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A priori estimates for finite-energy sequences of one-dimensional Cahn-Hilliard functional with non-standard multi-well potential

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Abstract. In this paper, we provide some results pertaining to asymptotic behaviour as $\varepsilon \longrightarrow 0$ of the finite-energy sequences of the one-dimensional Cahn-Hilliard functional

$$I_0^\varepsilon(u) = \int_0^1 \Big(\varepsilon^2 u'^2(s) + W(u(s))\Big) ds,$$

where $u \in \mathrm{H}^1(0,1)$, and where W is a multi-well potential endowed with a non-standard integrability condition. We introduce a new class of finite-energy sequences, recover its underlying geometric properties as $\varepsilon \longrightarrow 0$, and obtain the related a priori estimates.
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1. Introduction

We study asymptotic behaviour of the functional $I_0^\varepsilon : \mathrm{H}^1(0,1) \longrightarrow \mathbf{R}$ defined by

$$I_0^\varepsilon(u) := \int_0^1 \Big(\varepsilon^2 u'^2(s) + W(u(s))\Big) ds, \tag{1}$$

as a small parameter ε tends to zero, where W is a non-negative continuous function with suitable behaviour at infinity, which satisfies $W(\zeta) = 0$ if and only if $\zeta \in \{\alpha_1, \alpha_2, \dots, \alpha_l\}$, for some $\alpha_i \in \mathbf{R}$, $i = 1, \dots, l$, with $l \geq 2$ and $\alpha_i \neq \alpha_j$ for $i \neq j$ (in short, W is the multi-well potential). Functional (1) (cf. [1, 18, 20, 21]) is known as the Cahn-Hilliard functional (or the Modica-Mortola functional). To simplify the notation, we often omit to relabel subsequences, and by "a sequence (x_ε)" we mean a sequence defined only for countably many $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$. We say that (x_ε) is pre-compact in a metric space X if every subsequence of (x_ε) admits a further subsequence which converges in X. We recall that we say that a sequence (z_ε) in a metric space Z is a finite-energy sequence (or an FE sequence for short) for a sequence of functionals $F^\varepsilon : Z \longrightarrow [0, +\infty]$ if it holds that

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$\limsup_{\varepsilon\longrightarrow 0} F^\varepsilon(z_\varepsilon) < +\infty$. In Section 2 (Section 3, Section 4 and Section 5, resp.) of this paper, by (u_ε) we always denote a sequence of absolutely continuous functions $u_\varepsilon : \overline{J} \longrightarrow \mathbf{R}$, where $J \subseteq \mathbf{R}$ is a non-empty bounded open interval (an arbitrary FE sequence for $(\varepsilon^{-1}I_0^\varepsilon)$ in $\mathrm{H}^1(0,1)$, resp.), and use abbreviations $m_\varepsilon := \min\{|u_\varepsilon(s)| : s \in \overline{J}\}$, $M_\varepsilon := \max\{|u_\varepsilon(s)| : s \in \overline{J}\}$. Accordingly, throughout sections 2-5, the expression "an FE sequence" is reserved for an FE sequence (u_ε) for $(\varepsilon^{-1}I_0^\varepsilon)$, the letter R_0 stands for a real number which satisfies $R_0 > \max\{|\alpha_i| : i = 1,\ldots,l\} > 0$, while the letter M stands for a finite upper bound for the sequence $(\varepsilon^{-1}I_0^\varepsilon(u_\varepsilon))$, where $\varepsilon > 0$ is sufficiently small. As a consequence of the definitions above, FE sequences for the rescaled functional (1) do not develop internally created small scale oscillations, which makes them easier to handle (compare [25]-[38] for the study of similar functionals with an internally created small oscillatory scale). Such singular perturbation problems are studied within the framework of the gradient theory of phase transitions. In the case of functional (1), where $l = 2$, the term $\int_0^1 \varepsilon^2|u'|^2$ penalizes rapid changes of the density u in (1) and plays the role of an interfacial energy. A small positive quantity ε is the thickness of the transition layer separating two different phases or states of u within the domain $(0,1)$. Different phases develop as the result of the minimization process subject to a given mass constraint. As we pass to the limit as $\varepsilon \longrightarrow 0$, optimal configurations described by geometric properties of the minimizing sequence (u_ε) of (1) resemble more and more the optimal configuration of the system subject to classical assumptions in the theory of phase transitions, where it is assumed that the contact area between different phases of u is concentrated on the interfacial surface of thickness zero. Similar types of functionals appear in studying coherent solid-solid phase transformations and can be understood as a simplified one-dimensional model for a phase transition at a martensite-austenite interface (cf. [2, 22] and references therein). Extensive literature is available on a wider subject, and our list of references is by no means complete, nor does it attempt to cite the most important contributions (a more complete list is available in, for instance, [2, 6, 23]). Although many authors studied asymptotic behaviour of the functionals similar to (1), the analysis is usually done under rather strong growth conditions on W (cf. [2, 4, 8, 17, 24, 39]). In particular, the commonly used classical Fonseca-Tartar assumption (cf. [14]) requires that W grows at least linearly at infinity (in such a case we say that W is coercive). For such a choice of W, we immediately deduce equi-integrability of (u_ε) in $\mathrm{L}^1(0,1)$ of an arbitrary FE sequence (u_ε), which (by the Vitali convergence theorem) gives strong pre-compactness of (u_ε) in $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. Such compactness result is the prerequisite for the proof of Γ-convergence of $(\varepsilon^{-1}I_0^\varepsilon)$ on $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. One possibility of relaxing the assumptions on W is to consider the case of W which satisfies a suitable non-integrability condition

$$\int_0^{+\infty} \sqrt{V(\xi)}d\xi = +\infty, \tag{2}$$

where $V : [0,+\infty) \longrightarrow [0,+\infty)$ is defined by $V(\xi) := \min\{W(\zeta) : |\zeta| = \xi\}$. In a relatively recent paper [18], Leoni obtained, under the non-integrability assumption (2), strong pre-compactness of FE sequences (u_ε) for $(\varepsilon^{-1}I_0^\varepsilon)$ (equipped with the mass constraint $\int_0^1 u_\varepsilon = m$, or, equivalently, with the normality property introduced in Definition 1, (i)) in $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$, by showing that such FE sequences are bounded in $\mathrm{L}^\infty(0,1)$ as $\varepsilon \longrightarrow 0$ (cf. Theorem 1.3 in [18]). As a typical example of W which satisfies (2) ($\int_{\mathbf{R}} \sqrt{W(\zeta)}d\zeta < +\infty$, resp.), we consider W such that for $0 \leq q \leq 2$ ($q > 2$, resp.) and $R_0 > \max\{|\alpha_i| : i = 1, \ldots, l\}$, it holds that $\frac{c_0}{|\zeta|^q} \leq W(\zeta) \leq \frac{C_0}{|\zeta|^q}$ for every $|\zeta| \geq R_0$, where $0 < c_0 \leq C_0 < +\infty$. In this paper, we adopt a similar strategy, with particular emphasis placed on the optimality of the assumptions on W. We always assume that W satisfies $W(\zeta) = 0$ iff $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, $W \geq 0$. If no additional properties of W are assumed, we refer to such W as an arbitrary multi-well potential. Furthermore, if $0 < \rho < 1$, we introduce the non-integrability condition $\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty$, which is essential to our setting. In particular, our assumptions on W allow for consideration of the case $\liminf_{\xi \longrightarrow +\infty} V(\xi) = 0$ and/or $\limsup_{\xi \longrightarrow +\infty} V(\xi) = +\infty$. These assumptions constitute a non-standard behaviour of W as infinity, and are not covered well in the literature. We refer to the case of $0 < \rho < \frac{1}{2}$ ($\rho > \frac{1}{2}$, resp.) as the subcritical case (the supercritical case, resp.). Since the critical case $\rho = \frac{1}{2}$ is already covered in [18], our results should be primarily viewed as an expansion of the considerations in [18] (cf. Remark 3). While the author of [18] is focussed on compactness results, in this paper we mostly confine ourselves to proving some a priori estimates for a class of FE sequences as $\varepsilon \longrightarrow 0$. In doing so, we follow, as far as possible, the line of reasoning in Theorem 1.3 in [18]. However, we have not been able to use the aforementioned a priori estimates to recover compactness results for general FE sequences (compare Theorem 4). Instead, we offer a weaker conclusion, expressed in terms of control of oscillations of FE sequences at infinity as $\varepsilon \longrightarrow 0$. The relevance of such a conclusion can be inferred by comparison with the classical analysis of weakly (but not strongly) convergent sequences in L^p spaces. Indeed, by the Kakutani theorem (the Banach-Alaoglu theorem, resp.), sequences bounded in L^p, where $1 < p < +\infty$ (L^∞, resp.), are weakly (weakly*, resp.) sequentially pre-compact (cf. [7]). Generally speaking, there are two effects which prevent weakly convergent sequences to be strongly convergent: oscillation and concentration. Young measures and their variants provide the apparatus for studying oscillations in a weakly convergent sequence (cf. [5] or [23]), while H-measures and their variants are a tool which can capture concentrations in such a sequence (cf. [3, 40, 41]). The fundamental theorem of Young measures and the fundamental theorem of H-measures both require boundedness of the sequence in L^p norm (or in some similar, suitably chosen norm). In the case when a priori bounds in some norm are not available, it is necessary to look for a new approach to describe asymptotic behaviour of sequences as $\varepsilon \longrightarrow 0$ (cf. Remark 1). In this paper, we propose such an approach, and show how it can be applied to analyse certain underlying asymptotic properties of FE sequences of the one-dimensional Cahn-Hilliard functional $\varepsilon^{-1}I_0^\varepsilon$, where W is a multi-well potential endowed with the aforementioned non-integrability condition. The main technical tool in the proofs of our results is the area formula (cf. [19], Theorem 3.65, p. 100) and the reverse Hölder inequality. In geometric terms, the easiest case for comparison between the classical approach and our approach is the case $p = +\infty$. Let us consider a sequence (u_ε) in $\mathrm{L}^\infty(0,1)$. If the sequence (u_ε) is bounded in $\mathrm{L}^\infty(0,1)$, we can apply Young measures or H-measures to study its asymptotic properties as $\varepsilon \longrightarrow 0$. In the case when such a sequence is unbounded in $\mathrm{L}^\infty(0,1)$, given sufficiently large $R > 0$, we can consider $\limsup_{\varepsilon \longrightarrow 0} \|\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^\infty(R,+\infty)}$ as a quantity which measures oscillation and concentration of the sequence (u_ε) at infinity as $\varepsilon \longrightarrow 0$. Such a quantity illustrates how often $|u_\varepsilon|$ takes values greater than R, and therefore provides some information about the frequency of oscillation of $|u_\varepsilon|$ above the level R. As we pass to the limit superior as R tends to $+\infty$, we capture the behaviour of the sequence at infinity. More precisely, in Definition 2, we extend this reasoning from the case $q = +\infty$ to the case of finite q, including $q < 0$ and $0 < q < 1$. Although some a priori estimates in this paper are obtained under the assumption $0 < \rho < 1$, we deal mostly with the subcritical case. In particular, we focus on FE sequences which are not a priori bounded in $\mathrm{L}^\infty(0,1)$ as $\varepsilon \longrightarrow 0$. We mention that in this paper, the expression "regularity" that pertains to the behaviour of an FE sequence (u_ε) is reserved for a specific boundedness property in terms of L^∞-norm as $\varepsilon \longrightarrow 0$ (cf. Definition 1), and not to the smoothness property of (u_ε), which is more commonly the case in the literature (however, cf. Theorem 2 and Corollary 1, which provide the connection between L^∞-bounds and the smoothness property). To this end, in Section 2, we introduce the notation and terminology, including a new concept designed to capture asymptotic behaviour of a sequence of functions at infinity (cf. Definition 2). In Section 3 (and Section 4), we present a priori estimates for (u'_ε) (and (u_ε), respectively), as $\varepsilon \longrightarrow 0$ (cf. Lemma 1, Theorem 2, Corollary 1 and Corollary 2) (cf. Corollary 3, Theorem 3 and Theorem 4, resp.), where (u_ε) is a suitably chosen class of FE sequences for $(\varepsilon^{-1} I_0^\varepsilon)$. Lemma 1, Theorem 2 and Theorem 3 deal with the asymptotic properties of FE sequences which satisfy the condition of Definition 1(i), and which are not bounded in $\mathrm{L}^\infty(0,1)$ as $\varepsilon \longrightarrow 0$. Theorem 4 and Corollary 4 show that, subject to the appropriate integrability assumption on W and the conditions of Definition 1(i), and Definition 2, FE sequences are actually bounded in $\mathrm{L}^\infty(0,1)$ and strongly pre-compact in $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. Theorem 4 is the core regularity result of the paper, and its corollaries can be viewed as an L^q-version of Theorem 1.3 in [18] (cf. Remark 3). We were not able to find these results in widely available sources. In the appendix, we included some technical results used throughout the paper.

2. Notation and terminology

Following [2], we consider a compact metric space (K, d) (the space of patterns), which is the set of all measurable mappings $x : \mathbf{R} \longrightarrow [-\infty, +\infty]$ (modulo equivalence λ-almost everywhere, where λ is the one-dimensional Lebesgue measure), endowed with the metric d defined by

$$d(x_1, x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} \Big| \int_{\mathbf{R}} y_k \Big(\frac{2}{\pi} \arctan x_1 - \frac{2}{\pi} \arctan x_2 \Big) d\lambda \Big|,$$

where (y_k) is a sequence of bounded functions which are dense in $\mathrm{L}^1(\mathbf{R})$, such that the support of y_k is a subset of $(-k, k)$, with $\alpha_k := \|y_k\|_{\mathrm{L}^1} + \|y_k\|_{\mathrm{L}^\infty}$. As shown in [2], p. 806, $\mathrm{L}^p_{loc}(\mathbf{R})$ continuously embeds in K for every $p \in [1, +\infty]$. The notation $\mathrm{C}(K)$ ($\mathrm{C}_0(\mathbf{R})$, resp.) stands for the space of all continuous real functions on K (the space of all continuous real functions on $\mathbf{R}$ which vanish at infinity, resp.), whose dual is identified with the space of all real Radon measures on K (all real bounded

Radon measures on $\mathbf{R}$, resp.), denoted by $\mathcal{M}(K)$ ($\mathcal{M}_b(\mathbf{R})$, resp.), endowed with the corresponding weak-star topology. By $\mathcal{P}(K)$ ($\mathcal{P}(\mathbf{R})$, resp.) we denote the set of all probability measures in $\mathcal{M}(K)$ ($\mathcal{M}_b(\mathbf{R})$, resp.). By $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.) we denote the dual of $\mathrm{L}^1(\Omega;\mathrm{C}(K))$ ($\mathrm{L}^1(\Omega;\mathrm{C}_0(\mathbf{R}))$, resp.), where $\Omega \subseteq \mathbf{R}$ is a measurable set such that $0 < \lambda(\Omega) < +\infty$. The set of all K-valued ($\mathbf{R}$-valued, resp.) Young measures on Ω denoted by $YM(\Omega;K)$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{P}(\mathbf{R}))$, resp.) is the set of all $\boldsymbol{\nu} \in \mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\boldsymbol{\nu} \in \mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.) such that $\nu_s \in \mathcal{P}(K)$ ($\nu_s \in \mathcal{P}(\mathbf{R})$, resp.) for almost every $s \in \Omega$, where $\boldsymbol{\nu}(s) := \nu_s$, $s \in \Omega$, and it is always endowed with the weak-star topology of $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.). The elementary Young measure associated to a measurable map $u : \Omega \longrightarrow K$ ($u : \Omega \longrightarrow \mathbf{R}$, resp.) is the mapping $\underline{\delta}_u : \Omega \longrightarrow \mathcal{M}(K)$ ($\underline{\delta}_u : \Omega \longrightarrow \mathcal{M}_b(\mathbf{R})$, resp.) given by $\underline{\delta}_u(s) := \delta_{u(s)}$, $s \in \Omega$. Besides the fundamental theorem of Young measures which involves $\mathbf{R}$-valued Young measures (cf. [5] or [16]), we use the version of the theorem which involves K-valued Young measures (cf. [2]). The main advantage of the introduction of the notion of K-valued Young measures comes from the fact that compactness of $YM((0,1);K)$ is guaranteed by compactness of K (such compactness fails in the case of $\mathbf{R}$-valued Young measures). By $u \llcorner \omega$ we denote the restriction of the function $u : \Omega \longrightarrow \mathbf{R}$ on the set $\omega \subset \Omega$, by $\mathrm{Im} u$ we denote the image of the function u, while by $u^{\leftarrow}(\xi)$ we denote the pre-image of ξ with respect to u, i.e., $u^{\leftarrow}(\xi) := \{s \in \Omega : u(s) = \xi\}$. For simplicity of the notation, if $q > 0$ is arbitrary, we define $\|u\|_{\mathrm{L}^q(\Omega)} := \left(\int_\Omega |u(s)|^q ds\right)^{\frac{1}{q}}$, where $u : \Omega \longrightarrow \mathbf{R}$ is a measurable function. We recall that in the case $q = +\infty$, the uniform norm of such a function u is defined by $\|u\|_{\mathrm{L}^\infty(\Omega)} := \inf\{\alpha \geq 0 : \lambda\{s \in \Omega : u(s) > \alpha\} = 0\}$, and that $\|\cdot\|_{\mathrm{L}^q(\Omega)}$ is a norm iff $q \in [1,+\infty]$. By $\mathrm{card} S$ we denote the cardinality of a set S. Throughout the paper, we assume that every Sobolev function $u \in \mathrm{W}^{1,p}(0,1)$, where $1 \leq p \leq +\infty$, is already replaced by its absolutely continuous representative (cf. [9], Theorem 1, p. 163). If a measurable function $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ belongs to $\mathrm{L}^1_{loc}(\rho,+\infty)$, where $\rho \geq 0$, we define $\liminf_{\xi \longrightarrow +\infty} \psi(\xi) := \liminf_{\xi \longrightarrow +\infty} \psi_\star(\xi)$ ($\limsup_{\xi \longrightarrow +\infty} \psi(\xi) := \limsup_{\xi \longrightarrow +\infty} \psi_\star(\xi)$, resp.), where $\psi_\star$ denotes the precise representative of ψ (cf. [9], p. 46), which is well-defined for every $\xi \in (\rho,+\infty)$. By $\mathrm{C}^{0,\theta}[0,1]$, where $0 < \theta < 1$, we denote the set of all θ-Hölder continuous functions on $[0,1]$ (cf. [19], p. 335). If the number $0 < \sigma < 1$ is given, by σ' we denote its dual number according to the equality $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, whereby $-\infty < \sigma' < 0$. We complete this section by introducing some definitions related to the asymptotic properties of a sequence of functions (u_ε) as $\varepsilon \longrightarrow 0$.

Definition 1. *Consider a sequence (u_ε) in $\mathrm{C}(\overline{J})$, where $J \subseteq \mathbf{R}$ is a non-empty bounded open interval. We say that the sequence (u_ε) is:*

(i) *normal on J if there exists a sequence (c_ε) in $\overline{J}$ such that $(u_\varepsilon(c_\varepsilon))$ is bounded,*

(ii) *regular on J if it is a bounded sequence in $\mathrm{L}^\infty(J)$; otherwise, we say that (u_ε) is non-regular on J, which means that there exists a subsequence of (u_ε) (not relabeled) such that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon = +\infty$.*

In the language of Definition 1, the regularity means that the sequence (u_ε) does not exhibit any leakage to infinity as $\varepsilon \longrightarrow 0$. In particular, the regularity implies that (by continuity of u_ε) there exists a large enough $R > 0$ such that for a sufficiently small $\varepsilon_0 > 0$ and for every $0 < \varepsilon \le \varepsilon_0$ we get $\|\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^\infty(R,+\infty)} = 0$. On the other hand, in the following definition, we introduce a more general tail condition in terms of the pre-images of the sequence (u_ε), which allows controlled oscillation and concentration of a sequence (u_ε) to occur at infinity as $\varepsilon \longrightarrow 0$, and which is designed to study asymptotic behaviour of FE sequences (u_ε) as $\varepsilon \longrightarrow 0$.

Definition 2. *Consider $q \in (0,+\infty]$ ($q \in (-\infty, 0)$, resp.) and a sequence of absolutely continuous functions $u_\varepsilon : \overline{J} \longrightarrow \mathbf{R}$, where $J \subseteq \mathbf{R}$ is a non-empty bounded open interval. We define $D_\varepsilon(R) := (\min\{\max\{m_\varepsilon, R\}, M_\varepsilon\}, M_\varepsilon]$ and we set*

$$U_q(\overline{J}) := \limsup_{R \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \|\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^q(D_\varepsilon(R))} \tag{3}$$

$$\Big(U_q(\overline{J}) := \limsup_{R \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \int_{D_\varepsilon(R)} (\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^q d\xi, \ \textit{resp.}\Big). \tag{4}$$

We say that the sequence (u_ε) is q-upper pre-oscillatory (or q-UPO) on J if it holds that $U_q(\overline{J}) < +\infty$. In the case of $J = (0,1)$, we say that (u_ε) is a q-upper pre-oscillatory (or q-UPO) sequence, and we write U_q instead of $U_q([0,1])$.

We note that in (3) (and (4), resp.), $\mathrm{L}^q(D_\varepsilon(R))$ can be equivalently written as $\mathrm{L}^q(R,+\infty)$ ($U_q(\overline{J})$ is well-defined since we have $D_\varepsilon(R) \subseteq [m_\varepsilon, M_\varepsilon]$, resp.). If (u_ε) is not a normal sequence, then $D_\varepsilon(R)$ can be replaced by $(m_\varepsilon, M_\varepsilon]$ in (3) (and (4), resp.) without changing the value $U_q(\overline{J})$. Furthermore, if $q_1 \neq 0$, $q_2 \neq 0$ and $-\infty < q_2 < q_1 < +\infty$, then every q_1-UPO sequence on J is also a q_2-UPO sequence on J. In particular, if $-\infty < q_2 < q_1 < 0$, then it follows that $U_{q_2}(\overline{J}) \le U_{q_1}(\overline{J})$. By (3) (and (4), resp.) it follows that every subsequence of a q-UPO sequence on J is itself a q-UPO sequence on J. It is possible to avoid the introduction of the domain $D_\varepsilon(R)$ in (4). If, for $-\infty < q < 0$, we define $\psi_q : [0,+\infty) \longrightarrow [0,+\infty)$ by $\psi_q(t) := t^q$, if $t > 0$, and $\psi_q(0) := 0$, then $U_q(\overline{J})$ in (4) can be equivalently written as $U_q(\overline{J}) = \limsup_{R \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \int_R^{+\infty} \psi_q(\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}) d\xi$. We purposely chose to state Definition 2 only for absolutely continuous functions u_ε. The reasons for doing so are as follows. For one thing, we apply the definition only to the case of FE sequences (u_ε), which allow absolutely continuous representatives themselves, and for the other, the assumption of absolute continuity ensures that, for a sufficiently small $\varepsilon_0 > 0$, and for every $0 < \varepsilon \le \varepsilon_0$, $\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}$ is finite for all ξ, except possibly for points ξ which belong to a set of measure zero (cf. [19], Corollary 2.51, p. 71, and Corollary 3.9, p. 76). We recall that for every absolutely continuous function $u : [0,1] \longrightarrow \mathbf{R}$ it holds that $||u|'(s)| = |u'(s)|$ (a.e. $s \in (0,1)$) (cf. [42], Theorem 2.1.11, p. 48), so that, by the Banach Indicatrix Theorem (cf. [19], Theorem 2.47, p. 68), we have

$$\int_0^{+\infty} \mathrm{card}\{|u|^{\leftarrow}(\xi)\} d\xi = \int_0^1 |u'(s)| ds. \tag{5}$$

In particular, it follows that the mapping $\xi \mapsto \mathrm{card}\{|u|^{\leftarrow}(\xi)\}$ (known as the Banach indicatrix of $|u|$) is measurable and Lebesgue integrable, provided that u is absolutely continuous. Definition 2 can be stated in a more general setting, which is a discussion which we do not pursue here (for instance, under more general assumptions on

u, the measurability of the mapping $\xi \mapsto \operatorname{card}\{|u|^{\leftarrow}(\xi)\}$ was established in [10], while validity of the Banach Indicatrix Theorem was proved in [11]). In particular, $U_\infty < +\infty$ implies that we have $\limsup_{\xi \longrightarrow +\infty} \limsup_{\varepsilon \longrightarrow 0} \operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\} < +\infty$ (cf. Theorem 3). We also note that every sequence of constant functions satisfies $U_q = 0$ for every $q \in (-\infty, +\infty]\backslash\{0\}$. As the next observation in this section, in the case of general sequences of absolutely continuous functions u_ε, we provide a comparison between regularity in Definition 1, (ii), and some properties similar to (3) (and (4), resp.). In the proof of the next proposition, the continuity of (u_ε) on $[0,1]$ is used as an essential feature.

Proposition 1. *Consider a sequence (u_ε) of absolutely continuous functions on $[0,1]$. Then we have the following conclusions:*

(i) if (u_ε) is a normal and a non-regular sequence, then for every $0 < q < +\infty$ it holds that $U_q = +\infty$, i.e., (u_ε) is not a q-UPO sequence,

(ii) if (u_ε) is an ∞-UPO sequence which does not allow a subsequence of constant functions, and such that $U_\infty = 0$, then (u_ε) is a regular sequence,

(iii) if $1 \leq q < +\infty$, if (u_ε) is a normal sequence, and if it holds that

$$\limsup_{\varepsilon \longrightarrow 0} \|\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^q(0,+\infty)} < +\infty, \tag{6}$$

then (u_ε) is a regular sequence,

(iv) if $1 \leq q \leq +\infty$, if (u_ε) is a regular sequence which satisfies (6), then (u_ε) is bounded in $\mathrm{W}^{1,1}(0,1)$,

(v) if (u_ε) is a regular sequence, then, for every $q \in (-\infty, +\infty]\backslash\{0\}$, it holds that $U_q = 0$.

Proof. Regarding the proof of assertion (i), we argue as follows. By the intermediate value property of continuous function u_ε, we get $\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\} \geq 1$ iff $\xi \in [m_\varepsilon, M_\varepsilon]$. We consider a subsequence (not relabeled) such that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon = +\infty$. We choose $R > 0$ and $\varepsilon_0 > 0$, which satisfy that for every $0 < \varepsilon \leq \varepsilon_0$ we have $M_\varepsilon > R > m_\varepsilon$. In effect, it follows that $\int_R^{M_\varepsilon} \operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}^q d\xi \geq M_\varepsilon - R$. As we pass to the limit as $\varepsilon \longrightarrow 0$, we get assertion (i).

To prove assertion (ii), we assume the opposite. Then there exists a subsequence (not relabeled) such that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon = +\infty$. We consider an arbitrary $R > 0$ and a sufficiently small $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ we have $M_\varepsilon > R$. Since (up to a subsequence) (u_ε) is not a sequence of constant functions, it holds that $\|\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^\infty(R,M_\varepsilon)} \geq 1$. On the other hand, the assumption $U_\infty = 0$ implies that for every $0 < \delta < 1$ there exists a large enough $R_1 = R_1(\delta) > 0$ such that for a sufficiently small $0 < \varepsilon_1$ and every $0 < \varepsilon \leq \varepsilon_1$ we get $M_\varepsilon > R_1$ and $\|\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^\infty(R_1,M_\varepsilon)} < \delta$. Consequently, we have $\|\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^\infty(R_1,M_\varepsilon)} = 0$, and we get the contradiction. Assertion (iii) follows directly from the estimate

$$\limsup_{\varepsilon \longrightarrow 0}(M_\varepsilon - m_\varepsilon) \leq \limsup_{\varepsilon \longrightarrow 0} \int_{m_\varepsilon}^{M_\varepsilon} (\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^q d\xi < +\infty.$$

Next, we consider $1 \leq q' \leq +\infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. From (5) and from the Hölder inequality we deduce

$$M_\varepsilon - m_\varepsilon \leq \|u'_\varepsilon\|_{\mathrm{L}^1(0,1)} \leq \|\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\cdot)\}\|_{\mathrm{L}^q(m_\varepsilon, M_\varepsilon)}(M_\varepsilon - m_\varepsilon)^{\frac{1}{q'}},$$

which gives assertion (iv). Finally, we address the proof of assertion (v). We choose $R > \limsup_{\varepsilon \longrightarrow 0} M_\varepsilon$. It follows that for a sufficiently small $\varepsilon_0 > 0$ and every $0 < \varepsilon \leq \varepsilon_0$ we have $D_\varepsilon(R) = \emptyset$. Hence, for every $q \in (-\infty, +\infty] \backslash \{0\}$, we get $U_q = 0$. □

Remark 1. *In this paper, we primarily consider the case $q < 0$. Note that FE sequences for which we have $U_q = 0$ for every $q \in (-\infty, +\infty] \backslash \{0\}$ exist (obvious examples are constant sequences $u^i_\varepsilon(s) := \alpha_i$, $s \in [0, 1]$), but explicit construction of normal and non-regular FE sequences is a more difficult task, and such examples are not easily found in the literature. In particular, we mention that, while there are examples of non-regular FE sequences which are not bounded in $\mathrm{W}^{1,1}(0, 1)$ as $\varepsilon \longrightarrow 0$ (cf. [18], Example 1.4), normal and non-regular FE sequences can not be bounded in $\mathrm{W}^{1,1}(0, 1)$ as $\varepsilon \longrightarrow 0$. The latter assertion follows from the fundamental theorem of calculus for absolutely continuous functions (cf. [15], Theorem 6.52), whereby, for every $s \in [0, 1]$, we estimate $|u_\varepsilon(s)| \leq |u_\varepsilon(c_\varepsilon)| + \int_0^1 |u'_\varepsilon(\sigma)|d\sigma$. Hence, if the sequence (c_ε) is chosen such that the sequence $(u_\varepsilon(c_\varepsilon))$ is a bounded sequence as $\varepsilon \longrightarrow 0$, it follows that every normal sequence, which is also bounded in $\mathrm{W}^{1,1}(0, 1)$, is regular. As a consequence, taking into account (5) and Proposition 1(iii), the range of q relevant to the study of normal and non-regular FE sequences is $q < 0$ or $0 < q < 1$. Intuitively, the condition $U_q < +\infty$, where $q < 0$ ($0 < q < 1$, resp.), shows that, at infinity, oscillations of the sequence (u_ε) are fast enough (slow enough, resp.) to ensure finiteness of the quantity U_q, but possibly not fast enough (slow enough, resp.) to ensure finiteness of the quantity $U_{\tilde{q}}$, where $q < \tilde{q} < 0$ ($0 < q < \tilde{q} < 1$, resp.).*

One possibility for introducing a q-UPO property is to allow the integration in (4) over the entire domain $[m_\varepsilon, M_\varepsilon]$, which would restrict the class of q-UPO sequences. In fact, for the purposes of analysis of non-regular FE sequences, it is more natural to consider the limit as $R \longrightarrow +\infty$ in (4). It is easy to construct an example of sequence of absolutely continuous functions (u_ε) which is non-regular and for which the limit as $\varepsilon \longrightarrow 0$ in (4), with integration over the domain $[m_\varepsilon, M_\varepsilon]$, is strictly positive and finite for a given $q < 0$ (for every $q < 0$, resp.). Indeed, we consider a sequence of absolutely continuous functions (v_n) ((w_n), resp.), such that $\max_{[0,1]} |v_n| = n$, $\min_{[0,1]} |v_n| = 1$ ($\max_{[0,1]} |w_n| = n$, $\min_{[0,1]} |w_n| = 1$, resp.), with the following properties: $\mathrm{card}\{|v_n|^{\leftarrow}(\xi)\} = (2\lceil\xi\rceil - 1)^{\theta_q}$ (a.e. $1 \leq \xi \leq n$) ($\mathrm{card}\{|w_n|^{\leftarrow}(\xi)\} = (2\lceil\xi\rceil - 1)^{2\lceil\xi\rceil - 1}$ (a.e. $1 \leq \xi \leq n$), resp.), where, for a given $q < 0$, we choose $\theta_q \in \mathbf{N}$ such that $q\theta_q < -1$. Then, by applying of the integral test for convergence of series, it holds that

$$\limsup_{n \longrightarrow +\infty} \int_1^n (\mathrm{card}\{|v_n|^{\leftarrow}(\xi)\})^q d\xi \leq \sum_{n=1}^{+\infty} (2n - 1)^{q\theta_q} < +\infty \tag{7}$$

$$\Big(\limsup_{n \longrightarrow +\infty} \int_1^n (\mathrm{card}\{|w_n|^{\leftarrow}(\xi)\})^q d\xi \leq \sum_{n=1}^{+\infty} (2n - 1)^{(2n-1)q} < +\infty, \text{ resp.}\Big). \tag{8}$$

In this example, due to the convergence of series in (7) (and (8), resp.), it follows that the corresponding iterated limit in (4) is equal to zero. On the other hand, if, for a given $-1 \leq q < 0$, we choose $\theta_q := 1$, we get $-1 \leq q\theta_q < 0$, and it follows that the sequence (v_n) is not a q-UPO sequence, since the remainder $\sum_{n \geq R}^{+\infty} n^{q\theta_q}$ of a divergent series equals $+\infty$ for every $R \geq 1$.

In the sequel, we provide an explicit construction of the aforementioned sequence (v_n) in the case $\theta_q = 1$. The construction in the case $\theta_q \in \mathbf{N}$, as well as the construction of (w_n), is similar and it is left to the interested reader. To this end, for $n \geq 2$ and $k \geq 1$, we consider a piecewise affine continuous function $\omega_{n,k} : [0, \frac{1}{n-1}] \longrightarrow [0,1]$ with slope $\pm(2k+1)(n-1)$, and such that $\omega_{n,k}(t_{n,k}^{2i}) = 0$, $\omega_{n,k}(t_{n,k}^{2i+1}) = 1$, where $i = 0, \ldots, k$, with $t_{n,k}^{j} := \frac{j}{(2k+1)(n-1)}$, where $j = 0, \ldots, 2k+1$. We set $v_n(0) := 1$, $v_n(s) := \omega_{n,k}(s - \frac{k-1}{n-1}) + k$, $s \in J_{n,k}$, where $J_{n,k} := (\frac{k-1}{n-1}, \frac{k}{n-1}]$ and $k = 1, \ldots, n-1$. Now it is readily seen that we have $\operatorname{card}\{|v_n|^{\leftarrow}(\xi)\} = 2k+1$ (a.e. $\xi \in (k, k+1]$), $\max_{J_{n,k}} |v_n| = k+1$, $\min_{J_{n,k}} |v_n| = k$, where $k = 1, \ldots, n-1$.

Furthermore, we provide an example of the sequence (z_n) for which the iterated limit in (3) (and (4), resp.) is strictly positive and finite, while the expression in (3) (and (4), resp.) does not depend on R. We set $z_n(s) := s + n$, $s \in [0,1]$, $m_n := \min_{[0,1]} |z_n|$ and $M_n := \max_{[0,1]} |z_n|$. Then it holds that $\operatorname{card}\{|z_n|^{\leftarrow}(\xi)\} = 1$ (a.e. $n \leq \xi \leq n+1$), $m_n = n$, $M_n = n+1$. In effect, for every $q \in (-\infty, +\infty] \backslash \{0\}$ we have $U_q = 1$. In Section 4, we further investigate the connection between condition (4) and the regularity in the case of FE sequences (u_ε), getting a kind of the converse of assertion (v) (cf. Theorem 4). For convenience of the reader, we restate a corollary of the area formula (cf. [15], Theorem 6.81, p. 385, or [9], Theorem 3.9, p. 122), which will be used in the proofs of Lemma 1, Theorem 3 and Theorem 4.

Theorem 1. *Consider a measurable set $A \subseteq [0,1]$, a Lipschitz-continuous function $f : [0,1] \longrightarrow \mathbf{R}$, and a continuous and nonnegative function $g : \mathbf{R} \longrightarrow \mathbf{R}$. Then it holds that $\int_A g(f(s))|f'(s)|ds = \int_{f(A)} g(\xi)\operatorname{card}\{f^{\leftarrow}(\xi)\}d\xi$.*

3. A priori estimates for (u'_ε)

For further analysis of asymptotic behaviour of FE sequences as $\varepsilon \longrightarrow 0$, it makes sense to consider only non-regular FE sequences. To begin with, in this section, we obtain some a priori estimates for the first derivative of normal and non-regular FE sequences. In the first lemma of this section, we recover an a priori estimate, which shows that the bending of the graph of normal and non-regular FE sequences (u_ε) is conditioned by the integrability properties of W.

Lemma 1. *Consider an arbitrary W, an arbitrary normal and non-regular FE sequence (u_ε), $0 < \rho < 1$, and $\theta_\rho := \frac{\rho}{1-\rho}$. Then we have the following conclusions:*

(*i*) *If it holds that $\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty$, then it holds that*

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{\theta_\rho} \int_0^1 |u'_\varepsilon(s)|^{1+\theta_\rho} ds = +\infty, \tag{9}$$

(ii) If it holds that $\int_0^{+\infty} V^\rho(\xi)d\xi < +\infty$, then for every $\gamma \in (0,1)$ it holds that

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{\gamma\theta_\rho} \int_0^1 |u_\varepsilon'(s)|^{1+\theta_\rho} ds = +\infty. \tag{10}$$

Proof. We note that by our assumption, (u_ε) can not be a sequence of constant functions. Consider an FE sequence $(\overline{u}_\varepsilon)$ in $\mathrm{C}^1[0,1]$ such that provisions of Proposition 2 hold. From the normality and the non-regularity of the sequence $(\overline{u}_\varepsilon)$, we infer that $(\overline{u}_\varepsilon)$ itself is not a sequence of constant functions. Then the set $G_\varepsilon := \{s \in (0,1) : \overline{u}_\varepsilon'(s) \neq 0\}$ is an open set and a non-empty set, and so $\lambda(G_\varepsilon) > 0$. By the reverse Hölder inequality for a given $0 < \rho < 1$ and $\rho' < 0$ such that $\frac{1}{\rho} + \frac{1}{\rho'} = 1$ it holds that

$$\begin{aligned} M &\geq \varepsilon^{-1} \int_{G_\varepsilon} V(|\overline{u}_\varepsilon|) = \varepsilon^{-1} \int_{G_\varepsilon} V(|\overline{u}_\varepsilon|) |\overline{u}_\varepsilon'|^{\frac{1}{\rho}} |\overline{u}_\varepsilon'|^{-\frac{1}{\rho}} \\ &\geq \varepsilon^{-1} \Big(\int_{G_\varepsilon} V^\rho(|\overline{u}_\varepsilon|) |\overline{u}_\varepsilon'| \Big)^{\frac{1}{\rho}} \Big(\int_{G_\varepsilon} |\overline{u}_\varepsilon'|^{-\frac{\rho'}{\rho}} \Big)^{\frac{1}{\rho'}} \\ &\geq \Big(\int_{m_\varepsilon}^{M_\varepsilon} V^\rho(\xi) d\xi \Big)^{\frac{1}{\rho}} \Big(\varepsilon^{\frac{\rho}{1-\rho}} \int_0^1 |\overline{u}_\varepsilon'|^{\frac{1}{1-\rho}} \Big)^{\frac{\rho-1}{\rho}}, \end{aligned}$$

where, in the last inequality, we combined Theorem 1, the equality $|\overline{u}_\varepsilon'(s)| = ||\overline{u}_\varepsilon|'(s)|$ (a.e. $s \in (0,1)$) (cf. [42], Theorem 2.1.11, p. 48), and the fact that $\overline{u}_\varepsilon \in \mathrm{C}^1[0,1]$ implies that $|\overline{u}_\varepsilon|$ is a Lipschitz-continuous function. By the normality of $(\overline{u}_\varepsilon)$ it follows that there exist $m > 0$ and $\varepsilon_0 > 0$ such that $\sup_{0<\varepsilon\leq\varepsilon_0} m_\varepsilon \leq m < +\infty$. On the other hand, by the non-regularity of $(\overline{u}_\varepsilon)$, as we pass to the subsequence (not relabeled) we obtain $\lim_{\varepsilon\longrightarrow 0} M_\varepsilon = +\infty$, and, for sufficiently small $\varepsilon > 0$ we get

$$M \geq \Big(\int_m^{M_\varepsilon} V^\rho(\xi) d\xi \Big)^{\frac{1}{\rho}} \Big(\varepsilon^{\frac{\rho}{1-\rho}} \int_0^1 |\overline{u}_\varepsilon'|^{\frac{1}{1-\rho}} \Big)^{\frac{\rho-1}{\rho}}. \tag{11}$$

At this point we note that M provides a finite upper bound for the product of two sequences in (11), and that the boundedness of the product of these sequences, with one of the sequences divergent to $+\infty$, implies that the other sequence converges to zero. As we pass to the limit as $\varepsilon \longrightarrow 0$ in (11), it follows that

$$M \geq \Big(\int_m^{+\infty} V^\rho(\xi) d\xi \Big)^{\frac{1}{\rho}} \liminf_{\varepsilon\longrightarrow 0} \Big(\varepsilon^{\frac{\rho}{1-\rho}} \int_0^1 |\overline{u}_\varepsilon'|^{\frac{1}{1-\rho}} \Big)^{\frac{\rho-1}{\rho}}, \tag{12}$$

where, since ρ satisfies $0 < \rho < 1$, the integral in the second term in (12) appears under the negative exponent. Hence, by applying the observations above, $\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty$ ($\int_0^{+\infty} V^\rho(\xi)d\xi < +\infty$, resp.) implies (9) (and (10), resp.) with u_ε replaced by $\overline{u}_\varepsilon$. To complete the argument, we distinguish two cases. In the first case, we assume that $0 < \rho \leq \frac{1}{2}$, whereby it holds that $1 < \frac{1}{1-\rho} \leq 2$. Then we have $u_\varepsilon' \in \mathrm{L}^{\frac{1}{1-\rho}}(0,1)$. In the second case, we assume that $\frac{1}{2} < \rho < 1$, and it results that $2 < \frac{1}{1-\rho} < +\infty$. Without loss of generality, we can assume that there exists a subsequence (u_ε) (not relabeled) such that $u_\varepsilon' \in \mathrm{L}^{\frac{1}{1-\rho}}(0,1)$ (otherwise assertions (9) and (10) are obvious). Finally, we apply Corollary 8, getting (9) and (10) for an arbitrary normal and a non-regular FE sequence (u_ε) in $\mathrm{H}^1(0,1)$. □

To proceed, we consider the non-integrability condition $\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty$, $0 < \rho < \frac{1}{2}$, whereby we allow the case $\int_0^{+\infty} \sqrt{V(\xi)}d\xi < +\infty$.

Theorem 2 (L^∞-estimates for u_ε': the subcritical case)**.** *Consider* $0 < \rho < \frac{1}{2}$*. If* W *satisfies* $\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty$*, then every normal and non-regular FE sequence* (v_ε) *satisfies the following: for every* $\Delta > 0$ *and for every sequence* (ρ_ε) *of strictly positive numbers such that* $\liminf_{\varepsilon\longrightarrow 0} \varepsilon^{1+\Delta}\rho_\varepsilon > 1$*, it holds that*

$$\limsup_{\varepsilon\longrightarrow 0} \|\rho_\varepsilon^{-1} u_\varepsilon'\|_{\mathrm{L}^\infty(0,1)} \leq 1. \tag{13}$$

In particular, for every $p > 1$ *it holds that*

$$\limsup_{\varepsilon\longrightarrow 0} \|\varepsilon^p u_\varepsilon'\|_{\mathrm{L}^\infty(0,1)} = 0, \tag{14}$$

Moreover, for every normal FE sequence (u_ε) *such that* $\lim_{\varepsilon\longrightarrow 0} M_\varepsilon = +\infty$*, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \leq \varepsilon_0$ *we have* $u_\varepsilon \in \mathrm{W}^{1,\infty}(0,1)$*.*

Proof. We assume the opposite. Then there exists $\Delta_0 > 0$, a subsequence of strictly decreasing positive numbers (ε_j) such that $\lim_{j\longrightarrow+\infty} \varepsilon_j = 0$ and a sequence (ρ_{ε_j}) of strictly positive numbers such that $\lim_{j\longrightarrow+\infty} \rho_{\varepsilon_j} = +\infty$ and such that

$$\lambda(H_j) > 0 \text{ for every } j \in \mathbf{N}, \text{ where } H_j := \{s \in (0,1) : |u_{\varepsilon_j}'(s)| > \rho_{\varepsilon_j}\}, \tag{15}$$

whereby

$$\liminf_{j\longrightarrow+\infty} \varepsilon_j^{1+\Delta_0}\rho_{\varepsilon_j} > 1. \tag{16}$$

By the reverse Hölder inequality for $0 < \theta < 2$ and every $j \in \mathbf{N}$ we get

$$M \geq \varepsilon_j \int_{H_j} |u_{\varepsilon_j}'|^{2-\theta}|u_{\varepsilon_j}'|^\theta \geq \varepsilon_j \Big(\int_{H_j} |u_{\varepsilon_j}'|^{(2-\theta)r_0}\Big)^{\frac{1}{r_0}} \Big(\int_{H_j} |u_{\varepsilon_j}'|^{r_0'\theta}\Big)^{\frac{1}{r_0'}}, \tag{17}$$

where $\frac{1}{r_0} + \frac{1}{r_0'} = 1$, $0 < r_0 < 1$, $r_0' < 0$, using assumption (15). We choose $0 < \theta < 2$ and $0 < r_0 < 1$ such that $(2-\theta)r_0 = \frac{1}{1-\rho}$. Then we get $r_0\theta = 2r_0 - \frac{1}{1-\rho} > 0$, $1 > r_0 > \frac{\frac{1}{2}}{1-\rho}$ (note that since $0 < \rho < \frac{1}{2}$, such a choice of r_0 is possible) and $r_0'\theta = \frac{r_0\theta}{r_0-1} = (2r_0 - \frac{1}{1-\rho})\frac{1}{r_0-1}$. By (17) it follows that

$$M^{r_0} \geq \Big(\varepsilon_j^{\frac{\rho}{1-\rho}} \int_{H_j} |u_{\varepsilon_j}'|^{\frac{1}{1-\rho}}\Big)\Big(\varepsilon_j^{(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}} \int_{H_j} |u_{\varepsilon_j}'|^{(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}}\Big)^{r_0-1}.$$

From $M \geq \int_{(0,1)\backslash H_j} \varepsilon_j |u_{\varepsilon_j}'|^2$ we deduce $\lambda((0,1)\backslash H_j) \leq M\varepsilon_j^{-1}\rho_{\varepsilon_j}^{-2}$. Consequently, we estimate

$$\begin{aligned}\varepsilon_j^{\frac{\rho}{1-\rho}} \int_0^1 |u_{\varepsilon_j}'|^{\frac{1}{1-\rho}} &\leq \varepsilon_j^{\frac{\rho}{1-\rho}} \int_{H_j} |u_{\varepsilon_j}'|^{\frac{1}{1-\rho}} + \varepsilon_j^{\frac{\rho}{1-\rho}} \int_{(0,1)\backslash H_j} |u_{\varepsilon_j}'|^{\frac{1}{1-\rho}} \\ &\leq \varepsilon_j^{\frac{\rho}{1-\rho}} \int_{H_j} |u_{\varepsilon_j}'|^{\frac{1}{1-\rho}} + M(\varepsilon_j\rho_{\varepsilon_j})^{\frac{2\rho-1}{1-\rho}},\end{aligned}$$

By (16) and Lemma 1, (i), we recover $\lim_{j\longrightarrow+\infty}\varepsilon_j^{\frac{\rho}{1-\rho}}\int_{H_j}|u'_{\varepsilon_j}|^{\frac{1}{1-\rho}}=+\infty$, getting

$$\lim_{j\longrightarrow+\infty}\varepsilon_j^{(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}}\int_{H_j}|u'_{\varepsilon_j}|^{(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}}=+\infty,$$

where $(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}<0$ and $(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}<0$. It follows that

$$\lim_{j\longrightarrow+\infty}\varepsilon_j^{(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}}\lambda(H_j)\rho_{\varepsilon_j}^{(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}}=+\infty,$$

and so

$$\lim_{j\longrightarrow+\infty}\varepsilon_j^{(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}+\log_{\varepsilon_j}\rho_{\varepsilon_j}(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}}=+\infty.$$

Hence, there exists $j_0=j_0(r_0)\in\mathbf{N}$ such that for every $j\geq j_0$ it necessarily holds that

$$(r_0-\frac{\rho}{1-\rho})\frac{1}{r_0-1}+\log_{\varepsilon_j}\rho_{\varepsilon_j}(2r_0-\frac{1}{1-\rho})\frac{1}{r_0-1}<0,\ \text{i.e. },\rho_{\varepsilon_j}<(\varepsilon_j^{-1})^{\frac{r_0-\frac{\rho}{1-\rho}}{2r_0-\frac{1}{1-\rho}}},$$

where $\frac{\frac{1}{2}}{1-\rho}<r_0<1$ is arbitrary. We consider

$$\varphi_\rho(\xi):=\frac{\xi-\frac{\rho}{1-\rho}}{2\xi-\frac{1}{1-\rho}}.$$

Then φ_ρ is strictly decreasing on $(\frac{\frac{1}{2}}{1-\rho},1)$, and $\varphi_\rho(1-\delta)\searrow 1$ as $\delta\searrow 0$. For a given $\Delta_0>0$, we choose $\delta_0(\Delta_0)>0$ such that for every $0<\delta\leq\delta_0(\Delta_0)$ it holds that $\varphi_\rho(1-\delta)\leq 1+\Delta_0$. In effect, it follows that $\lambda(H_j)>0$ implies $\rho_{\varepsilon_j}\leq\varepsilon_j^{-1-\Delta_0}$ for every $j\geq j_0(\delta)$. We conclude that it holds that $\limsup_{j\longrightarrow+\infty}\varepsilon_j^{1+\Delta_0}\rho_{\varepsilon_j}\leq 1$, which contradicts the initial assumption (16). Thus, we proved (13). To prove (14), we choose an arbitrary $1<p<+\infty$, $0<\Delta<p-1$ and we set $\rho_\varepsilon:=\varepsilon^{-p}$. Then we have $\lim_{\varepsilon\longrightarrow 0}\varepsilon^{1+\Delta}\rho_\varepsilon=+\infty$, and so $\limsup_{\varepsilon\longrightarrow 0}\|\varepsilon^p u'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}\leq 1$. Since $p>1$ was arbitrary, by re-writing the latter estimate as $\limsup_{\varepsilon\longrightarrow 0}\varepsilon^{-\theta}\|\varepsilon^{p+\theta}u'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}\leq 1$, where $\theta>0$ is arbitrary, we recover (14). □

By the Sobolev embedding (cf. [19], Theorem 11.34, p. 335 and exercise 11.38, p. 339), we immediately get the following corollary:

Corollary 1. *Under the assumptions of Theorem 2, it holds that* $u_\varepsilon\in\mathrm{C}^{0,\theta}[0,1]$, *where* $\theta=1-\frac{1}{p}$, *and* $p>1$ *is arbitrary. Besides, by (9) (and (14), resp.), for every* $0\leq\theta\leq\frac{\rho}{1-\rho}$ *(*$\theta>1$*, resp.) it holds that* $\limsup_{\varepsilon\longrightarrow 0}\|\varepsilon^\theta u'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}=+\infty$ *(*$\lim_{\varepsilon\longrightarrow 0}\|\varepsilon^\theta u'_\varepsilon\|_{\mathrm{L}^\infty(0,1)}=0$*, resp.).*

If W is an arbitrary multi-well potential, we have $\limsup_{\varepsilon\longrightarrow 0}\|\varepsilon^{\frac{1}{2}}u'_\varepsilon\|_{\mathrm{L}^2(0,1)}<+\infty$. As a consequence, by the Hölder inequality, for every $1\leq p\leq 2$, it follows that

$$\limsup_{\varepsilon\longrightarrow 0}\|\varepsilon^{\frac{1}{2}}u'_\varepsilon\|_{\mathrm{L}^p(0,1)}<+\infty,\tag{18}$$

$$\lim_{\varepsilon \longrightarrow 0} \|\varepsilon^{\theta} u'_{\varepsilon}\|_{\mathrm{L}^p(0,1)} = 0, \quad \text{where } \theta > \tfrac{1}{2}. \tag{19}$$

As a further corollary, we deduce the estimate which is the counterpart of (19) in the case $p > 2$.

Corollary 2. *Under assumptions of Theorem 2, for every $p > 2$ and every $\Delta_0 > 0$ it holds that*

$$\lim_{\varepsilon \longrightarrow 0} \|\varepsilon^{1-\frac{1}{2}\frac{1}{p}+\Delta_0} u'_{\varepsilon}\|_{\mathrm{L}^p(0,1)} = 0. \tag{20}$$

Proof. We recall that the Riesz-Thorin interpolation theorem (cf. [12], Proposition 6.10, p. 185) provides that for arbitrary $0 < p_0 < p_1 \leq +\infty$ and $0 < \theta < 1$ we have $\|g\|_{\mathrm{L}^{p_\theta}} \leq \|g\|^{\theta}_{\mathrm{L}^{p_0}} \|g\|^{1-\theta}_{\mathrm{L}^{p_1}}$, where $g \in \mathrm{L}^{p_0} \cap L^{p_1}$ and $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$. We set $g := u'_{\varepsilon}$, $p_0 := 1$, $p_1 := +\infty$ and $\theta := \frac{1}{p}$. It follows $\|u'_{\varepsilon}\|_{\mathrm{L}^p(0,1)} \leq \|u'_{\varepsilon}\|^{\frac{1}{p}}_{\mathrm{L}^1(0,1)} \|u'_{\varepsilon}\|^{1-\frac{1}{p}}_{\mathrm{L}^\infty(0,1)}$, i.e.,

$$\|\varepsilon^{\frac{1}{2}\frac{1}{p}+\rho_0(1-\frac{1}{p})} u'_{\varepsilon}\|_{\mathrm{L}^p(0,1)} \leq \|\varepsilon^{\frac{1}{2}} u'_{\varepsilon}\|^{\frac{1}{p}}_{\mathrm{L}^1(0,1)} \|\varepsilon^{\rho_0} u'_{\varepsilon}\|^{1-\frac{1}{p}}_{\mathrm{L}^\infty(0,1)},$$

where $\rho_0 > 1$ is arbitrarily chosen. In turn, we get

$$\lim_{\varepsilon \longrightarrow 0} \|\varepsilon^{\frac{1}{2}\frac{1}{p}+\rho_0(1-\frac{1}{p})} u'_{\varepsilon}\|_{\mathrm{L}^p(0,1)} = 0. \tag{21}$$

Finally, as we pass to the limit as $\varepsilon \longrightarrow 0$ in (21), we complete the argument by applying (18) and Theorem 2, and by choosing $\rho_0 = 1 + \Delta$ and $\Delta_0 := (1 - \frac{1}{p})\Delta$. $\square$

Remark 2. *The estimate (20) is true for $1 < p \leq 2$, but it is weaker than (19), since $\rho_0 > 1$ gives $\frac{1}{2}\frac{1}{p} + \rho_0(1 - \frac{1}{p}) \geq \frac{1}{4} + \frac{1}{2}\rho_0 > \frac{3}{4}$.*

4. A priori estimates for (u_ε)

As a consequence of Lemma 1, we can deduce the result which was already proved by Leoni in [18] (cf. Theorem 1.3 therein).

Corollary 3 (Basic properties of FE sequences: the critical case)**.** *If a multi-well potential W satisfies $\int_0^{+\infty} \sqrt{V(\xi)}d\xi = +\infty$, then the following properties hold:*

- *(i) (Regularity) Every normal FE sequence (u_ε) is regular,*
- *(ii) (Identification of cluster points of regular FE sequences) Every subsequence of a regular FE sequence (u_ε) allows a further subsequence (not relabeled) which satisfies $\underline{\delta}_{u_\varepsilon} \xrightarrow{\;*\;} \sum_{i=1}^{l} \theta_i \delta_{\alpha_i}$ in $YM((0,1);K)$ as $\varepsilon \longrightarrow 0$, where $\underline{\delta}_{u_\varepsilon}(s) := \delta_{u_\varepsilon(s)}$ (a.e. $s \in (0,1)$), and $\theta_i = \theta_i(s)$, $s \in (0,1)$, $i = 1, \ldots, l$, are measurable functions (which depend on the chosen subsequence) such that $0 \leq \theta_i(s) \leq 1$ (a.e. $s \in (0,1)$) and $\sum_{i=1}^{l} \theta_i(s) = 1$ (a.e. $s \in (0,1)$),*
- *(iii) (Tightness of non-regular FE sequences) If there exists an FE sequence (u_ε) which is non-regular, then it is not a normal FE sequence, and there exists a subsequence of (u_ε) (not relabeled) and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ each u_ε is either strictly positive or strictly negative on $(0,1)$,*

(*iv*) *(Identification of cluster points of non-regular FE sequences) Every non-regular FE sequence* (u_ε) *allows a subsequence (not relabeled) such that* (u_ε) *charges infinity on* $(0,1)$*. More precisely, there exists a subsequence of* (u_ε) *(not relabeled) such that: every subsequence of* (u_ε) *allows a further subsequence (not relabeled) which satisfies either* $\underline{\delta}_{u_\varepsilon} \xrightarrow{\ *\ } \delta_{-\infty}$ *or* $\underline{\delta}_{u_\varepsilon} \xrightarrow{\ *\ } \delta_{+\infty}$ *in* $YM((0,1);K)$ *as* $\varepsilon \longrightarrow 0$*.*

Proof. Claim (i) is a direct consequence of Lemma 1. It is enough to assume the opposite to choose $\rho := \frac{1}{2}$ in Lemma 1(i), and then to take into account that (u_ε) is an FE sequence. Regarding (ii), we note that by the results of Leoni in [18], it holds that (u_ε) is strongly pre-compact in $\mathrm{L}^1(0,1)$ and so, by the fundamental theorem of Young measures (cf. [5] or [16]), it follows that $(\underline{\delta}_{u_\varepsilon})$ is pre-compact in $\mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R}))$ as $\varepsilon \longrightarrow 0$. Hence, assertion (ii) follows by the embedding $\mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R})) \hookrightarrow YM((0,1);K)$. Next, to prove (iii), we observe that by (i), every non-regular FE sequence (u_ε) allows a subsequence (not relabeled) which is not a normal sequence, and therefore it follows that $\lim_{\varepsilon\longrightarrow 0} m_\varepsilon = \lim_{\varepsilon\longrightarrow 0} M_\varepsilon = +\infty$. Consider $L > 0$ and $\varepsilon_0(L) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0(L)$ it holds that $m_\varepsilon > L$ and $M_\varepsilon > L$. We choose any such ε and note that if the opposite were true, it would follow that there exists $c_\varepsilon^{(1)}, c_\varepsilon^{(2)} \in (0,1)$ such that $u_\varepsilon(c_\varepsilon^{(1)}) \leq 0$ and $u_\varepsilon(c_\varepsilon^{(2)}) \geq 0$, where without loss of generality, we can assume $c_\varepsilon^{(1)} < c_\varepsilon^{(2)}$. By the intermediate value property of the continuous function u_ε on $[c_\varepsilon^{(1)}, c_\varepsilon^{(2)}]$, it follows that there exists $c_\varepsilon \in [c_\varepsilon^{(1)}, c_\varepsilon^{(2)}]$ such that $u_\varepsilon(c_\varepsilon) = 0$. It follows that $m_\varepsilon = 0$, which is not possible. Finally, assertion (iv) follows from the existence of the subsequence of (u_ε) (not relabeled) which satisfies $\lim_{\varepsilon\longrightarrow 0} m_\varepsilon = +\infty$. Indeed, we recall that by compactness of K, $YM((0,1);K)$ itself is a compact metric space. Then every subsequence of (v_ε) allows a further subsequence (not relabeled) such that for some $\theta_0 \in \mathrm{L}^\infty(0,1)$, $0 \leq \theta_0(s) \leq 1$ (a.e. $s \in (0,1)$), we have $\underline{\delta}_{u_\varepsilon} \xrightarrow{\ *\ } \theta_0\delta_{-\infty} + (1-\theta_0)\delta_{+\infty}$ in $YM((0,1);K)$ as $\varepsilon \longrightarrow 0$. According to (ii), without loss of generality, we can assume that u_ε is strictly positive (strictly negative, resp.) on $(0,1)$. Then we get $\lim_{\varepsilon\longrightarrow 0} u_\varepsilon(c_\varepsilon) = +\infty$ ($\lim_{\varepsilon\longrightarrow 0} u_\varepsilon(c_\varepsilon) = -\infty$, resp.), and therefore $\mathrm{arctg}(u_\varepsilon) \longrightarrow \frac{\pi}{2}$ ($\mathrm{arctg}(u_\varepsilon) \longrightarrow -\frac{\pi}{2}$, resp.) uniformly on $(0,1)$ as $\varepsilon \longrightarrow 0$, where $\min_{[0,1]} u_\varepsilon = u_\varepsilon(c_\varepsilon)$ ($\max_{[0,1]} u_\varepsilon = u_\varepsilon(c_\varepsilon)$, resp.). It follows that $\theta_0(s) = 0$ ($\theta_0(s) = 1$, resp.) (a.e. $s \in (0,1)$). □

In the next theorem, we provide a priori estimates for asymptotic behaviour as $\xi \longrightarrow +\infty$ of the quantity $\liminf_{\varepsilon\longrightarrow 0} \mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}$, subject to appropriate integrability assumptions on W.

Theorem 3. *Suppose that there exist* $0 < \rho \leq \frac{1}{4}$ *(*$\frac{1}{4} < \rho < \frac{1}{2}$*, resp.) such that*

$$\int_0^{+\infty} V^\rho(\xi)d\xi = +\infty. \tag{22}$$

Then every normal and non-regular FE sequence (u_ε) *satisfies*

$$\liminf_{\xi\longrightarrow+\infty} \frac{\liminf_{\varepsilon\longrightarrow 0} \mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}}{\xi^{\frac{1}{2\rho}}} = 0 \tag{23}$$

$$\Big(\liminf_{\xi\longrightarrow+\infty}\frac{\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}}{\xi}=0,\ \textit{resp.}\Big). \tag{24}$$

Proof. If (u_ε) is also a regular FE sequence, the assertion is obvious. Given a non-regular FE sequence (u_ε), by passing to the subsequence (not relabeled) we get $\lim_{\varepsilon\longrightarrow 0} M_\varepsilon=+\infty$. By the normality of (u_ε), there exist $R\geq R_0>0$ and $\varepsilon_0(R)>0$ such that for every $0<\varepsilon\leq\varepsilon_0(R)$ we get $m_\varepsilon<R<M_\varepsilon$. By the arithmetic-geometric means inequality and Theorem 1 it follows that $M\geq\int_0^1\sqrt{V(|u_\varepsilon(s)|)}|u_\varepsilon'(s)|ds\geq\int_R^{M_\varepsilon}\sqrt{V(\xi)}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}d\xi$. As we pass to the limit as $\varepsilon\longrightarrow 0$ in the last inequality, by Fatou's Lemma, it follows that the mapping $\xi\mapsto\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}$ belongs to $\mathrm{L}^1((R,+\infty);d\mu)$, with $d\mu=\sqrt{V}d\lambda$. Since for every $R_1>R$ we have $\min\{\sqrt{V(\xi)}:\xi\in[R,R_1]\}>0$, it follows that the aforementioned mapping also belongs to $\mathrm{L}^1_{loc}(R,+\infty)$, and its precise representative is well-defined on $(R,+\infty)$. In particular, its limit inferior at $+\infty$ is also well-defined. Furthermore, by the intermediate value property for continuous functions, for every $\xi\in[m_\varepsilon,M_\varepsilon]$ it holds that $\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}\geq 1$, and, consequently, $\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})\geq 1$ for every $\xi\in[R,+\infty)$. We set $r:=2\rho$. By the reverse Hölder inequality (with $0<r<1$ and $r'<0$, where $\frac{1}{r}+\frac{1}{r'}=1$), we deduce

$$\begin{aligned}M&\geq\int_R^{+\infty}\sqrt{V(\xi)}\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}d\xi\\&\geq\Big(\int_R^{+\infty}\big(\sqrt{V(\xi)}\big)^r d\xi\Big)^{\frac{1}{r}}\Big(\int_R^{+\infty}(\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}d\xi\Big)^{\frac{1}{r'}}.\end{aligned}$$

By (22), we recover $\int_R^{+\infty}(\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}d\xi=+\infty$. On the other hand, for $0<r\leq\frac{1}{2}$ ($\frac{1}{2}<r<1$, resp.) it holds that $\int_R^{+\infty}\xi^{r'-1}d\xi<+\infty$ ($\int_R^{+\infty}\xi^{r'}d\xi<+\infty$, resp.). Arguing by contradiction, it follows that we have

$$\liminf_{\xi\longrightarrow+\infty}\frac{\xi^{r'-1}}{(\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}}=0$$

$$\Big(\liminf_{\xi\longrightarrow+\infty}\frac{\xi^{r'}}{(\liminf_{\varepsilon\longrightarrow 0}\operatorname{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}}=0,\ \text{resp.}\Big),$$

which amounts to (23) (and (24), resp.). □

Theorem 4 (Regularity in the subcritical case)**.** *Consider $0<\rho<\frac{1}{2}$ and an arbitrary multi-well potential W such that $\int_0^{+\infty}V^\rho(\xi)d\xi=+\infty$. Then every normal and q_ρ-UPO FE sequence is regular and strongly pre-compact in $\mathrm{L}^1(0,1)$ as $\varepsilon\longrightarrow 0$, where $q_\rho:=\frac{2\rho}{2\rho-1}<0$. In particular, for every normal FE sequence and every $q\in[q_\rho,+\infty]\backslash\{0\}$ it follows that $U_q<+\infty$ implies $U_q=0$.*

Proof. We choose $r:=2\rho$. We assume the opposite. Then there exists a subsequence of (u_ε) (not relabeled) such that $\lim_{\varepsilon\longrightarrow 0}M_\varepsilon=+\infty$. Similarly to the proof of Theorem 3, for a sufficiently small $\varepsilon_0>0$, for every $0<\varepsilon\leq\varepsilon_0$ we get $m_\varepsilon<R<M_\varepsilon$

and $D_\varepsilon(R)=(R,M_\varepsilon]$, where $D_\varepsilon(R)$ is defined as in Definition 2. Furthermore, if $\frac{1}{r}+\frac{1}{r'}=1$ and $0<\varepsilon\le\varepsilon_0$, it follows that

$$\begin{aligned} M &\ge \int_R^{M_\varepsilon}\sqrt{V(\xi)}\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}d\xi \\ &\ge \Big(\int_R^{M_\varepsilon}\big(\sqrt{V(\xi)}\big)^r d\xi\Big)^{\frac{1}{r}}\Big(\int_R^{M_\varepsilon}(\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}d\xi\Big)^{\frac{1}{r'}}. \end{aligned}$$

Consequently, as we pass to the limit as $\varepsilon\longrightarrow 0$ in the last inequality, it follows that $\limsup_{\varepsilon\longrightarrow 0}\int_R^{M_\varepsilon}(\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\})^{r'}d\xi=+\infty$, which gives a contradiction. By combining the fundamental theorem of Young measures (cf. [23], Theorem 3.1) and the Vitali convergence theorem (cf. [13], Theorem 2.24), as in the proof of Theorem 1.3 in [18], strong pre-compactness of (u_ε) in $\mathrm{L}^1(0,1)$ as $\varepsilon\longrightarrow 0$ can now be obtained by a classical argument. The second assertion follows by the regularity and Proposition 1(v). $\square$

Remark 3. *We note that Theorem 1.3 in [18] (Theorem 4, resp.) shows that in the critical case (in the subcritical case, resp.), every normal FE sequence is regular (normality and a q_ρ-UPO property imply regularity, resp.), whereby we have $U_q=0$ for every $q\in(-\infty,+\infty]\backslash\{0\}$ (cf. Proposition 1(v)). Therefore, both theorems provide the same type of conclusion in terms of a q-UPO property. In the case of Theorem 1.3 in [18], regularity ultimately leads to strong pre-compactness in $\mathrm{L}^1(0,1)$ of normal FE sequences as $\varepsilon\longrightarrow 0$. By contrast, in the case of Theorem 4, strong pre-compactness in $\mathrm{L}^1(0,1)$ follows for normal and q_ρ-UPO FE sequences, which is a more narrow class of FE sequences. On the other hand, in the subcritical case, normal and non-regular FE sequences necessarily satisfy $U_{q_\rho}=+\infty$, which means that only sufficiently slow oscillations at infinity can occur. We do not know if the result of Theorem 4 is sharp.*

Corollary 4. *Suppose that a multi-well potential W satisfies the following: there exist $q>2$ and $R\ge R_0$ such that it holds that*

$$V(\xi)\ge\frac{c_0}{\xi^q}\ \text{for every}\ \xi\ge R, \tag{25}$$

where $c_0>0$. Then every normal and $\frac{2}{2-q}$-UPO FE sequence is regular. In particular, if (25) is satisfied for some $q>2$, and if $\int_0^{+\infty}\sqrt{V(\xi)}d\xi<+\infty$, then the same conclusion applies.

Proof. We note that W satisfies $\int_R^{+\infty}\Big(\sqrt{V(\xi)}\Big)^r d\xi=+\infty$, provided that we choose r such that $0<r\le\frac{2}{q}<1$, and the assumptions of Theorem 4 are fulfilled. Hence, we have $\frac{2}{2-q}\le r'<0$, where $\frac{1}{r}+\frac{1}{r'}=1$. Finally, we apply the observation that for every $-\infty<q_2<q_1<0$, we have $U_{q_2}\le U_{q_1}$. $\square$

Remark 4. *Corollary 4 establishes the regularity of normal and q-UPO FE sequences for a given $q<0$, as long as the decay of V at infinity is not too fast. However, if the decay is fast enough, we conjecture that the regularity result is no* *longer true. The multi-well potential W chosen such that for every $\xi \geq R \geq R_0$ we have $V(\xi) \leq \alpha e^{-\beta\xi}$, where $\alpha, \beta > 0$, is an example of the function which satisfies $\int_0^{+\infty} \left(\sqrt{V(\xi)}\right)^r d\xi < +\infty$ for every $0 < r < 1$. Similar observations hold if V satisfies $V(\xi) \leq \alpha e^{-\beta\xi^{\frac{1}{n}}}$, $\xi \geq R \geq R_0$, for some $n \in \mathbf{N}$, since it holds that $\int_0^{+\infty} e^{-\xi^{\frac{1}{n}}} d\xi = n!$.*

In the last corollary, we mention an extension of some of the previous results to the case of a variable power $\rho = \rho(\xi)$. Further generalizations to the variable power setting are possible and are left to the interested reader.

Corollary 5. *Consider $0 < \rho_1 < \rho_2 < \frac{1}{2}$ and a multi-well potential W which satisfies $\int_R^{+\infty} V^{\rho(\xi)}(\xi) d\xi = +\infty$, where $\rho : (R, +\infty) \longrightarrow (\rho_1, \rho_2)$ is a continuous function, where $R \geq R_0$. Then every normal and q_0-UPO FE sequence is regular, where $q_0 := \max\{q_{\rho_1}, q_{\rho_2}\}$ and $q_{\rho_i} := \frac{2\rho_i}{2\rho_i - 1}$, $i = 1, 2$.*

Proof. By $\mathcal{C}^{\rho(\cdot)}$ we denote the set of all continuous and strictly positive functions $V : (R, +\infty) \longrightarrow (0, +\infty)$ which satisfy $\int_R^{+\infty} V^{\rho(\xi)}(\xi) d\xi = +\infty$. Then the canonical representation $V^{\rho(\xi)}(\xi) = Z_1(\xi) + Z_2(\xi)$, $Z_1(\xi) := V^{\rho(\xi)}(\xi) \chi_{\{\eta \in (R,+\infty) : V(\eta) < 1\}}(\xi)$, $Z_2(\xi) := V^{\rho(\xi)}(\xi) \chi_{\{\eta \in (R,+\infty) : V(\eta) > 1\}}(\xi)$, provides the estimate $V^{\rho(\xi)}(\xi) \leq V^{\rho_1}(\xi) + V^{\rho_2}(\xi)$, which, in turn, yields the inclusion $\mathcal{C}^{\rho(\cdot)} \subseteq \mathcal{C}^{\rho_1} \cup \mathcal{C}^{\rho_2}$. The assertion now follows from Theorem 4. □

5. Appendix

In the appendix we have gathered some technical results used in the previous sections. We begin by the following definition:

Definition 3. *Given $1 \leq p < +\infty$, and a sequence (u_ε) in $\mathrm{L}^p(0,1)$, we say that (u_ε) is*

- (i) *p-equi-integrable, if there exists $0 < \varepsilon_0 < 1$ with the following property: for every $\eta > 0$ there exists $\delta > 0$ such that for every measurable set $E \subseteq (0,1)$ we have that $\lambda(E) \leq \delta$ implies $\sup_{0<\varepsilon\leq\varepsilon_0} \int_E |u_\varepsilon|^p \leq \eta$,*
- (ii) *uniformly integrable, if there exists $0 < \varepsilon_0 < 1$ with the property*

$$\lim_{R \longrightarrow +\infty} \sup_{0<\varepsilon\leq\varepsilon_0} \int_{\{|u_\varepsilon| > R\}} |u_\varepsilon| = 0.$$

If (u_ε) is 1-equi-integrable, we say that (u_ε) is equi-integrable.

Proposition 2. *Consider an arbitrary multi-well potential W. Then for every FE sequence (u_ε) in $\mathrm{H}^1(0,1)$ there exists an FE sequence $(\overline{u}_\varepsilon)$ in $\mathrm{C}^1[0,1]$ such that*

$$|\varepsilon^{-1} I_0^\varepsilon(u_\varepsilon) - \varepsilon^{-1} I_0^\varepsilon(\overline{u}_\varepsilon)| \leq \varepsilon, \quad \|u_\varepsilon - \overline{u}_\varepsilon\|_{\mathrm{H}^1(0,1)} \leq \varepsilon, \;\; \|u_\varepsilon - \overline{u}_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \leq \varepsilon,$$

$$\lambda\{u_\varepsilon \neq \overline{u}_\varepsilon\} \leq \varepsilon, \quad \lambda\{u'_\varepsilon \neq \overline{u}'_\varepsilon\} \leq \varepsilon,$$

where ε is an arbitrary real number which satisfies $0 < \varepsilon \leq \varepsilon_0$, for a sufficiently small $0 < \varepsilon_0 << 1$.

Proof. If $\varepsilon > 0$ is given, then by Corollary 6.6.2 in [9], p. 256, for every $\delta > 0$ there exists $w_{\varepsilon,\delta} \in \mathrm{C}^1[0,1]$ with the properties

$$\|u_\varepsilon - w_{\varepsilon,\delta}\|_{\mathrm{H}^1(0,1)} \le \delta, \quad \lambda\{u_\varepsilon \ne w_{\varepsilon,\delta}\} \le \delta, \quad \lambda\{u'_\varepsilon \ne w'_{\varepsilon,\delta}\} \le \delta.$$

By the Sobolev embedding, we have $\|u_\varepsilon - w_{\varepsilon,\delta}\|_{\mathrm{L}^\infty(0,1)} \le C\|u_\varepsilon - w_{\varepsilon,\delta}\|_{\mathrm{H}^1(0,1)} \le C\delta$, for a constant $C > 0$ which is independent of W and δ. Hence, we can choose $0 < \delta << \frac{1}{C}$. On the other hand, for every $\zeta_1, \zeta_2 \in [-\|u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} - 1, \|u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} + 1]$ it holds that $|W(\zeta_2) - W(\zeta_1)| \le \omega_\varepsilon(|\zeta_2 - \zeta_1|)$, where ω_ε denotes the modulus of continuity of the restriction of W on $[-2\|u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} - 2, 2\|u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} + 2]$. We recall that without loss of generality, we can assume that $\omega_\varepsilon : [0, 4\|u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} + 4) \longrightarrow [0, +\infty)$ is a strictly increasing and invertible function. Therefore, it follows that

$$\varepsilon^{-1}\int_0^1 |W(u_\varepsilon(s)) - W(w_{\varepsilon,\delta}(s))|ds \le \varepsilon^{-1}\omega_\varepsilon(\|u_\varepsilon - w_{\varepsilon,\delta}\|_{\mathrm{L}^\infty(0,1)}) \le \varepsilon^{-1}\omega_\varepsilon(C\delta).$$

Next, we estimate

$$\begin{aligned} |\|u'_\varepsilon\|^2_{\mathrm{L}^2} - \|w'_{\varepsilon,\delta}\|^2_{\mathrm{L}^2}| &= |\|u'_\varepsilon\|_{\mathrm{L}^2(0,1)} - \|w'_{\varepsilon,\delta}\|_{\mathrm{L}^2}|\Big(\|u'_\varepsilon\|_{\mathrm{L}^2} + \|w'_{\varepsilon,\delta}\|_{\mathrm{L}^2}\Big) \\ &\le |\|u'_\varepsilon\|_{\mathrm{L}^2} - \|w'_{\varepsilon,\delta}\|_{\mathrm{L}^2}|\Big(2\|u'_\varepsilon\|_{\mathrm{L}^2} + \|w'_{\varepsilon,\delta} - u'_\varepsilon\|_{\mathrm{L}^2}\Big) \\ &\le \|u'_\varepsilon - w'_{\varepsilon,\delta}\|_{\mathrm{L}^2}\Big(2\|u'_\varepsilon\|_{\mathrm{L}^2} + \delta\Big) \\ &\le \delta\Big(2\|u'_\varepsilon\|_{\mathrm{L}^2} + \delta\Big), \end{aligned}$$

whereby $|\varepsilon\|u'_\varepsilon\|^2_{\mathrm{L}^2} - \varepsilon\|w'_{\varepsilon,\delta}\|^2_{\mathrm{L}^2}| \le \delta\varepsilon^{\frac{1}{2}}\Big(2\varepsilon^{\frac{1}{2}}\|u'_\varepsilon\|_{\mathrm{L}^2} + \delta\varepsilon^{\frac{1}{2}}\Big) \le \delta\varepsilon^{\frac{1}{2}}\Big(2(M+1)^{\frac{1}{2}} + \delta\varepsilon^{\frac{1}{2}}\Big)$. Thus, we have

$$|\varepsilon^{-1}I_0^\varepsilon(u_\varepsilon) - \varepsilon^{-1}I_0^\varepsilon(w_{\varepsilon,\delta})| \le \delta\varepsilon^{\frac{1}{2}}\Big(2(M+1)^{\frac{1}{2}} + \delta\varepsilon^{\frac{1}{2}}\Big) + \varepsilon^{-1}\omega_\varepsilon(C\delta).$$

By the construction we can choose $\delta_\varepsilon^{(1)} > 0$ ($\delta_\varepsilon^{(2)}$, resp.) such that it holds that $\delta_\varepsilon^{(1)} \le \min\{\frac{\varepsilon}{C}, \frac{1}{C}\omega_\varepsilon^{-1}(\frac{\varepsilon^2}{2})\}$ ($\delta_\varepsilon^{(2)} \le \varepsilon^{\frac{1}{2}}2^{-1}\Big(2(M+1)^{\frac{1}{2}} + 1\Big)^{-1}$, resp.), and the proof is completed provided that we define $\overline{u}_\varepsilon := w_{\varepsilon,\delta_\varepsilon}$, where $\delta_\varepsilon := \min\{\delta_\varepsilon^{(1)}, \delta_\varepsilon^{(2)}, \varepsilon\}$. $\square$

Corollary 6. *Consider sequences (u_ε) and $(\overline{u}_\varepsilon)$ as in Proposition 2. Then the following holds:*

(i) (u_ε) *is normal iff* $(\overline{u}_\varepsilon)$ *is normal,*

(ii) (u_ε) *is regular iff* $(\overline{u}_\varepsilon)$ *is regular.*

Moreover, for a given $1 \le p < +\infty$ *we have the following:*

(iii) (u_ε) *is p-equi-integrable on* $(0,1)$ *iff* $(\overline{u}_\varepsilon)$ *is p-equi-integrable on* $(0,1)$,

(iv) (u_ε) *is bounded (strongly pre-compact, resp.) in* $\mathrm{L}^p(0,1)$ *iff* $(\overline{u}_\varepsilon)$ *is bounded (strongly pre-compact, resp.) in* $\mathrm{L}^p(0,1)$.

Proof. Since it holds that $\|u_\varepsilon - \overline{u}_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \le \varepsilon$ ($|u_\varepsilon(c_\varepsilon) - \overline{u}_\varepsilon(c_\varepsilon)| \le \max\{|u_\varepsilon(s) - \overline{u}_\varepsilon(s)| : s \in [0,1]\} \le \varepsilon$, where (c_ε) is chosen as in Definition 1, resp.), it follows that the sequence (u_ε) is regular (normal, resp.) iff the sequence $(\overline{u}_\varepsilon)$ is regular (normal, resp.). The same argument applies to assertions (iii) and (iv). $\square$

Corollary 7. *Consider sequences* (u_ε) *and* $(\overline{u}_\varepsilon)$ *as in Proposition 2. Then the following holds:*

(i) (u_ε) *is weakly pre-compact in* $\mathrm{L}^p(0,1)$ *iff* $(\overline{u}_\varepsilon)$ *is weakly pre-compact in* $\mathrm{L}^p(0,1)$*, where* $1 < p < +\infty$*,*

(ii) (u_ε) *is weakly* pre-compact in* $\mathrm{L}^\infty(0,1)$ *iff* $(\overline{u}_\varepsilon)$ *is weakly* pre-compact in* $\mathrm{L}^\infty(0,1)$*,*

(iii) (u_ε) *is weakly pre-compact in* $\mathrm{L}^1(0,1)$ *iff* $(\overline{u}_\varepsilon)$ *is weakly pre-compact in* $\mathrm{L}^1(0,1)$.

Proof. Assertion (i) follows from the estimate $\|u_\varepsilon - \overline{u}_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \le \varepsilon$ and weak pre-compactness of bounded sets in $\mathrm{L}^p(0,1)$, where $1 < p < +\infty$. Quite in the same way, we infer conclusion (ii), taking into account weak* pre-compactness of bounded sets in $\mathrm{L}^\infty(0,1)$. Finally, assertion (iii) follows from Corollary 6(iv), from equivalence of equi-integrability and uniform integrability for bounded sets in $\mathrm{L}^p(0,1)$(cf. [13], Theorem 2.29), from the Dunford-Pettis Theorem (cf. [13], Theorem 2.54) and from the estimate $\|u_\varepsilon - \overline{u}_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \le \varepsilon$. $\square$

Corollary 8. *If a subsequence of* (u_ε) *(not relabeled) satisfies* $u'_\varepsilon \in \mathrm{L}^{\frac{1}{1-\rho}}(0,1)$ *for some* $0 < \rho < 1$*, then under the assumptions of Proposition 2, for a sufficiently small* $\varepsilon_0 > 0$ *and for every* $0 < \varepsilon \le \varepsilon_0$*, the sequence* $(\overline{u}_\varepsilon)$ *satisfies* $\overline{u}'_\varepsilon \in \mathrm{L}^{\frac{1}{1-\rho}}(0,1)$*,* $\|u'_\varepsilon - \overline{u}'_\varepsilon\|_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)} \le \varepsilon$ *and* $\int_0^1 |\overline{u}'_\varepsilon|^{\frac{1}{1-\rho}} \le 2^{\frac{\rho}{1-\rho}}\left(\varepsilon^{\frac{1}{1-\rho}} + \int_0^1 |u'_\varepsilon|^{\frac{1}{1-\rho}}\right)$.

Proof. By Corollary 6.6.2 in [9], p. 256, we can achieve $\|u'_\varepsilon - \overline{u}'_\varepsilon\|_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)} \le \varepsilon$, so that

$$\left|\|u'_\varepsilon\|_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)} - \|\overline{u}'_\varepsilon\|_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)}\right| \le \varepsilon.$$

As a result, we obtain

$$\|\overline{u}'_\varepsilon\|^{\frac{1}{1-\rho}}_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)} \le \left(\varepsilon + \|u'_\varepsilon\|_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)}\right)^{\frac{1}{1-\rho}} \le 2^{\frac{\rho}{1-\rho}}\left(\varepsilon^{\frac{1}{1-\rho}} + \|u'_\varepsilon\|^{\frac{1}{1-\rho}}_{\mathrm{L}^{\frac{1}{1-\rho}}(0,1)}\right).$$

$\square$

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