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On the preconditioned APSS iterative method for singular coupled saddle point problems

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Abstract. Salkuyeh et al. (D.K. Salkuyeh, H. Aslani, Z.Z. Liang, An alternating positive semi-definite splitting preconditioner for the three-by-three block saddle point problems, Math. Commun. 26 (2021) 177-195) has recently established an alternating positive semi-definite splitting (APSS) method for nonsymmetric block three-by-three nonsingular saddle point problems arising from the Picard iteration method for a class of mixed finite element scheme. In this work, we analyse the semi-convergence of the APSS method for solving a class of nonsymmetric block three-by-three singular saddle point problems. The APSS induced preconditioner is applied to improve the semi-convergence rate of the flexible GMRES (FGMRES) method. Numerical results are designated to support the theoretical results. These results show that the served preconditioner is efficient compared with FGMRES without a preconditioner.

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1. Introduction

We are interested in solving the following large and sparse block three-by-three saddle point system by iteration methods:

$$\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{l\times m}$. Here $f \in \mathbb{R}^n, g \in \mathbb{R}^m$ and $h \in \mathbb{R}^l$ are known and $\mathbf{x} = (x; y; z)$ is an unknown vector that has to be determined. The coefficient matrix of system (1) is of order $\mathbf{n} = m + n + l$. Many practical applications produce linear systems of the form (1); for example, the application of the Picard iteration method for a class of mixed finite element scheme for stationary magnetohydrodynamic models [17] and the finite element method to solve the time-dependent Maxwell equations having discontinuous coefficients in polyhedral

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domains with a Lipschitz boundary. References [26, 14] provide additional instances and references therein.

The aim of the paper is to solve the following system:

$$\mathcal{A}\mathbf{x} \equiv \begin{pmatrix} A & B^T & 0 \\ -B & 0 & -C^T \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ -g \\ h \end{pmatrix} \equiv \mathbf{b}, \tag{2}$$

which is an equivalent form of (1). Note that the coefficient matrix in (1) is symmetric, however, $\mathcal{A}$ is nonsymmetric. Nevertheless, $\mathcal{A}$ has some good properties. For instance, $\mathcal{A}$ is positive semi-definite. It means that $\mathcal{A} + \mathcal{A}^T$ is Symmetric Positive Semi-Definite (SPSD). This property can greatly improve the performance of the GMRES method for solving the system. Some other features are available in [20].

Consider the two-by-two block saddle point problem:

$$\tilde{\mathcal{A}}\tilde{\mathbf{x}} \equiv \begin{pmatrix} \tilde{A} & \tilde{B}^T \\ -\tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} \equiv \tilde{\mathbf{b}}. \tag{3}$$

The matrix $\tilde{A} \in \mathbb{R}^{p\times p}$ is assumed to be a Symmetric Positive Definite (SPD), $\tilde{B} \in \mathbb{R}^{p\times q}$ with $p > q$ is a matrix that $rank(B) = r < q$ (i.e., $\tilde{B}$ is a rank-deficient matrix), $\tilde{f} \in \mathbb{R}^p$ and $\tilde{g} \in \mathbb{R}^q$. Thereby, linear system (3) leads to a two-by-two singular saddle point problem. In the context of the two-by-two singular saddle point problem, Bai [4] established the HSS method for a singular saddle point problem and also derived some conditions for guaranteeing the semi-convergence of the HSS method. Li et al. [22] generalized the HSS method for solving non-Hermitian, positive semi-definite and singular linear systems. They studied semi-convergence analysis of the Generalized HSS (GHSS) method. In addition, an upper bound for the semi-convergence factor was derived. In [12], a Generalized Preconditioned Hermitian and skew-Hermitian splitting method (GPHSS) was considered to solve singular saddle point problems. The semi-convergence of the GPHSS scheme was proved under some conditions. In addition, the authors obtained the induced preconditioner and discussed the eigenvalues of the preconditioned matrix. Then, the Local Hermitian and skew-Hermitian (LHSS) method and the Modified LHSS (MLHSS) method were established in [20]. They also gave the semi-convergence conditions. Motivated by the Uzawa method, Chao et al. [11] designed the Uzawa-SOR method for singular saddle point problems. Generalization of the Uzawa-SOR method was investigated in [31]. The authors established the Uzawa-AOR scheme to solve (3). The distribution of the eigenvalues of the iteration matrix and the semi-convergence properties were given. Numerical results indicate that the Uzawa-AOR method outperforms the Uzawa method [2], the parameterized Uzawa method [35], and the Uzawa-SOR method [11]. Some other efficient methods for the singular saddle point problem of the form (3) were studied in [24, 34, 33].

Note that some partitioning techniques can be employed to represent the three-

by-three coefficient matrix in (2) as:

$$\begin{pmatrix} A & B^T & \vdots & 0 \\ B & 0 & \vdots & C^T \\ \cdots & \cdots & \vdots & \cdots \\ 0 & C & \vdots & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \vdots & B^T & 0 \\ \cdots & \vdots & \cdots & \cdots \\ B & \vdots & 0 & C^T \\ 0 & \vdots & C & 0 \end{pmatrix},$$

which are the standard two-by-two saddle point problem.

Many available preconditioning schemes for (3) can not be implemented for solving (1). This is because of different properties in two problems. Note that the matrix $\mathcal{A}$ described in (2) is nonsingular if A is SPD and the matrices B and C are of full row rank (see [18, 19, 28]). In recent years, many researchers have considered the three-by-three saddle point problem. Huang et al. [20] arranged a block diagonal preconditioner for solving the nonsingular system of the form (2). The exact and inexact versions of the preconditioner were also studied. Then, in [10], the shift splitting (SS) and the relaxed shift splitting (RSS) method were designed. Xie et al. [30] considered three efficient preconditioners. The authors analyzed the eigenvalues of the corresponding preconditioned matrices. Aslani et al. [3] presented a new method for solving (2) when A is SPD and B, C are full row rank matrices. Convergence properties of the method were derived. Moreover, the spectral properties of the preconditioned matrix were discussed. Abdolmaleki et al. [1] proposed a block three-by-three diagonal preconditioner from another way for (2). A suitable estimation strategy for lower and upper bounds of eigenvalues of the preconditioned matrix was considered. In [29], the authors proposed an exact parameterized block SPD preconditioner and its inexact version for a class of block three-by-three saddle point problems. They also estimated the eigenvalue bounds for the preconditioned matrix.

In [21], Liang and Zhang proposed the Alternating Positive Semi-definite Splitting (APSS) iteration method for double saddle point problems. Using the idea of [21], Salkuyeh et al. applied in [27] the APSS method for solving problem (2) and proved its convergence. In the case of C being rank-deficient, the coefficient matrix (2) is singular. Accordingly, linear system (2) is labeled as a singular three-by-three saddle point problem. In this work, the three-by-three large, sparse, and singular saddle point problem is considered and the semi-convergence analysis of the APSS method is discussed.

The structure of this paper is as follows. The paper starts with a review of the APSS method and its corresponding induced preconditioner. In Section 2, we focus on the semi-convergence properties of the APSS method for solving (2). Unconditional semi-convergence for the APSS iteration method are derived in Section 3. A strategy is given to estimate the parameter of the adopted method in Section 4. Section 5 is devoted to giving some numerical tests to support the theoretical results. Some succinct conclusions will be included at the end of the work.

2. Review of the APSS method

Let us first give a brief overview of the APSS method. Consider the following decomposition for the coefficient matrix $\mathcal{A}$ in (2):

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \tag{4}$$

where

$$\mathcal{A}_1 = \begin{pmatrix} A & B^T & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -C^T \\ 0 & C & 0 \end{pmatrix}. \tag{5}$$

Let $\alpha > 0$ be a given parameter. Based on the decomposition (4), the following splittings for the matrix $\mathcal{A}$ can be stated:

$$\mathcal{A} = (\alpha\mathcal{I} + \mathcal{A}_1) - (\alpha\mathcal{I} - \mathcal{A}_2) = (\alpha\mathcal{I} + \mathcal{A}_2) - (\alpha\mathcal{I} - \mathcal{A}_1),$$

where $\mathcal{I}$ is the identity matrix of order $\mathbf{n}$. Now, by using these splittings, the APSS method can be written as

$$\begin{cases} (\alpha\mathcal{I} + \mathcal{A}_1)x^{(k+\frac{1}{2})} = (\alpha\mathcal{I} - \mathcal{A}_2)x^{(k)} + b, \\ (\alpha\mathcal{I} + \mathcal{A}_2)x^{(k+1)} = (\alpha\mathcal{I} - \mathcal{A}_1)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where $x^{(0)} \in \mathbb{R}^{\mathbf{n}}$ is an initial guess. By eliminating $x^{(k+\frac{1}{2})}$, the iteration scheme can be rewritten as the stationary form

$$x^{k+1} = \mathcal{T}_\alpha x^k + f, \tag{6}$$

with

$$\mathcal{T}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2), \tag{7}$$

and

$$f = 2\alpha(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}b.$$

Similarly to the Hermitian and Skew-Hermitian splitting (HSS) iteration method [5], if we set

$$\tilde{\mathcal{M}}_\alpha = \frac{1}{2\alpha}(\alpha\mathcal{I} + \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_2), \quad \tilde{\mathcal{N}}_\alpha = \frac{1}{2\alpha}(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} - \mathcal{A}_2),$$

then $\mathcal{A} = \tilde{\mathcal{M}}_\alpha - \tilde{\mathcal{N}}_\alpha$ and

$$\mathcal{T}_\alpha = {\tilde{\mathcal{M}}_\alpha}^{-1}\tilde{\mathcal{N}}_\alpha = \mathcal{I} - {\tilde{\mathcal{M}}_\alpha}^{-1}\mathcal{A}.$$

From now on, we use $\mathcal{M}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_2)$ as the APSS preconditioner since the pe-factor $\frac{1}{2\alpha}$ has no effect on the preconditioned matrix. So, the saddle point system (2) can be preconditioned from the left as ${\mathcal{M}_\alpha}^{-1}\mathcal{A}\mathbf{x} = {\mathcal{M}_\alpha}^{-1}\mathbf{b}$. In this case, we have

$${\mathcal{M}_\alpha}^{-1}\mathcal{A}\mathbf{x} = (\mathcal{I} - {\mathcal{M}_\alpha}^{-1}\mathcal{N}_\alpha)\mathbf{x} = {\mathcal{M}_\alpha}^{-1}\mathbf{b}, \tag{8}$$

where

$$\mathcal{N}_\alpha = (\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} - \mathcal{A}_2) = \alpha\mathcal{I}.\alpha\mathcal{I} - \alpha(\mathcal{A}_1 + \mathcal{A}_2) + \mathcal{A}_1\mathcal{A}_2. \tag{9}$$

3. The semi-convergence of the APSS iteration method

In this section, we will analyze the semi-convergence properties of the APSS iteration method for solving the double saddle point problem (2). First, let us give some related definitions, lemmas and theorems.

Definition 1. *The iteration method* (6) *is said to be semi-convergent if for any initial guess* $\mathbf{x}^{(0)}$*, the iteration sequence* $\{\mathbf{x}^{(\mathbf{k})}\}$ *produced by* (6) *converges to a solution* $\mathbf{x}^{(*)}$ *of* (2)*. Moreover,*

$$\mathbf{x}^{(*)} = (\mathcal{I} - \mathcal{T}_\alpha)^D f + [\mathcal{I} - (\mathcal{I} - \mathcal{T}_\alpha)^D(\mathcal{I} - \mathcal{T}_\alpha)]\mathbf{x}^{(0)},$$

where $(\mathcal{I} - \mathcal{T}_\alpha)^D$ *denotes the Drazin inverse of* $(\mathcal{I} - \mathcal{T}_\alpha)$ *[8].*

Definition 2. *The index of* A *is the smallest nonnegative integer* i *such that*

$$rank(A^i) = rank(A^{i+1}),$$

and it is expressed by the expression $i = index(D)$*.*

Theorem 1 ([7])**.** *The iteration method* (6) *is semi-convergent if and only if*

$$index(\mathcal{I} - \mathcal{T}_\alpha) = 1 \quad and \quad \vartheta(\mathcal{T}_\alpha) < 1,$$

where

$$\vartheta(\mathcal{T}_\alpha) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}_\alpha), \lambda \neq 1\} < 1,$$

in which $\vartheta(\mathcal{T}_\alpha)$ *is called the pseudo-spectral radius of* $\mathcal{T}_\alpha$*.*

Lemma 1 (Kellogg's lemma, [25])**.** *If* $A \in \mathbb{C}^{n\times n}$ *is positive semi-definite, then*

$$\left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2 \leq 1,$$

for all $\alpha > 0$*. Moreover, if* $A \in \mathbb{C}^{n\times n}$ *is positive definite, then*

$$\left\|(\alpha I + A)^{-1}(\alpha I - A)\right\|_2 < 1,$$

for all $\alpha > 0$*.*

Remark 1. *For the saddle point matrix of the form*

$$\mathcal{A} = \begin{pmatrix} A & B^T & 0 \\ -B & 0 & -C^T \\ 0 & C & 0 \end{pmatrix},$$

if $\mathcal{A}$ *is singular, we can easily see that at least one of the sets* $null(B^T) \cap null(C)$ *and* $null(C^T)$ *is nontrivial, i.e., the dimension of one of the two sets is at least one.*

Noticing that

$$(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1} = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1), \tag{10}$$

$$(\alpha\mathcal{I} - \mathcal{A}_2)(\alpha\mathcal{I} + \mathcal{A}_2)^{-1} = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2), \tag{11}$$

it is easy to see that the matrix $\mathcal{T}_\alpha$ is similar to

$$\mathcal{L}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2). \tag{12}$$

Now, since the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are both positive semi-definite, then using Kellogg's lemma we have

$$\|\mathcal{L}_\alpha\|_2 \leq \|(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)\|_2 \|(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)\|_2 \leq 1.$$

Thus, it holds that $\mathcal{L}_\alpha \mathbf{x} = \mathbf{x}$ if and only if $\mathcal{L}_\alpha^* \mathbf{x} = \mathbf{x}$, for any $\mathbf{x} \in \mathbb{C}^{n\times n}$ [4]. As a result, the index of $(\mathcal{I} - \mathcal{L}_\alpha)$ is equal to 1. Eventually, since two similar matrices have the same index, we see that

$$\text{index}(\mathcal{I} - \mathcal{T}_\alpha) = 1. \tag{13}$$

Next, we give the conditions for $\vartheta(\mathcal{T}_\alpha) < 1$.

Theorem 2. *Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ is full row rank, and $C \in \mathbb{R}^{l\times m}$ is rank-deficient. Then, for decomposition* (4) *of the matrix $\mathcal{A}$, the pseudo-spectral radius of $\mathcal{T}_\alpha$ is less than one, i.e., $\vartheta(\mathcal{T}_\alpha) < 1, \forall \alpha > 0$.*

Proof. First of all, note that the matrix $\alpha\mathcal{I} - \mathcal{A}_2$ is non-singular, and equalities (10) and (11) hold. Hence, the matrix $\mathcal{T}_\alpha$ is similar to

$$\mathcal{P}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1),$$

So, $\vartheta(\mathcal{T}_\alpha) = \vartheta(\mathcal{P}_\alpha)$. Let $x \in \mathbb{C}^n$ be any eigenvector of the matrix $\mathcal{P}_\alpha$ and λ the eigenvalue of matrix $\mathcal{P}_\alpha$ corresponding to eigenvector x, i.e., $\mathcal{P}_\alpha x = \lambda x$. Without loss of generality, we assume that $\|x\|_2 = 1$. Now, we prove the theorem according to the following four cases:

Case 1. $x \in null(\mathcal{A}_1) \cap null(\mathcal{A}_2)$. So, we have $\mathcal{A}_1 x = \mathcal{A}_2 x = 0$ and it follows that

$$(\alpha\mathcal{I} + \mathcal{A}_1)x = (\alpha\mathcal{I} - \mathcal{A}_1)x \Longrightarrow x = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)x, \tag{14}$$

and

$$(\alpha\mathcal{I} + \mathcal{A}_2)x = (\alpha\mathcal{I} - \mathcal{A}_2)x \Longrightarrow x = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)x. \tag{15}$$

By combining (14) and (15), we have $\mathcal{P}_\alpha x = x$. Thus, $\lambda = 1$.

Case 2. $x \in null(\mathcal{A}_2)$, but $x \notin null(\mathcal{A}_1)$. It means that $\mathcal{A}_1 x \neq 0$ and $\mathcal{A}_2 x = 0$. From $\mathcal{P}_\alpha x = \lambda x$, by easy computations we can obtain

$$\lambda x = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)x. \tag{16}$$

Since $\mathcal{A}_1$ is positive semi-definite, it follows from Kellogg's lemma that

$$\|(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)\|_2 \leq 1.$$

Therefore $|\lambda| \leq 1$. In what follows, we further prove that $|\lambda| < 1$ for any $\alpha > 0$. We will argue it by contradiction. If $|\lambda| = 1$, then there exists $\theta \in (-\pi, \pi]$ such that

$$(\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}x = e^{\imath\theta}x. \tag{17}$$

If we set $\mathcal{V}_\alpha = (\alpha\mathcal{I} - \mathcal{A}_1)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}$, then we can rewrite (17) as follows:

$$\mathcal{V}_\alpha x = e^{\imath\theta} x.$$

It then follows that

$$\|\mathcal{V}_\alpha x\|_2 = \|x\|_2. \tag{18}$$

Now, letting

$$w := (u; v; p) = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1} x, \quad \text{with} \quad u \in \mathbb{C}^n, \quad v \in \mathbb{C}^m, \text{ and } \quad p \in \mathbb{C}^l,$$

we get

$$\|(\alpha\mathcal{I} - \mathcal{A}_1)w\|_2 = \|(\alpha\mathcal{I} + \mathcal{A}_1)w\|_2. \tag{19}$$

From (19), it holds that

$$w^*(\mathcal{A}_1 + \mathcal{A}_1^*)w = 0,$$

or equivalently,

$$\begin{pmatrix} u^* & v^* & p^* \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0,$$

which leads to $u = 0$ due to the symmetric positive definiteness of A. On the one hand, from (17) we have

$$(\alpha\mathcal{I} - \mathcal{A}_1)w = e^{\imath\theta}(\alpha\mathcal{I} + \mathcal{A}_1)w,$$

which gives

$$\begin{pmatrix} \alpha I - A & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & \alpha I \end{pmatrix} \begin{pmatrix} 0 \\ v \\ p \end{pmatrix} = e^{\imath\theta} \begin{pmatrix} \alpha I + A & B^T & 0 \\ -B & \alpha I & 0 \\ 0 & 0 & \alpha I \end{pmatrix} \begin{pmatrix} 0 \\ v \\ p \end{pmatrix},$$

and so

$$\begin{pmatrix} -B^T v \\ \alpha v \\ \alpha p \end{pmatrix} = e^{\imath\theta} \begin{pmatrix} B^T v \\ \alpha v \\ \alpha p \end{pmatrix}.$$

It leads to

$$\begin{cases} B^T v = -e^{\imath\theta} B^T v, & (20) \\ \alpha v = e^{\imath\theta} \alpha v, & (21) \\ \alpha p = e^{\imath\theta} \alpha p. & (22) \end{cases}$$

Here, if $B^T v = 0$, then we see that $v = 0$, because B is a full row rank matrix. It can be immediately concluded that $\mathcal{A}_1 x = 0$, which contradicts the assumption $x \notin null(\mathcal{A}_1)$. Thereby, from (20) it holds $e^{\imath\theta} = -1$. Substituting the equality $e^{\imath\theta} = -1$ into (21) and (22) results in $v = 0$ and $p = 0$, respectively. So $w = 0$. Consequently, we have $x = (\alpha\mathcal{I} + \mathcal{A}_1)w = 0$, which contradicts the fact that x is an eigenvector. Therefore, $|\lambda| < 1$.

Case 3. $x \in null(\mathcal{A}_1)$, but $x \notin null(\mathcal{A}_2)$. So, we have

$$\mathcal{A}_1 x = 0 \Longrightarrow (\alpha\mathcal{I} + \mathcal{A}_1)x = \alpha x \Longrightarrow (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}x = \frac{1}{\alpha}x.$$

Hence, using $\mathcal{A}_1 x = 0$ and the above equation, we get

$$\begin{aligned}\mathcal{P}_\alpha x &= \alpha(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)(\alpha\mathcal{I} + \mathcal{A}_1)^{-1}x \\ &= (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)x.\end{aligned}$$

Therefore, using the fact that $\mathcal{P}_\alpha x = \lambda x$, the positive semi-definiteness of $\mathcal{A}_2$ and Kellogg's lemma, gives

$$|\lambda| \leq \|\mathcal{P}_\alpha x\| \leq \|(\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)\|_2 \leq 1. \tag{23}$$

Moreover, we claim that $|\lambda| = 1$ never happens. By contradiction, we assume that $|\lambda| = 1$. So, there exists $\theta \in (-\pi, \pi]$ so that

$$(\alpha\mathcal{I} - \mathcal{A}_2)x = e^{\imath\theta}(\alpha\mathcal{I} + \mathcal{A}_2)x,$$

which is equivalent to

$$\begin{cases} \alpha x_1 = e^{\imath\theta}\alpha x_1, & (24) \\ \alpha x_2 + C^T x_3 = e^{\imath\theta}(\alpha x_2 - C^T x_3), & (25) \\ -Cx_2 + \alpha x_3 = e^{\imath\theta}(Cx_2 + \alpha x_3). & (26) \end{cases}$$

From Eq. (24), either $e^{\imath\theta} = 1$ or $x_1 = 0$. First, suppose that $e^{\imath\theta} = 1$. From $\mathcal{A}_1 x = 0$, we have

$$\begin{cases} Ax_1 + B^T x_2 = 0, & (27) \\ Bx_1 = 0. & (28) \end{cases}$$

Substituting $x_1 = -A^{-1}B^T x_2$, which is deduced from (27) into (28), gives $BA^{-1}B^T x_2 = 0$, consequently, $x_2 = 0$, and then by (27), $x_1 = 0$. Now, from Eq. (25), we have $C^T x_3 = 0$. Thus, $\mathcal{A}_2 x = 0$, which contradicts the assumption. Therefore, $e^{\imath\theta} = 1$ never happens. Now, we discuss $x_1 = 0$. Clearly, the assumption $\mathcal{A}_1 x = 0$, gives

$$\begin{pmatrix} A & B^T & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} B^T x_2 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Since B has full row rank, we deduce that $x_2 = 0$. Substituting $x_2 = 0$ into (26) gives $x_3 = 0$. So, $x = (x_1; x_2; x_3) = 0$, which is impossible. Thereupon, $|\lambda| = 1$ is unacceptable.

Case 4. $x \notin null(\mathcal{A}_1), x \notin null(\mathcal{A}_2)$. For this case, we have $\mathcal{A}_1 x \neq 0$ and $\mathcal{A}_2 x \neq 0$. If we define $\mathcal{Q}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_2)^{-1}(\alpha\mathcal{I} - \mathcal{A}_2)$ and $\mathcal{V}_\alpha = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1}(\alpha\mathcal{I} - \mathcal{A}_1)$, we can see that $\mathcal{P}_\alpha = \mathcal{Q}_\alpha\mathcal{V}_\alpha$. Since $\mathcal{Q}_\alpha$ is a unitary matrix, we see that $\|\mathcal{Q}_\alpha\|_2 = 1$. On

the other hand, since $\mathcal{A}_1$ is a positive semi-definite matrix from Kellogg's lemma, we have $\|\mathcal{V}_\alpha\|_2 \leq 1$ for any $\alpha > 0$. It leads to

$$\|\mathcal{P}_\alpha\|_2 \leq \|\mathcal{Q}_\alpha\|_2 \|\mathcal{V}_\alpha\|_2 = \|\mathcal{V}_\alpha\|_2 \leq 1, \quad \forall \alpha > 0.$$

Thus, $|\lambda| \leq 1$. In what follows, we further prove that $|\lambda| < 1$, for any $\alpha > 0$. We will argue it by contradiction. If $|\lambda| = 1$, then there exists $\theta \in (-\pi, \pi]$ such that

$$\mathcal{P}_\alpha x = e^{\imath\theta} x,$$

which is equivalent to

$$\mathcal{V}_\alpha x = e^{\imath\theta} \mathcal{Q}_\alpha^* x. \tag{29}$$

Consequently,

$$\|\mathcal{V}_\alpha x\|_2 = \|x\|_2. \tag{30}$$

Substituting $\mathcal{V}_\alpha$ into (30) and using the change of variable

$$w := (u; v; p) = (\alpha\mathcal{I} + \mathcal{A}_1)^{-1} x, \quad \text{with} \quad u \in \mathbb{C}^n \quad , v \in \mathbb{C}^m \text{ and } \quad p \in \mathbb{C}^l,$$

gives

$$\|(\alpha\mathcal{I} - \mathcal{A}_1)w\|_2 = \|(\alpha\mathcal{I} + \mathcal{A}_1)w\|_2. \tag{31}$$

From (31), it holds

$$w^*(\mathcal{A}_1 + \mathcal{A}_1^*)w = 0,$$

or equivalently,

$$\begin{pmatrix} u^* & v^* & p^* \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0,$$

which leads to $u = 0$ due to the symmetric positive definiteness of A. On the one hand, from (29) we have

$$(\alpha\mathcal{I} - \mathcal{A}_1)w = e^{\imath\theta} \mathcal{Q}_\alpha^* (\alpha\mathcal{I} + \mathcal{A}_1)w,$$

which results in

$$\begin{pmatrix} \alpha I - A & -B^T & 0 \\ B & \alpha I & 0 \\ 0 & 0 & \alpha I \end{pmatrix} \begin{pmatrix} 0 \\ v \\ p \end{pmatrix} = e^{\imath\theta} \mathcal{Q}_\alpha^* \begin{pmatrix} \alpha I + A & B^T & 0 \\ -B & \alpha I & 0 \\ 0 & 0 & \alpha I \end{pmatrix} \begin{pmatrix} 0 \\ v \\ p \end{pmatrix}.$$

So

$$\begin{pmatrix} -B^T v \\ \alpha v \\ \alpha p \end{pmatrix} = e^{\imath\theta} \mathcal{Q}_\alpha^* \begin{pmatrix} B^T v \\ \alpha v \\ \alpha p \end{pmatrix}.$$

The above equality can be rewritten as

$$(\alpha\mathcal{I} - \mathcal{A}_2) \begin{pmatrix} -B^T v \\ \alpha v \\ \alpha p \end{pmatrix} = e^{\imath\theta} (\alpha\mathcal{I} + \mathcal{A}_2) \begin{pmatrix} B^T v \\ \alpha v \\ \alpha p \end{pmatrix}.$$

It leads to

$$\begin{pmatrix} \alpha I & 0 & 0 \\ 0 & \alpha I & C^T \\ 0 & -C & \alpha I \end{pmatrix} \begin{pmatrix} -B^T v \\ \alpha v \\ \alpha p \end{pmatrix} = e^{\imath\theta} \begin{pmatrix} \alpha I & 0 & 0 \\ 0 & \alpha I & -C^T \\ 0 & C & \alpha I \end{pmatrix} \begin{pmatrix} B^T v \\ \alpha v \\ \alpha p \end{pmatrix},$$

which implies that

$$\begin{cases} B^T v = -e^{\imath\theta} B^T v, & (32) \\ \alpha v + C^T p = e^{\imath\theta}(\alpha v - C^T p), & (33) \\ -Cv + \alpha p = e^{\imath\theta}(Cv + p). & (34) \end{cases}$$

If $B^T v = 0$, then we can easily see that $v = 0$. IIt can be immediately concluded that $\mathcal{A}_1 x = 0$, which contradicts the consideration $x \notin null(\mathcal{A}_1)$. Thereby, from (32) it holds $e^{\imath\theta} = -1$. Substituting the identity $e^{\imath\theta} = -1$ into (33) and (34), derives $v = 0$ and $p = 0$, respectively. So $w = 0$. Consequently, we have $x = (\alpha\mathcal{I} + \mathcal{A}_1)w = 0$, which contradicts the fact that x is a non-zero vector. Therefore, $|\lambda| < 1$.

In summary, from *cases 1-4*, we see that $\vartheta(\mathcal{T}_\alpha) < 1$. □

Theorem 3. *Suppose that the assumptions of Theorem 2 hold. Then, the APSS iteration method* (6) *is semi-convergent for any* $\alpha > 0$.

Proof. The proof immediately follows from Eq. (13) and Theorem 2. □

4. Estimation strategy for the parameter α

Finding the optimal parameter α of the APSS method is generally difficult. In this section, we adopt an appropriate strategy for estimating α in the APSS method, which has been studied by Cao [9].

Notably, from (8) it is anticipated that $\mathcal{M}_\alpha$ is as close as possible to $\mathcal{A}$ when $\mathcal{N}_\alpha \approx 0$. In this way, having Eq. (9) in mind, the function

$$\begin{aligned} \Psi(\alpha) &= \alpha\|\mathcal{I}\|_F \cdot \alpha\|\mathcal{I}\|_F - \alpha\left(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F\right) + \|\mathcal{A}_1\|_F \|\mathcal{A}_2\|_F \\ &= (n+m+l)\alpha^2 - \alpha\left(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F\right) + \|\mathcal{A}_1\|_F \|\mathcal{A}_2\|_F \\ &= \mathbf{n}\alpha^2 - \alpha\left(\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F\right) + \|\mathcal{A}_1\|_F \|\mathcal{A}_2\|_F \end{aligned}$$

can be characterized. Minimizing $\Psi(\alpha)$ with respect to α leads to the estimation parameter α_{est} set to

$$\alpha_{est} = \frac{\|\mathcal{A}_1\|_F + \|\mathcal{A}_2\|_F}{2\mathbf{n}}.$$

Since the matrix A is SPD (in this case, $\mathcal{A}_1 \neq 0$), one can conclude that $\alpha_{est} > 0$. In the following section, the efficiency of this choice will be verified.

5. Numerical experiments

To test the efficiency of the APSS preconditioner, we conduct some numerical tests. We provide two examples, and in each example, symmetric diagonal scaling for the matrix $\mathcal{A}$ is used initially. In order to do this, the coefficient matrix A is replaced by the matrix $\mathcal{D}^{-\frac{1}{2}}\mathcal{A}\mathcal{D}^{-\frac{1}{2}}$, in which $\mathcal{D} = \text{diag}(\|\mathcal{A}_1\|_2, \ldots, \|\mathcal{A}_\mathbf{n}\|_2)$. In addition, the jth column of the matrix $\mathcal{A}$ is represented by $\mathcal{A}_j$.

For all examples, a zero vector was used as the initial guess, and the iteration was terminated once

$$Res = \frac{\left\|\mathbf{b} - \mathcal{A}\mathbf{x}^{(k)}\right\|_2}{\|\mathbf{b}\|_2} \leq 10^{-7},$$

or the specified number of iteration steps, $maxit = 2000$, was exceeded. Note that $\mathbf{x}^{(k)}$ stands for the computed solution at the kth iteration. The right-hand side vector $\mathbf{b}$ was chosen such that the exact solution of (2) was a vector of all ones.

In what follows, we apply complete version of the FGMRES method with the right preconditioning technique in conjunction with the APSS preconditioner $\mathcal{M}_\alpha$. For the APSS preconditioner, we need to solve two SPD linear systems including $\alpha I + A + \frac{1}{\alpha}B^TB$ and $\alpha^2 I + CC^T$ as subtasks. To solve these linear systems, we employ the Conjugate Gradient (CG) method without preconditioning and the iteration is stopped when the residual 2-norm is reduced by a factor of 10^3 or the maximum number of inner iiterations reaches 200.

In the tables below, we use "CPU", "IT" and "RES" to represent the elapsed CPU time to converge in second, iteration counts and relative residual, respectively. In addition, degree of freedom (DOF) is defined as $DOF = n + m + l = \mathbf{n}$. Finally, a dagger (†), means that more than the maximum number of iterations is needed to converge. All examples are performed in MATLAB-R2019A. All numerical results are obtained by means of a laptop with the following features:

intel (R), Core(TM) i5-8265U, CPU @ 1.60 GHz, 8 GB.

Example 1. Consider the two-dimensional leaky lid-driven cavity problem:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = 0, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \end{cases} \tag{35}$$

in which suitable boundary conditions are applied on the side and bottom points. $\mathbf{u}$ and p symbolize the velocity vector field and the pressure scalar field, respectively. In addition, Δ and ∇ refer to the vector Laplacian in $\mathbb{R}^2$ and the gradient, respectively. This problem is called the Stokes problem. For the saddle point problem (2), the matrices A and B come from the Stokes problem. To obtain these matrices, we use the IFISS package by Elman et al. [15]. The stablized $Q_1 - P_0$ and $Q_2 - P_1$ finite element method (FEM) is employed to discretize the Stokes equation (35). The grid parameters are chosen as $h = \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{256}$, for all uniform or stretched grid points. Now, the matrix C is taken to be of the form:

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^{(l+2)\times m},$$

where $C_1 = \big(\text{diag}\{1, 3, 5, \ldots, 2l-1\}\ \text{randn}(l, m-l)\big) \in \mathbb{R}^{l\times m}$, and $c_1 = \big(e^T,\, 0^T\big)\, C_1$, $c_2 = \big(0^T,\, e^T\big)\, C_1$, $e^T = \big(1,\, 1,\, 1, \ldots,\, 1\big) \in \mathbb{R}^{\frac{l}{2}}$. Note that $l = m - 2$. Here, $\text{randn}(l, m-l)$ is a normally distributed random matrix of order $l \times m - l$.

This example is a technical variant of Example 1 in [29].

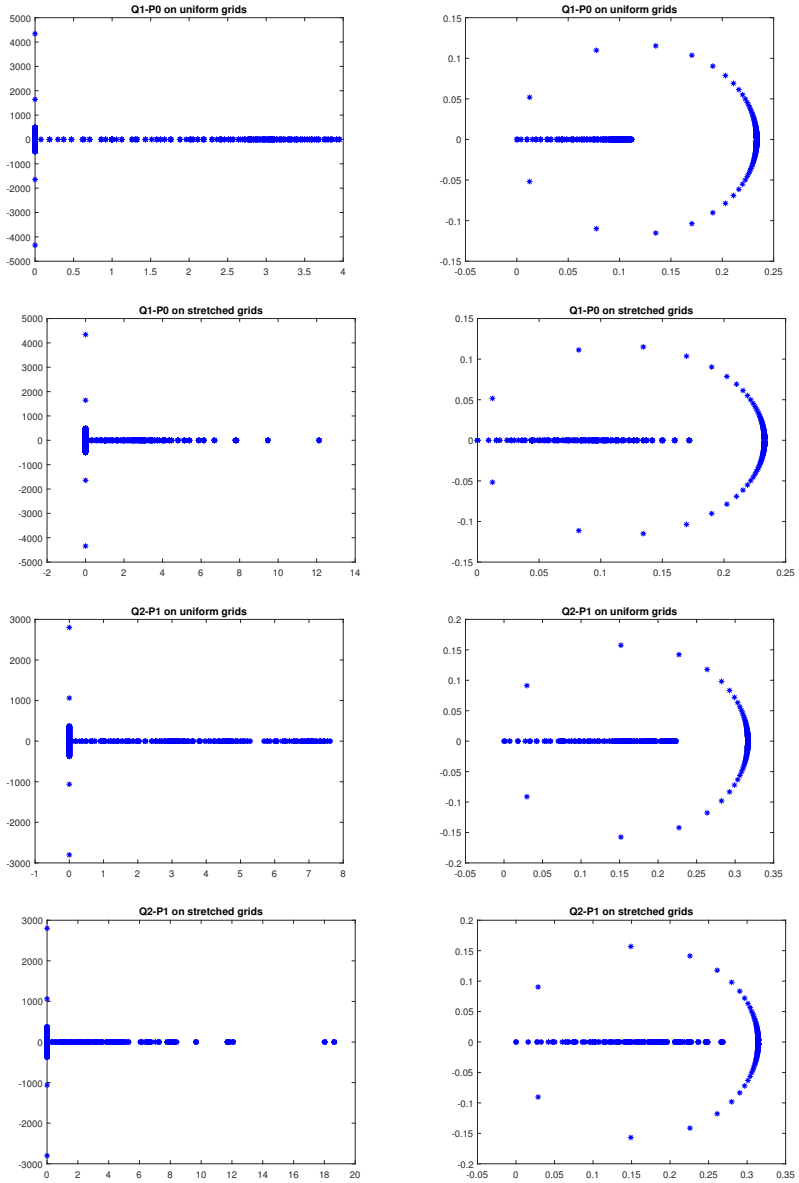

Figure 1: *Eigenvalue distributions of* $\mathcal{A}$ *and* $\mathcal{M}_\alpha^{-1}\mathcal{A}$ *(from the left to the right) for* $h = \frac{1}{16}$ *for Example 1.*

The results are given in tables 1-4. From these tables, we can observe that the FGMRES method in conjunction with the APSS preconditioner $\mathcal{M}_\alpha$ is strongly more efficient than the FGMRES without a preconditioner. As we can see from Figure 1, the preconditioner $\mathcal{M}_\alpha$ is efficient to cluster the eigenvalues of the original coefficient matrix. Another observation which can be posed here is that the number of iterations of the FGMRES method in conjuction with the APSS preconditioner remains almost constant as the problem size increases, whereas this is not the case for FGMRES without preconditioning.

Prec.	$h(DOF)$	$\frac{1}{8}$ (286)	$\frac{1}{16}$(1086)	$\frac{1}{32}$(4222)	$\frac{1}{64}$(16638)	$\frac{1}{128}$(66046)	$\frac{1}{256}$(263166)
$\mathcal{I}$	IT	49	103	552	188	474	1492
	CPU	0.0416	0.0600	0.4866	0.5810	4.1456	135.3632
	RES	2.2e-08	9.9e-08	1.0e-07	9.3e-08	9.8e-08	1.1e-07
$\mathcal{M}_\alpha$	α_{est}	0.0396	0.0201	0.0101	0.0051	0.0025	0.0013
	IT	11	13	11	9	10	11
	CPU	0.0795	0.1009	0.0915	0.1807	0.6871	6.6438
	RES	9.9e-08	4.6e-08	8.9e-08	3.1e-08	2.3e-08	6.6e-08

Table 1: *Numerical results for FGMRES to solve Example 1 with $Q_1 - P_0$ FEM on uniform grids.*

Prec.	$h(DOF)$	$\frac{1}{8}$ (286)	$\frac{1}{16}$(1086)	$\frac{1}{32}$(4222)	$\frac{1}{64}$(16638)	$\frac{1}{128}$(66046)	$\frac{1}{256}$(263166)
$\mathcal{I}$	IT	48	133	93	187	406	974
	CPU	0.0391	0.0625	0.1089	0.5716	3.5570	118.1586
	RES	3.2e-08	7.5e-08	9.9e-08	9.9e-08	9.9e-08	1.0e-07
$\mathcal{M}_\alpha$	α_{est}	0.3962	0.2001	0.1011	0.0051	0.0025	0.0013
	IT	11	13	7	7	7	9
	CPU	0.0739	0.0846	0.0807	0.1851	0.5726	8.6066
	RES	6.4e-08	3.5e-08	3.1e-08	6.5e-08	9.0e-08	3.1e-08

Table 2: *Numerical results for FGMRES to solve Example 1 with $Q_1 - P_0$ FEM on stretched grids.*

Prec.	$h(DOF)$	$\frac{1}{8}$(254)	$\frac{1}{16}$(958)	$\frac{1}{32}$(3710)	$\frac{1}{64}$(14590)	$\frac{1}{128}$(57854)	$\frac{1}{256}$(230398)
$\mathcal{I}$	IT	51	130	186	253	947	†
	CPU	0.0391	0.0653	0.1494	0.6904	7.8228	
	RES	8.8e-08	9.0e-08	9.8e-08	9.9e-08	1.0e-07	
$\mathcal{M}_\alpha$	α_{est}	0.0419	0.0214	0.0108	0.0054	0.0027	0.0013
	IT	11	14	13	10	12	12
	CPU	0.0737	0.0771	0.0978	0.2462	1.0375	9.0643
	RES	3.5e-08	2.6e-08	7.7e-08	7.3e-08	5.8e-08	9.0e-08

Table 3: *Numerical results for FGMRES to solve Example 1 with $Q_2 - P_1$ FEM on uniform grids.*

Prec.	$h(DOF)$	$\frac{1}{8}(254)$	$\frac{1}{16}(958)$	$\frac{1}{32}(3710)$	$\frac{1}{64}(14590)$	$\frac{1}{128}(57854)$	$\frac{1}{256}(230398)$
$\mathcal{I}$	IT	55	131	121	198	512	1137
	CPU	0.0380	0.0782	0.1123	0.5466	4.1606	129.2754
	RES	6.5-08	1.0e-07	1.0e-07	9.5e-08	9.9e-08	1.0e-07
$\mathcal{M}_\alpha$	α_{est}	0.4195	0.0214	0.0108	0.0054	0.027	0.014
	IT	11	13	8	9	9	10
	CPU	0.0732	0.0722	0.0833	0.2363	0.9544	10.7704
	RES	9.6e-08	9.5e-08	9.8e-08	2.1e-08	4.2e-08	8.0e-08

Table 4: *Numerical results for FGMRES to solve Example 1 with $Q_2 - P_1$ FEM on stretched grids.*

Example 2. Consider a technical modification of Example 1 in [20, 30] as follows:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},$$

$$B = (I \otimes F \quad F \otimes I) \in \mathbb{R}^{p^2 \times 2p^2},$$

and

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^{(p^2+2) \times p^2},$$

where

$$C_1 = E \otimes F \in \mathbb{R}^{p^2 \times p^2},$$

and

$$c_1 = \left(e^T, 0^T \right) C_1, \qquad c_2 = \left(0^T, e^T \right) C_1, \qquad e^T = \left(1, 1, 1, \ldots, 1 \right) \in \mathbb{R}^{\frac{p^2}{2}},$$

where

$$T = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h} \cdot \text{tridiag}(0, 1, -1) \in \mathbb{R}^{p \times p},$$

and $E = \text{diag}\left(1, p+1, 2p+1, \ldots, p^2 - p + 1\right)$ in which the Kronecker product is denoted by $\otimes$, while the discretization mesh size is represented by $h = 1/(p+1)$.

Table 5 reports the result of FGMRES method and the FGMRES in conjunction with the proposed preconditioner $\mathcal{M}_\alpha$ with respect to IT, CPU and RES. As seen, the suggested preconditioner requires significantly less iteration numbers and CPU time than FGMRES without a preconditioner. We also see that the number of iterations of FGMRES without preconditioning increases drastically by increasing the size of the problem, whereas for FGMRES with the preconditioner $\mathcal{M}_\alpha$ increases moderately. Figure 2 displays the eigenvalues of the original cofficient matrix and the preconditioned matrix. It can be seen that the proposed preconditioner works well in eigenvalue clustering.

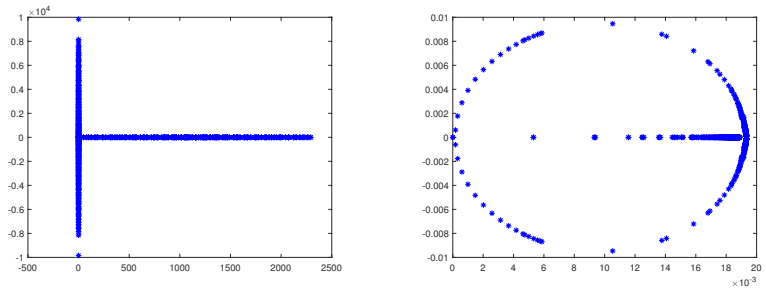

Figure 2: *Eigenvalue distributions of $\mathcal{A}$ and $\mathcal{M}_\alpha^{-1}\mathcal{A}$ (from the left to the right) for $h = \frac{1}{16}$ for Example 2.*

Prec.	$p(DOF)$	8 (258)	16 (1026)	32 (4098)	64 (16386)	128 (65538)
$\mathcal{I}$	IT	659	1999	†	†	†
	CPU	0.1371	0.5270			
	RES	9.9e-08	1.0e-07			
$\mathcal{M}_\alpha$	α_{est}	0.0434	0.0219	0.0110	0.0055	0.0027
	IT	13	14	15	17	27
	CPU	0.0700	0.0857	0.1448	0.5156	2.5788
	RES	2.4e-08	3.6e-08	4.5e-08	8.3e-08	7.3e-08

Table 5: *Numerical results for FGMRES to solve Example 2.*

6. Conclusion

In this paper, the APSS method was employed to solve a class of nonsymmetric three-by-three singular saddle point problems. We have applied the induced preconditioner, $\mathcal{M}_\alpha$, to improve the semi-convergence rate, when it is conjugated with FGMRES. We have proved that if C is rank-deficient, the APSS method is unconditionally semi-convergent. Numerical tests prove our theoretical claims.

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