \cup L_1) \cap (B \cup L_2) = \emptyset$ and $a \in A$, $b \in B$ (note that (3) and $a, b \in K$ imply $a \notin J_\mu$ and $b \notin J_\mu$).

We get the same conclusion if $(S \cup L) \setminus (U_a \cup U_b) = \emptyset$. Namely, we can define $S' = S \cup L \cup K$ and then

$$S' = A \cup B \cup L_1 \cup L_2 \cup K,$$

where $A = S \cap U_a$, $B = S \cap U_b$, $L_1 = L \cap U_a$ and $L_2 = L \cap U_b$. Similarly as before, we conclude that A and B are semicomputable, L_1 and L_2 are compact, $(A \cup L_1) \cap (B \cup L_2) = \emptyset$ and $a \in A$, $b \in B$.

Let $\mathcal{C}$ and $\mathcal{D}$ be the subsets of $\mathbb{N}^2$ from Theorem 1 and let $f : \mathbb{N} \to \mathbb{N}$ be a fixed computable function such that $I_i = J_{f(i)}$ for each $i \in \mathbb{N}$ (such a function certainly exists).

Suppose $i \in \mathbb{N}$ is such that $I_i \cap K \neq \emptyset$. We claim that there exists $x \in I_i \cap K$, $x \neq a, b$. Namely, if $I_i \cap K \subseteq \{a, b\}$, then $I_i \cap K$ is finite and therefore closed in K. Also, $I_i \cap K$ is open in K. Together with the fact that K is connected, we have that $I_i \cap K = K$. Since K is finite and Hausdorff, it is discrete, which contradicts the fact that K is connected and $\mathrm{card}(K) \geq 2$.

So, there exists $x \in I_i \cap (K \setminus \{a, b\})$. Choose r so that

$$0 < r < \min\{d(a, x), d(b, x)\}$$

and

$$B(x, r) \subseteq I_i \cap K \subseteq I_i = J_{f(i)}. \tag{4}$$

Furthermore, since (K, d) is a continuum chainable from a to b, there is a compact r-chain $K_0, \dots, K_n$ in (K, d) which covers K and such that $a \in K_0$ and $b \in K_n$.

Let $p \in \{0, \dots, n\}$ be such that $x \in K_p$. Because of (4) and $\mathrm{diam}(K_p) < r$, we have $K_p \subseteq I_i$, hence

$$K_p \subseteq J_{f(i)}. \tag{5}$$

Since $r < d(x, a), d(x, b)$, we have $p \neq 0, n$.

Let us denote

$$F = A \cup L_1 \cup K_0 \cup \ldots \cup K_{p-1} \text{ and } G = B \cup L_2 \cup K_{p+1} \cup \ldots \cup K_n.$$

Note that

$$S' = F \cup K_p \cup G. \tag{6}$$

We claim that F and G are disjoint. Obviously, $A \cap B = \emptyset$ and since $A \subseteq S$, $A \subseteq U_a$ and $S \cap K = \{a, b\}$ (by the assumption of the theorem), we have $A \cap K \subseteq \{a\}$. However, $a \notin K_j$, for $j \in \{p+1, \ldots, n\}$ because $a \in K_0$, $p+1 \geq 2$ and $K_0, \ldots, K_n$ is a chain, so $A \cap K_{p+1} = \emptyset, \ldots, A \cap K_n = \emptyset$. Similarly, $B \cap K_0 = \cdots = B \cap K_{p-1} = \emptyset$. Moreover, $A \cap L_2 = \emptyset$ because $L_2 \subseteq U_b$ and $A \subseteq U_a$. Similarly, $B \cap L_1 = \emptyset$. Hence $F \cap G = \emptyset$.

The sets F, K_p and G are compact in $(X, \mathcal{T})$, F and G are disjoint and we have (5), so according to Theorem 1, there are $u, v, w \in \mathbb{N}$ such that $F \subseteq J_u$, $K_p \subseteq J_v$, $G \subseteq J_w$, $(u, w) \in \mathcal{D}$ and $(v, f(i)) \in \mathcal{C}$. It follows from (6) that $S' \subseteq J_u \cup J_v \cup J_w$. By the definitions of F and G we have $A \subseteq J_u$ and $B \subseteq J_w$.

So, if $i \in \mathbb{N}$ is such that $I_i \cap K \neq \emptyset$, then there exist $u, v, w \in \mathbb{N}$ such that:

(i) $S' \subseteq J_u \cup J_v \cup J_w$;

(ii) $A \subseteq J_u$;

(iii) $B \subseteq J_w$;

(iv) $(u, w) \in \mathcal{D}$;

(v) $(v, f(i)) \in \mathcal{C}$.

Let Ω be the set of all $(i, u, v, w) \in \mathbb{N}^4$ for which statements (i)-(v) hold.

We have proved the following: if $i \in \mathbb{N}$ is such that $I_i \cap K \neq \emptyset$, then there exist $u, v, w \in \mathbb{N}$ such that $(i, u, v, w) \in \Omega$.

Conversely, let us suppose that $i \in \mathbb{N}$ is such that there exist $u, v, w \in \mathbb{N}$ such that $(i, u, v, w) \in \Omega$. We claim that $I_i \cap K \neq \emptyset$.

Suppose the opposite, i.e., $I_i \cap K = \emptyset$. Since $J_v \subseteq I_i$ by (v), we have $J_v \cap K = \emptyset$, and since by (i) it holds $K \subseteq J_u \cup J_v \cup J_w$, we have $K \subseteq J_u \cup J_w$. Because $A \subseteq J_u$, it holds $a \in J_u$, and because $B \subseteq J_w$, it holds $b \in J_w$. So, the sets J_u and J_w are open in $(X, \mathcal{T})$, they are disjoint, their union contains K and each of them intersects K. This implies that K is not connected, which is impossible. Therefore, $I_i \cap K \neq \emptyset$.

So we have:

$$I_i \cap K \neq \emptyset \text{ if and only if there exist } u, v, w \in \mathbb{N} \text{ such that } (i, u, v, w) \in \Omega. \tag{7}$$

Since S', A and B are semicomputable sets, by Proposition 1 we have that Ω is a c.e. set. It follows now from (7) that the set $\{i \in \mathbb{N} \mid I_i \cap K \neq \emptyset\}$ is c.e. Hence K is a c.e. set in $(X, \mathcal{T}, (I_i))$. □

Let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions such that $\{(\sigma(j, 0), \dots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all nonempty finite sequences in $\mathbb{N}$. Instead of $\sigma(i, j)$ we will write $(i)_j$ and $\overline{j}$ instead of $\eta(j)$. So $\{((j)_0, \dots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$ is the set of all nonempty finite sequences in $\mathbb{N}$.

The function $\mathbb{N} \to \mathcal{F}(\mathbb{N})$, $i \mapsto \{(j)_0, \dots, (j)_{\overline{j}}\}$, is computable and its range is the set of all nonempty finite subsets of $\mathbb{N}$. Therefore, we may assume (without any loss of generality) that

$$[j] = \{(j)_0, \dots, (j)_{\overline{j}}\}$$

for each $j \in \mathbb{N}$.

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Let $\mathcal{C}$ and $\mathcal{D}$ be from Theorem 1.

For $l \in \mathbb{N}$ we define

$$\mathcal{H}_l = (J_{(l)_0}, \dots, J_{(l)_{\overline{l}}}).$$

We say that $\mathcal{H}_l$ is a formal circular chain if the following holds:

$$((l)_i, (l)_j) \in \mathcal{D} \text{ for all } i, j \in \{0, \dots, \overline{l}\} \text{ such that } 1 < |i - j| < \overline{l}.$$

Note that this is a property of the number l. (More precisely, we can say that "l represents a formal circular chain"; it is possible that $\mathcal{H}_l = \mathcal{H}_{l'}$, l represents a formal circular chain, but l' does not – so $\mathcal{H}_l$ is a formal circular chain and $\mathcal{H}_{l'}$ is not.)

The following proposition can be proved similarly to propositions 32 and 34 in [12].

Proposition 6. *1. The set $\{l \in \mathbb{N} \mid \mathcal{H}_l$ is a formal circular chain$\}$ is c.e.*

2. Let S be a semicomputable set in $(X, \mathcal{T}, (I_i))$. Then the set $\{l \in \mathbb{N} \mid \mathcal{H}_l$ covers $S\}$ is c.e.

Lemma 1. *Let (K, d) be a connected metric space. Suppose $\epsilon > 0$ and $C_0, \dots, C_m$ are open sets in (K, d) which cover K, whose diameters are less than ϵ and such that $C_i \cap C_j = \emptyset$ for each $i, j \in \{0, \dots, m\}$ such that $|i - j| > 1$. Then there exists an open ε-chain in (K, d) which covers K.*

Proof. Let

$$v = \min\{i \in \{0, \dots, m\} \mid C_i \neq \emptyset\}$$

and

$$w = \max\{i \in \{0, \dots, m\} \mid C_i \neq \emptyset\}.$$

Then the finite sequence $C_v, \dots, C_w$ covers K. We claim that $C_v, \dots, C_w$ is an open ϵ-chain in (K, d). It suffices to prove that $C_i \neq \emptyset$ for each $i \in \{v, \dots, w\}$ and $C_i \cap C_{i+1} \neq \emptyset$ for each $i \in \{v, \dots, w - 1\}$.

Suppose $C_i = \emptyset$ for some $i \in \{v, \dots, w\}$. By definition of v and w we have $C_v \neq \emptyset$ and $C_w \neq \emptyset$, so $v < i < w$. Let $U = C_v \cup \dots \cup C_{i-1}$ and $V = C_{i+1} \cup \dots \cup C_w$. Then U and V are disjoint open sets in (K, d). Since $K = C_v \cup \dots \cup C_w$ and $C_i = \emptyset$, we have $K = U \cup V$ and $U \neq \emptyset$, $V \neq \emptyset$. This means that (U, V) is a separation of (K, d), which is impossible since (K, d) is connected.

Similarly, we see that $C_i \cap C_{i+1} \neq \emptyset$ for each $i \in \{v, \dots, w - 1\}$. So $C_v, \dots, C_w$ is an open ϵ-chain in (K, d) which covers K. $\square$

Now, we have a result similar to Proposition 5.

Proposition 7. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Suppose K, as a subspace of $(X, \mathcal{T})$, is a continuum which is circulary chainable but not chainable, and let $a \in X$. Let $S \subseteq X$ be such that $S \cap K = \{a\}$ and let $L \subseteq X$ be a compact set in $(X, \mathcal{T})$ such that $L \cap K \subseteq \{a\}$ (see Figure 4). If S and $S \cup L \cup K$ are semicomputable sets in $(X, \mathcal{T}, (I_i))$, then K is c.e. in $(X, \mathcal{T}, (I_i))$.*

Proof. Firstly, since K is not chainable, we have $\operatorname{card} K \geq 2$.

Similary as before, let d be the metric on K which induces the topology on K, i.e., the relative topology on K in $(X, \mathcal{T})$.

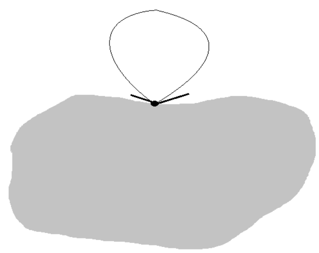

Figure 4: *$S \cup L \cup K$: the grey set is S, the union of the short straight lines is L and the circle above S is K.*

Let

$$S' = S \cup L \cup K. \tag{8}$$

By assuming the proposition S' and S are semicomputable and L is compact. Clearly, $a \in S$ and

$$(S \cup L) \cap K = \{a\}. \tag{9}$$

Since (K, d) is not chainable, there exists $\epsilon_0 > 0$ such that there exists no open ϵ_0-chain in (K, d) which covers K. Since K is compact and for each $z \in K$ and $\epsilon > 0$ there is $j \in \mathbb{N}$ such that $z \in J_j$ and $\operatorname{diam}(J_j \cap K) < \epsilon$, there are $a_0, \ldots, a_m \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{i=0}^{m} J_{a_i}$$

and

$$\operatorname{diam}(J_{a_i} \cap K) < \frac{\epsilon_0}{3}, \text{ for each } i \in \{0, \ldots, m\}. \tag{10}$$

Let $\lambda > 0$ be a Lebesgue number of the open cover

$$\{J_{a_0} \cap K, \ldots, J_{a_m} \cap K\} \tag{11}$$

of (K, d).

Since $S \cup L$ is compact in $(X, \mathcal{T})$, there exists $\alpha \in \mathbb{N}$ such that

$$S \cup L \subseteq J_\alpha \text{ and } \operatorname{diam}(J_\alpha \cap K) < \frac{\epsilon_0}{3}. \tag{12}$$

Namely, choose $r \in \mathbb{R}$ such that $0 < r < \frac{\epsilon_0}{8}$. The sets $S \cup L$ and $K \setminus B(a,r)$ are disjoint by (9) and they are clearly compact. Thus there exists $\alpha \in \mathbb{N}$ such that

$$S \cup L \subseteq J_\alpha \text{ and } (K \setminus B(a,r)) \cap J_\alpha = \emptyset.$$

It follows that $J_\alpha \cap K \subseteq B(a,r)$ and so $\operatorname{diam}(J_\alpha \cap K) \leq 2r \leq \frac{\epsilon_0}{4} < \frac{\epsilon_0}{3}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be as in Theorem 1 and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function such that $I_i = J_{f(i)}$ for each $i \in \mathbb{N}$.

Suppose $i \in \mathbb{N}$ is such that $I_i \cap K \neq \emptyset$. Then there exists $x \in I_i \cap K$ such that $x \neq a$. Otherwise, we would have $I_i \cap K = \{a\}$, which would imply that $\{a\}$ is open in K; however, this is impossible since $\{a\}$ is closed, K is connected and $\operatorname{card}(K) \geq 2$.

Since $x \in I_i \cap K$, there is $0 < r < \min\{\frac{1}{2}d(a,x), \lambda\}$ such that

$$B(x,r) \subseteq I_i \cap K \subseteq I_i = J_{f(i)}. \tag{13}$$

Now, since (K,d) is a circulary chainable continuum, there exists a compact r-circular chain $K_0, \dots, K_n$ in (K,d) which covers K. For each $l \in \{1, \dots, n\}$ the finite sequence $K_l, \dots, K_n, K_0, \dots, K_{l-1}$ is also an r-circular chain which covers K, so we may assume $a \in K_0$. Furthermore, without loss of generality, we may assume that $a \notin K_j$ for $j \neq 0$. Indeed, we have $a \notin K_j$ for each $j \notin \{n, 0, 1\}$ since $K_0, \dots, K_n$ is a circular chain and so we can replace $K_0, \dots, K_n$ by the circular chain $K_n \cup K_0 \cup K_1, K_2, \dots, K_{n-1}$ (which is an r-circular chain if $K_0, \dots, K_n$ is an $\frac{r}{3}$-circular chain).

Let $p \in \{0, \dots, n\}$ be such that $x \in K_p$. It follows from (13) and $\operatorname{diam}(K_p) < r$ that

$$K_p \subseteq I_i = J_{f(i)}. \tag{14}$$

Since $r < d(x,a)$, one has $p \neq 0$.

For each $j \in \{0, \dots, n\}$ we have $\operatorname{diam}(K_j) < \lambda$, so there exists $k_j \in \{0, \dots, m\}$ such that

$$K_j \subseteq J_{a_{k_j}} \tag{15}$$

(recall that λ is a Lebesgue number of the open cover (11)).

For each $j \in \{1, \dots, n\}$ we have $a \notin K_j$ and it follows from (9) that

$$(S \cup L) \cap K_j = \emptyset.$$

Also, for all $j, j' \in \{0, \dots, n\}$ such that $1 < |j - j'| < n$ we have $K_j \cap K_{j'} = \emptyset$. Using this, (14), (12), (15) and Theorem 1 we conclude that there are $u_0, \dots, u_n, u \in \mathbb{N}$ such that

$K_j \subseteq J_{u_j}$, for each $j \in \{0, \dots, n\}$,

$S \cup L \subseteq J_u$,

$(u_j, u_{j'}) \in \mathcal{D}$ for all $j, j' \in \{0, \dots, n\}$ such that $1 < |j - j'| < n$

$(u, u_j) \in \mathcal{D}$ for each $j \in \{1, \dots, n\}$,

$(u_p, f(i)) \in \mathcal{C}$ and $(u, \alpha) \in \mathcal{C}$,

$(u_j, a_{k_j}) \in \mathcal{C}$, for each $j \in \{0, \ldots, n\}$.

By (8) we have $S' = S \cup L \cup \bigcup_{j=0}^{n} K_j$, which implies $S' \subseteq J_u \cup J_{u_0} \cup \cdots \cup J_{u_n}$.

Choose $l \in \mathbb{N}$ so that $((l)_0, \ldots, (l)_{\bar{l}}) = (u_0, \ldots, u_n)$. Then the following holds:

(i) $S' \subseteq \bigcup \mathcal{H}_l \cup J_u$;

(ii) $S \subseteq J_u$;

(iii) $\mathcal{H}_l$ is a formal circular chain;

(iv) $(u, (l)_j) \in \mathcal{D}$, for each $j \in \{1, \ldots, \bar{l}\}$;

(v) $1 \leq p \leq \bar{l}$ and $((l)_p, f(i)) \in \mathcal{C}$;

(vi) $(u, \alpha) \in \mathcal{C}$;

(vii) for each $j \in \{0, \ldots, \bar{l}\}$ there exists $j' \in \{0, \ldots, m\}$ such that $((l)_j, a_{j'}) \in \mathcal{C}$.

Let Ω be the set of all $(i, l, u, p) \in \mathbb{N}^4$ such that (i)-(vii) hold. We have proved the following: if $i \in \mathbb{N}$ is such that $I_i \cap K \neq \emptyset$, then there exist $l, u, p \in \mathbb{N}$ such that $(i, l, u, p) \in \Omega$.

Conversely, let us suppose that $i \in \mathbb{N}$ is such that there exist $l, u, p \in \mathbb{N}$ such that $(i, l, u, p) \in \Omega$. So statements (i)-(vii) hold. We want to prove that $I_i \cap K \neq \emptyset$.

Suppose the opposite, i.e. $I_i \cap K = \emptyset$. So $J_{f(i)} \cap K = \emptyset$ and by (v) we have $J_{(l)_p} \subseteq J_{f(i)}$. This implies that $J_{(l)_p} \cap K = \emptyset$. It follows from (i) that $K \subseteq \bigcup \mathcal{H}_l \cup J_u$ and therefore

$$K \subseteq J_{(l)_0} \cup \ldots \cup J_{(l)_{p-1}} \cup J_{(l)_{p+1}} \cup \ldots \cup J_{(l)_{\bar{l}}} \cup J_u,$$

i.e.,

$$K \subseteq J_{(l)_{p+1}} \cup \ldots \cup J_{(l)_{\bar{l}}} \cup (J_{(l)_0} \cup J_u) \cup J_{(l)_1} \ldots \cup J_{(l)_{p-1}}.$$

It follows that K is the union of the following sets:

$$(J_{(l)_{p+1}} \cap K), \ldots, (J_{(l)_{\bar{l}}} \cap K), (((J_{(l)_0} \cap K) \cup (J_u \cap K)), (J_{(l)_1} \cap K), \ldots, (J_{(l)_{p-1}} \cap K). \tag{16}$$

Let M be the union of the following sets:

$$(J_{(l)_{p+1}} \cap K), \ldots, (J_{(l)_{\bar{l}}} \cap K), (J_{(l)_0} \cap K), (J_{(l)_1} \cap K), \ldots, (J_{(l)_{p-1}} \cap K). \tag{17}$$

By (vi) we have $J_u \subseteq J_\alpha$, so $J_u \cap K \subseteq J_\alpha \cap K$ and it follows from (12) that

$$\operatorname{diam}(J_u \cap K) < \frac{\epsilon_0}{3}. \tag{18}$$

In the same way, using (vii) and (10), we conclude that

$$\operatorname{diam}(J_{(l)_j} \cap K) < \frac{\epsilon_0}{3} \tag{19}$$

for each $j \in \{0, \ldots, \bar{l}\}$.

We claim that

$$(J_{(l)_0} \cap K) \cap (J_u \cap K) \neq \emptyset. \tag{20}$$

Otherwise, if $J_u \cap K$ and $J_{(l)_0} \cap K$ are disjoint, then $J_u \cap K$ is disjoint with each of the sets in (17) (this follows from (iv)) and thus $J_u \cap K$ and M are disjoint. By the definition of M we have $K = M \cup (J_u \cap K)$ and this means that $(M, J_u \cap K)$ is a separation of K: M and $J_u \cap K$ are clearly open in K, $J_u \cap K$ is nonempty since $a \in J_u$ by (ii) and M is nonempty since $M = \emptyset$ implies $K = J_u \cap K$ and this, together with (18), implies that there exists a (trivial) open ϵ_0-chain in K which covers K which is impossible by the choice of ϵ_0.

So (20) holds. Using this, (20), (18) and (19) we conclude that

$$\operatorname{diam}\big((J_{(l)_0} \cap K) \cup (J_u \cap K)\big) < \frac{\epsilon_0}{3} + \frac{\epsilon_0}{3} < \epsilon_0. \tag{21}$$

Let us consider the finite sequence of sets in (16). Nonadjacent sets in this sequence are disjoint, which follows from (iii) and (iv). These sets are open in K and their diameters, by (19) and (21), are less than ϵ_0. It follows from Lemma 1 that there exists an open ϵ_0-chain in K which covers K, but this is impossible by the choice of ϵ_0.

Hence, $I_i \cap K \neq \emptyset$.

We have proved the following:

$$I_i \cap K \neq \emptyset \;\Leftrightarrow\; \text{there exist } l, u, p \in \mathbb{N} \text{ such that } (i, l, u, p) \in \Omega. \tag{22}$$

It is not hard to conclude that Ω is a c.e. set (see e.g. the proofs of propositions 32 and 34 in [12]). Now (22) implies that the set $\{i \in \mathbb{N} \mid I_i \cap K \neq \emptyset\}$ is c.e. and thus K is a c.e. set in $(X, \mathcal{T}, (I)_i)$. $\square$

The following result generalizes both Proposition 5 and Proposition 7.

Theorem 2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $S \subseteq X$ be a computable set in $(X, \mathcal{T}, (I_i))$. Suppose $(K_0, \{a_0, b_0\}), \dots (K_n, \{a_n, b_n\})$ is a finite sequence of pairs, where each K_i, as a subspace of $(X, \mathcal{T})$, is either a continuum chainable from a_i to b_i, where $a_i, b_i \in K_i$ are such that $a_i \neq b_i$, or a continuum which is circulary chainable, but not chainable, where $a_i, b_i \in K_i$ are such that $a_i = b_i$.*

Suppose the following holds:

(*i*) *$K_i \cap S = \{a_i, b_i\}$ for each $i \in \{0, \dots, n\}$;*

(*ii*) *$K_i \cap K_j \subseteq S$ for all $i, j \in \{0, \dots, n\}$ such that $i \neq j$.*

Let

$$T = S \cup K_0 \cup \cdots \cup K_n.$$

See Figure 5. Suppose T is a semicomputable set in $(X, \mathcal{T}, (I_i))$. Then T is computable.

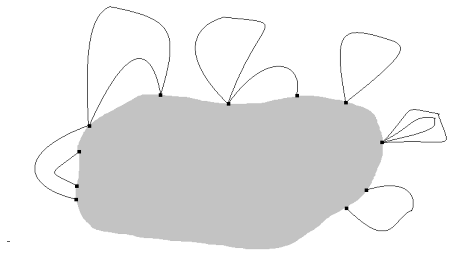

Figure 5: *The set T: the grey set is S.*

Proof. Let $i \in \{0, \dots, n\}$. Let

$$L = \bigcup_{j \neq i} K_j.$$

Then $T = S \cup L \cup K_i$ and so we have that S and $T = S \cup L \cup K_i$ are semicomputable sets. It follows from (i) and (ii) that $L \cap K_i \subseteq \{a_i, b_i\}$ and this, together with (i) and propositions 5 and 7, implies that the set K_i is c.e. in $(X, \mathcal{T}, (I_i))$.

Therefore T, as a finite union of c.e. sets, is also a c.e. set. Together with the fact that T is semicomputable, T is computable. □

Let X be a topological space and let $\mathcal{F}$ be a partition of the set X. Let $p : X \to \mathcal{F}$ be a (unique) function such that $x \in p(x)$ for each $x \in X$ (such a p will be called the quotient map). We topologize $\mathcal{F}$ by declaring that $V \subseteq \mathcal{F}$ is open if $p^{-1}(V)$ is open in X. This topology is called the quotient topology and $\mathcal{F}$, with this topology, is called a quotient space of X. Clearly, $p : X \to \mathcal{F}$ is a continuous surjection.

Remark 2. *The following facts are well-known (see e.g. [21]).*

(*i*) *Let $\mathcal{F}$ be the partition of $[0, 1]$ given by $\mathcal{F} = \{\{x\} \mid 0 < x < 1\} \cup \{\{0, 1\}\}$. If we take the Euclidean topology on $[0, 1]$ and the quotient topology on $\mathcal{F}$, then $\mathcal{F}$ is homeomorphic to the unit circle S^1.*

(*ii*) *Let X and Y be topological spaces such that X is compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous surjection. Let $X/f = \{f^{-1}(\{y\}) \mid y \in Y\}$. Then X/f (the given quotient topology) and Y are homeomorphic.*

Suppose A and B are topological spaces, C is a subspace of B and $f : C \to A$ is a function. Let us consider the topological space $A \sqcup B$ – the disjoint union of A and B, i.e., $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ (we identify A with $A \times \{1\}$ and B with $B \times \{2\}$), the topology on $A \sqcup B$ given by $U \subseteq A \sqcup B$ is open if $U \cap A$ is open in A and $U \cap B$ is open in B.

We have the partition $\mathcal{F}$ of $A \sqcup B$ given by

$$\mathcal{F} = \{\{a\} \mid a \in A \setminus f(C)\} \cup \{\{a\} \cup f^{-1}(\{a\}) \mid a \in f(C)\} \cup \{\{b\} \mid b \in B \setminus C\}.$$

Then $\mathcal{F}$, together with the quotient topology, is called an adjunction space obtained by adjoining A and B by way of f. This adjunction space is denoted by $A \cup_f B$.

Example 3. *Let X be a Hausdorff space and let A, B and C be compact sets in X such that $A \cap B = C$. Let $f : C \to A$ be defined by $f(x) = x$. Then $A \cup_f B$ is homeomorphic to $A \cup B$.*

Indeed, we have the obvious function $g : A \sqcup B \to A \cup B$ and we have that $(A \sqcup B)/g$ and $A \cup B$ are homeomorphic by Remark 2. However, $(A \sqcup B)/g = A \cup_f B$.

Remark 3. *If A and B are topological spaces, C is a closed subspace of B and $f : C \to A$ is a continuous function, we can identify A with an obvious subspace of $A \cup_f B$: this subspace is the image of A by the composition*

$$A \xrightarrow{i} A \sqcup B \xrightarrow{p} A \cup_f B,$$

where i is the inclusion, and p the quotient map. It is not hard to check (see [21]) that this subspace is actually homeomorphic to A.

Suppose $n \in \mathbb{N}$ and $I_0, \dots, I_n$ is the finite sequence of topological spaces defined by $I_i = [0,1]$ for each $i \in \{0, \dots, n\}$. For $i \in \{0, \dots, n\}$ let $\partial I_i = \{0,1\}$. We have the subspace $\partial I_0 \sqcup \dots \sqcup \partial I_n$ of the disjoint union $I_0 \sqcup \dots \sqcup I_n$.

Let A be a topological space and let $f : \partial I_0 \sqcup \dots \sqcup \partial I_n \to A$ be any function. Consider the adjunction space

$$A \cup_f (I_0 \sqcup \dots \sqcup I_n). \tag{23}$$

Suppose A has a computable type. Does (23) then has a computable type? The following simple example shows that in general the answer is negative.

Example 4. *Let $A = \{0,1\}$. Then A has a computable type (see Proposition 2). Let $f : \{0,1\} \to A$ be the identity and let us consider the adjunction space $A \cup_f [0,1]$. By Example 3 $A \cup_f [0,1]$ is homeomorphic to $[0,1]$ and $[0,1]$ does not have a computable type (recall that there exists $\gamma > 0$ such that $[0,\gamma]$ is semicomputable but not computable). So $A \cup_f [0,1]$ does not have a computable type.*

Nevertheless, we have the following result.

Theorem 3. *Let A be a topological space, let $I_0, \dots, I_n$ be such that $I_i = [0,1]$ for each $i \in \{0, \dots, n\}$ and let $f : \partial I_0 \sqcup \dots \sqcup \partial I_n \to A$ be a function. Suppose A has a computable type. Then*

$$(A \cup_f (I_0 \sqcup \dots \sqcup I_n), A)$$

has a computable type (where A is identified with a subspace of $A \cup_f (I_0 \sqcup \dots \sqcup I_n)$ as in Remark 3).

Proof. Suppose $(X, \mathcal{T}, (I_i))$ is a computable topological space and T and S are semicomputable sets in this space such that there exists a homeomorphism $g : A \cup_f (I_0 \sqcup \dots \sqcup I_n) \to T$ which maps A to S. More precisely, we have $g(p(i(A)) = S$, where $i : A \to A \sqcup (I_0 \sqcup \dots \sqcup I_n)$ is the inclusion and $p : A \sqcup (I_0 \sqcup \dots \sqcup I_n) \to A \cup_f (I_0 \sqcup \dots \sqcup I_n)$ is the quotient map. We will identify A and I_i with corresponding images by inclusions

$$A \to A \sqcup (I_0 \sqcup \dots \sqcup I_n) \text{ and } I_i \to A \sqcup (I_0 \sqcup \dots \sqcup I_n).$$

We want to prove that T is a computable set in $(X, \mathcal{T}, (I_i))$. In order to apply Theorem 2, we have to show that T "looks like" as in Figure 5. For that purpose, we have to show that $A \cup_f (I_0 \sqcup \ldots \sqcup I_n)$ "looks like" as in Figure 5 (since g is homeomorphism).

We have that T is a Hausdorff space (as a subspace of $(X, \mathcal{T})$), so $A \cup_f (I_0 \sqcup \ldots \sqcup I_n)$ is also a Hausdorff space. Obviously, $A \sqcup (I_0 \sqcup \ldots \sqcup I_n)$ is compact.

Let $i \in \{0, \ldots, n\}$. Then $p(I_i)$ is compact in $A \cup_f (I_0 \sqcup \ldots \sqcup I_n)$. Since p is a surjection, we have

$$A \cup_f (I_0 \sqcup \ldots \sqcup I_n) = p(A) \cup p(I_1) \cup \cdots \cup p(I_n).$$

By the definition of an adjunction space we have that p is injective on $I_i \setminus \{1\}$ and p maps the points $0, 1 \in I_i$ to the same point in the adjunction space $A \cup_f (I_0 \sqcup \ldots \sqcup I_n)$ if and only if $f(0) = f(1)$. So the function

$$p|_{I_i} : I_i \to p(I_i)$$

is a continuous surjection which is either injective (in particular, $p(0) \neq p(1)$) or it is injective on $I_i \setminus \{1\}$ and $p(0) = p(1)$. In the first case, we have that $p|_{I_i}$ is a homeomorphism (since I_i is compact and $p(I_i)$ is Hausdorff), so $p(I_i)$ is homeomorphic to $[0, 1]$. In the second case, it follows from Remark 2 that $p(I_i)$ is homeomorphic to S^1. Hence, $p(I_i)$ is either an arc or a topological circle.

The function f is continuous since $\partial I_0 \sqcup \ldots \sqcup \partial I_n$ is a discrete space and this space is also closed in $I_0 \sqcup \ldots \sqcup I_n$. So, as noted earlier, A and $p(A)$ are homeomorphic. Furthermore, $p(A)$ and S are homeomorphic (the homeomorphism is a restriction of g) and it follows that A and S are homeomorphic. This, together with the fact that A has a computable type, implies that S is computable in $(X, \mathcal{T}, (I_i))$.

We have that $A \cup_f (I_0 \sqcup \ldots \sqcup I_n)$ is the union of the sets $p(A)$, $p(I_0)$, …, $p(I_n)$. For each $i \in \{0, \ldots, n\}$ there exist $x_i, y_i \in p(A)$ such that $p(I_i) \cap p(A) = \{x_i, y_i\}$ and $p(I_i)$ is either an arc with endpoints x_i and y_i, $x_i \neq y_i$, or $p(I_i)$ is a topological circle and $x_i = y_i$. Furthermore, if $i, j \in \{0, \ldots, n\}$ are such that $i \neq j$, then $p(I_i) \cap p(I_j) \subseteq p(A)$.

From this and the fact that $g : A \cup_f (I_0 \sqcup \ldots \sqcup I_n) \to T$ is a homeomorphism, we conclude that for the finite sequence $(K_i, \{a_i, b_i\})_{0 \leq i \leq n}$ defined by $K_i = g(p(I_i))$, $a_i = g(x_i)$, $b_i = g(y_i)$ we have $T = S \cup K_0 \cup \cdots \cup K_n$ and the assumptions of Theorem 2 hold. Thus, by Theorem 2, T is computable. □

References

[1] D. E. Amir, M. Hoyrup, *Computability of finite simplicial complexes*, in: *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*, (M. Bojańczyk et al., Eds.), Dagstuhl Publishing, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, Germany, 2022, 1–16.

[2] V. Brattka, *Plottable real number functions and the computable graph theorem*, SIAM J. Comput. **38**(2008), 303–328.

[3] V. Brattka, G. Presser, *Computability on subsets of metric spaces*, Theor. Comput. Sci. **305**(2003), 43–76.

[4] V. BRATTKA, K. WEIHRAUCH, *Computability on subsets of Euclidean space I: Closed and compact subsets*, Theor. Comput. Sci. **219**(1999), 65–93.

[5] C. E. BURGESS, *Chainable continua and indecomposability*, Pac. J. Math. **9**(1959), 653–659.

[6] K. BURNIK, Z. ILJAZOVIĆ, *Computability of 1-manifolds*, Log. Methods Comput. Sci. **10**(2014), 1–28.

[7] V. ČAČIĆ, M. HORVAT, Z. ILJAZOVIĆ, *Computable subcontinua of semicomputable chainable Hausdorff continua*, Theor. Comput. Sci. **892**(2021), 155–169.

[8] M. ČELAR, Z. ILJAZOVIĆ, *Computability of products of chainable continua*, Theory Comput. Syst. **65**(2021), 410–427.

[9] M. ČELAR, Z. ILJAZOVIĆ, *Computability of glued manifolds*, J. Log. Comput. **32**(2022), 65–97.

[10] E. ČIČKOVIĆ, Z. ILJAZOVIĆ, L. VALIDŽIĆ, *Chainable and circularly chainable semicomputable sets in computable topological spaces*, Arch. Math. Log. **58**(2019), 885–897.

[11] M. HORVAT, Z. ILJAZOVIĆ, B. PAŽEK, *Computability of pseudo-cubes*, Ann. Pure Appl. Log. **171**(2020), 1–21.

[12] Z. ILJAZOVIĆ, *Chainable and circularly chainable co-c.e. sets in computable metric spaces*, J Univers. Comput Sci. **15**(2009), 1206–1235.

[13] Z. ILJAZOVIĆ, *Co-c.e. spheres and cells in computable metric spaces*, Log. Methods Comput. Sci. **7**(2011), 1–21.

[14] Z. ILJAZOVIĆ, *Compact manifolds with computable boundaries*, Log. Methods Comput. Sci. **9**(2013), 1–22.

[15] Z. ILJAZOVIĆ, *Computability of graphs*, Math. Log. Q. **66**(2020), 51–64.

[16] Z. ILJAZOVIĆ, B. PAŽEK, *Computable intersection points*, Computability **7**(2018), 57–99.

[17] Z. ILJAZOVIĆ, B. PAŽEK, *Warsaw discs and semicomputability*, Topol. Appl. **239**(2018), 308–323.

[18] Z. ILJAZOVIĆ, I. SUŠIĆ, *Semicomputable manifolds in computable topological spaces*, J. Complex. **45**(2018), 83–114.

[19] T. KIHARA, *Incomputability of simply connected planar continua*, Computability **1**(2012), 131–152.

[20] J. S. MILLER, *Effectiveness for embedded spheres and balls*, Electron. Notes Theor. Comput. Sci. **66**(2002), 127–138.

[21] J. R. MUNKRES, *Topology, Featured Titles for Topology*, Prentice Hall, New Jersey, 2000.

[22] M. B. POUR-EL, J. I. RICHARDS, *Computability in Analysis and Physics*, Springer, Berlin, 1989.

[23] E. SPECKER, *Der Satz vom Maximum in der rekursiven Analysis*, in: *Constructivity in Mathematics*, (A. Heyting, Ed.), North Holland Publ. Comp., Amsterdam, 1959, 254–265.

[24] H. TANAKA, *On a* Π_1^0 *set of positive measure*, Nagoya Math. J. **38**(1970), 139–144.

[25] A. M. TURING, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. **42**(1936), 230–265.

[26] K. WEIHRAUCH, *Computability on computable metric spaces*, Theor. Comput. Sci. **113**(1993), 191–210.

[27] K. WEIHRAUCH, *Computable Analysis*, Springer, Berlin, 2000.

[28] K. WEIHRAUCH, *Computable separation in topology, from* T_0 *to* T_2, J Univers. Comput Sci. **16**(2010), 2733–2753.

[29] K. WEIHRAUCH, T. GRUBBA, *Elementary computable topology*, J Univers. Comput Sci. **15**(2009), 1381–1422.