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An accurate numerical algorithm based on the generalized Narayana polynomials to solve a class of Caputo-Fabrizio and Liouville-Caputo fractional-order delay differential equations

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Abstract. A spectral collocation-based approximate algorithm is adapted for the numerical evaluation of a class of fractional-order delay differential equations. The involved fractional operators are defined as the Caputo-Fabrizio and Liouville-Caputo derivatives. A novel family of polynomials called Narayana polynomials and their generalization forms are utilized in our collocation procedure. We study the convergent analysis of the Narayana polynomials in a weighted L_2 norm and obtain an upper bound for their series expansion form. The performance of the present matrix collocation is justified by solving three test examples using both fractional operators and various fractional orders. The outcomes are compared with two existing numerical approaches, i.e., the modified operational matrix method (MOMM) and the operational matrix of integration relied on Taylor bases (OMTB). A comparison study reveals that our results are more accurate than these two methods and thus the presented matrix algorithm is superior in terms of efficacy and applicability.

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1. Introduction

The notion of fractional calculus can be interpreted as the integration and differentiation of an arbitrary order. The history of the invention of fractional calculation goes back about three centuries ago when the meaningful interpretation of one-half order derivative was asked by L'Hospital from Leibniz [35]. However, reuse of the

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concept of fractional integral and derivative goes back to the last few decades, which has been considered in mathematical modeling of many natural and physical phenomena [12]. Classical forms of fractional integration and differentiation are defined by the Riemann-Liouville (RL) fractional integral operator. A limitation of the RL derivative operator is that the derivative of a constant is not zero. Later on, the Caputo fractional derivative or more appropriately, the Liouville-Caputo (LC) operator was introduced to overcome this limitation [21, 29]. Note that in both RL and LC derivatives, the involving integral operator is singular. In order to have a regular kernel, the concept of the Caputo-Fabrizio (CF) fractional operator is introduced in [10]. In addition to these fractional operators, let us mention some other fractional derivative and integral operators such as Grünwald-Letnikov, Riesz, Prabhakar, Hadamard, and AtanganaBaleanu, to name a few. Although all of these fractional operators are useful, each of them has some disadvantages.

The delay and pantograph differential equations (DEs) have been widely used to model systems arising in diverse disciplines of applied life science including population dynamics, epidemiology, immunology, physiology, and neural networks, see [27, 3, 33]. In delay DEs, the evolution of the state variable is not only dependent on itself at a certain time but also at some past times. This implies that the function values depend on their history or memory in such DEs. On the other hand, modeling DEs with fractional-order derivatives provides the concept of memory in maintaining the fundamental properties of the understudied events from the origin of time to the desired time. We also note that the fractional-order derivatives and time-delay have similar properties in the models [32]. Therefore, delay fractional DEs with such two kinds of memories in underlying models will become definitely more complicated and in return contain more information about the models. In what follows, we are interested in acquiring approximate solutions of a class of delay differential equations with fractional order given by [37]:

$$\begin{cases} D_\star^\rho\, w(z) = w(z) + q\, w(qz) + g(z), \\ w(0) = w_0, \end{cases} \qquad z \in [0,1], \tag{1}$$

where $\rho, q \in (0,1)$, the initial value w_0 is a given constant and the function $g(z)$ is known. Here, $D_\star^\rho$ signifies the fractional operator interpreted in the sense of Liouville-Caputo denoted by ${}^{LC}D_z^\rho$ or represents the Caputo-Fabrizio fractional derivative (CFFD) denoted by ${}^{CF}D_z^\rho$. For brevity, we also write the LCFD for the Liouville-Caputo fractional derivative. Utilizing the fixed point theorems and in the case of the LCFD, the question of the existence of such delay and pantograph equations was answered in [8]. Moreover, in [9], the authors investigated the stability analysis of a class of delay differential equations, while the existence of positive solutions were discussed in [22]. On the other hand, the existence and uniqueness of solutions related to a class of implicit fractional differential equations with the CFFD were established in [11] via the fixed point theorems of Krasnoselskii and Banach. Considering a similar delay and pantograph utilizing the Atangana-Baleanu-Caputo, Hilfer-Hadamard, and fractional-fractal derivatives can be found in [1, 38, 31].

This model problem considering the LCFD was first studied in [37]. The authors solve (1) computationally with a matrix method called the modified operational matrix method (MOMM) as the only available numerical method (to the best of our knowledge). A lot of research attention has been devoted to the delay and pantograph DEs. In this respect, one can find many numerical and analytical approaches to the closely related models of (1) in the literature. As some examples, we may mention the differential transform approach [20], the multi-wavelet Galerkin scheme [25], the Vieta-Fibonacci wavelet technique [6], the Haar wavelet procedure [14], the collocation-based procedures using the orthoexponential funtion [7], Bernoulli [2], Bessel [15], Taylor [5], alternative Laguerre [16], Lucas [39], Vieta-Lucas [18], and Chelyshkov polynomials [19]. Other recent approaches are those related to neural network methodologies, see [13, 30].

The main features of the current work to solve the delay model (1) can be mentioned as follows. In addition to the LCFD, we consider the CFFD for this model (for the first time) and find the approximate solutions through an efficient matrix collocation algorithm with high-order accuracy compared to existing well-established methods in the literature. On the other hand, we use a novel family of polynomial functions called the Narayana polynomials (NPs) in our proposed matrix algorithm. The main characteristic of NPs is that their coefficients are all positive in comparison with the traditional set of (orthogonal) polynomials. Additionally, a generalized form of these polynomials named GNPs is introduced to generate more accurate results as the output of the proposed collocation matrix algorithm. We also establish the convergence of GNPs in a rigorous manner. The presented GNPs collocation procedure not only produces high-order accurate outcomes but also is simple in terms of implementation as described in Algorithm 2 below.

This research paper is organized as follows. We provide supplementary results from fractional calculus that will be useful for the rest of the paper in Section 2. Section 3 consists of a detailed description of the novel NPs polynomials and their generalized form. After that, we establish the convergence properties of GNPs series in a weighted L_2 norm. The main steps of the GNPs matrix collocation algorithm are given in Section 4 and all steps are summarized in Algorithm 2. The performance of the GNPs algorithm is tested by solving three test cases and validate through comparisons with two existing well-established operational matrix methods in Section 5. Finally, the conclusion of the study is given in Section 6.

2. Fractional Liouville-Caputo and Caputo-Fabrizio derivatives

In the subsequent sections, we need some facts of fractional calculus. In this respect, we review definitions of both the Liouville-Caputo and the Caputo-Fabrizio fractional operators. Some of their properties are also given. For more information related to CFFD and LCFD, we refer to [10, 21, 29, 34].

Definition 1. *The LCFD ${}^{LC}D_z^\rho$ of order $\rho > 0$ of function $w(z)$ is defined as follows:*

$$ {}^{LC}D_z^\rho\, w(z) = \begin{cases} \mathcal{I}^{s-\rho} w^{(s)}(z) & \text{if } s-1 < \rho < s, \\ w^{(s)}(z), & \text{if } \rho = s, \quad s \in \mathbb{N}, \end{cases} $$

where $\mathcal{I}^\rho w(z) = \frac{1}{\Gamma(\rho)} \int_0^z \frac{w(r)}{(z-r)^{1-\rho}}\, dr$ for $z > 0$. Here, we assumed that $w(z)$ is an s-times continuously differentiable function.

The linearity of the LCFD is an important property that will be used below. The other frequently used properties are as follows:

$$ {}^{LC}D_z^{\rho}(c) = 0, \quad (c \text{ is a constant}), \tag{2} $$

$$ {}^{LC}D_z^{\rho}\, z^k = \begin{cases} 0, & \text{for } k \in \mathbb{N}_0 \text{ and } k < \lceil \rho \rceil, \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\rho)} z^{k-\rho}, & \text{for } k \in \mathbb{N}_0 \text{ and } k \geq \lceil \rho \rceil, \text{ or } k \notin \mathbb{N}_0 \text{ and } k > \lfloor \rho \rfloor. \end{cases} \tag{3} $$

Definition 2. *Let us suppose that $w \in H^1(c,d)$, $c > d$, is a given function. The CFFD ${}^{CF}D_z^{\rho}$ of order $0 < \rho < 1$ of function $w(z)$ is defined as follows:*

$$ {}^{CF}D_z^{\rho}\, w(z) = \frac{K(\rho)}{1-\rho} \int_c^z w'(r)\, e^{-\eta_\rho (z-r)}\, dr, $$

where $\eta_\rho := \frac{\rho}{1-\rho}$ and $w'(z) := \frac{d}{dz} w(z)$ is the standard first-order derivative. Here, by $K(\rho)$ we denote a normalized function satisfying $K(0) = K(1) = 1$.

In addition to the linear property of the CFFD, we utilize the following two of its important properties as follows, see [24, 4]:

$$ {}^{CF}D_z^{\rho}(c) = 0, \quad (c \text{ is a constant}), \tag{4} $$

$$ {}^{CF}D_z^{\rho}\,(z^k) = $$
$$ \begin{cases} \frac{1}{\rho}\left(1 - e^{-\eta_\rho\, z}\right), & \text{if } k = 1, \\ \frac{1}{1-\rho} \sum_{j=1}^{k-1} (-1)^{j+1} \frac{k!}{(k-j)!} \left(\frac{1}{\eta_\rho}\right)^j z^{k-j} + (-1)^{k-1} k! \left(\frac{1}{\eta_\rho}\right)^{k-1} {}^{CF}D_z^{\rho}\,(z), & \text{if } k > 1, \end{cases} \tag{5} $$

where $k \in \mathbb{N}$. We next set

$$ C_m^k := \binom{m}{k}, \quad \phi(\rho) := \frac{1-\rho}{K(\rho)}, \quad \psi(\rho) := \frac{\rho}{K(\rho)}. $$

To proceed, we need the following result related to the CFFD, proofs of which can be found in [40].

Theorem 1. *Let $0 \leq \rho \leq 1$ be a real number. Assume further that ${}^{CF}D_z^{\ell\rho} w(z) \in C([c,d])$ for $\ell = 0, 1, \ldots, M \in \mathbb{N}$. Then, the function $w(z)$ has the following fractional power series form:*

$$ w(z) := \mathcal{W}_M(z) + \mathcal{R}_M(z), \tag{6} $$

where

$$ \mathcal{W}_M(z) := \sum_{\ell=0}^{M-1} \left({}^{CF}D_z^{\ell\rho} w(c)\right) \left[\sum_{j=0}^{\ell} C_\ell^j\, \phi^{\ell-j}(\rho)\, \psi^j(\rho) \frac{(z-c)^{j\rho}}{\Gamma(1+j\rho)} \right], $$
$$ \mathcal{R}_M(z) := \left({}^{CF}D_z^{M\rho} w(\eta)\right) \sum_{j=0}^{M} C_M^j\, \phi^{M-j}(\rho)\, \psi^j(\rho) \frac{(z-c)^{j\rho}}{\Gamma(1+j\rho)}, $$

for a $c \leq \eta \leq z$ and $z \in [c,d]$. Here, we have ${}^{CF}D_z^{\ell\rho} = {}^{CF}D_z^{\rho}\,{}^{CF}D_z^{\rho} \dots {}^{CF}D_z^{\rho}$ (ℓ times). The next result can be stated as follows:

Corollary 1 (Fractional Taylor inequality)**.** *Under assumptions of Theorem 1, let $U_1, U_2 > 0$ and $M \in \mathbb{N}$ be three numbers such that $\left|{}^{CF}D_z^{M\rho} w(z)\right| \leq U_1$ and $U_2 := \min_{\rho\in[0,1]} K(\rho)$. Then, the remainder error of the Taylor series* (6) *satisfies*

$$|\mathcal{R}_M(z)| = |w(z) - \mathcal{W}_M(z)| \leq U_1 \left(\frac{2}{U_2}\right)^M \sum_{j=0}^{M} \frac{(z-c)^{j\rho}}{\Gamma(1+j\rho)}, \quad \forall z \in [c,d].$$

Proof. Clearly, we have $0 \leq 1-\rho \leq 1$. By considering the following facts:

$$\phi(\rho), \psi(\rho) \leq \frac{1}{U_2}, \quad \text{and} \quad C_\ell^j \leq 2^\ell, \quad \forall\ j,$$

the proof can be easily done. □

3. Generalized Narayana polynomials and their convergence results

Let us first consider the definition of Narayana polynomials. Some aspects of these polynomials are then reviewed and their generalized forms are introduced. We finally give a detailed information related to the convergence analysis of the generalized Narayana polynomials below.

3.1. Narayana polynomials

Narayana polynomials (NPs) are related to Nayarana numbers with many great applications in combinatorial analysis. Although Nayarana numbers were first introduced by MacMahon [26], they were later retrieved by Narayana [23] and named in his honor [28]. The sequence of Narayana numbers $\{N_{m,j}\}_{m,j\in\mathbb{N}_0}$, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ is defined by [28]

$$N_{m,j} = \frac{1}{m}\binom{m}{j-1}\binom{m}{j}, \quad m, j \in \mathbb{N},$$

and with $N_{0,j} = N_{j,0} = \delta_{j0}$ for $j \in \mathbb{N}_0$ and δ_{rs} standing for the Kronecker delta function. Based on this sequence of numbers, the associated NPs are defined as follows:

Definition 3. *The (modified) Narayana polynomial $\mathcal{N}_m(s)$ of degree m is defined by*

$$\mathcal{N}_m(s) = \sum_{j=0}^{m} N_{m+1,j+1}\, s^j = \sum_{j=0}^{m} \frac{1}{m+1}\binom{m+1}{j}\binom{m+1}{j+1} s^j, \quad m \in \mathbb{N}_0. \tag{7}$$

The list of the first five NPs is obtained as follows: $\mathcal{N}_0(s) = 1$ and

$$\begin{aligned} &\mathcal{N}_1(s) = 1 + s, && \mathcal{N}_2(s) = 1 + 3s + s^2,\\ &\mathcal{N}_3(s) = 1 + 6s + 6s^2 + s^3, && \mathcal{N}_4(s) = 1 + 10s + 20s^2 + 10s^3 + s^4. \end{aligned}$$

One can obviously observe that we have $\mathcal{N}_m(0) = 1$ for all $m \geq 0$. Let $M \geq 1$ be a given integer. In the next lemma we provide an expression for the vector of NPs.

Lemma 1. *The vector of NPs is denoted by* $\boldsymbol{N}_M(s) := [\mathcal{N}_0(s) \quad \mathcal{N}_1(s) \quad \dots \quad \mathcal{N}_M(s)]$ *and is written as*

$$\boldsymbol{N}_M(s) = \boldsymbol{S}_M(s)\, \boldsymbol{D}_M, \tag{8}$$

where the vector of monomials $\boldsymbol{S}_M(s)$ *is*

$$\boldsymbol{S}_M(s) = \begin{bmatrix} 1 & s & s^2 & \dots & s^M \end{bmatrix},$$

and the upper-triangular matrix $\boldsymbol{D}_M$ *is*

$$\boldsymbol{D}_M = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & N_{3,2} & \dots & N_{M,2} & N_{M+1,2} \\ 0 & 0 & 1 & \dots & N_{M,3} & N_{M+1,3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & N_{M+1,M} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Proof. The proof can be deduced by induction on $M \in \mathbb{N}$ in an easy way. $\square$

Note that the elements of the first row of $\boldsymbol{D}_M$ are all one. Also, $N_{j,j} = 1$ for $j = 0, \dots, M + 1$ as they appeared on the diagonal of $\boldsymbol{D}_M$. In addition, the matrix $\boldsymbol{D}_M$ is non-singular owing to the fact that $\det(\boldsymbol{D}_M) = 1$.

3.2. Generalized NPs

We begin with a generalization of NPs, which are very useful in our applications below.

Definition 4. *Let* $\lambda \in (0, 1)$. *Generalized Narayana polynomials (GNPs)* $\mathcal{N}^{\lambda}_m(z)$ *of degree* m *are obtained by changing variable* $s = z^{\lambda}$ *in the NPs given by*

$$\mathcal{N}^{\lambda}_m(z) = \mathcal{N}_m(z^{\lambda}), \quad m \in \mathbb{N}_0. \tag{9}$$

Owing to (7) and using this transformation in (9), the explicit form of GNPs is

$$\mathcal{N}^{\lambda}_m(z) = \sum_{j=0}^{m} \frac{1}{m+1} \binom{m+1}{j} \binom{m+1}{j+1} z^{\lambda\, j}, \quad m \in \mathbb{N}_0.$$

Similarly, the vector of GNPs $\boldsymbol{N}_M^\lambda(z) := \begin{bmatrix}\mathcal{N}_0^\lambda(z) & \mathcal{N}_1^\lambda(z) & \dots & \mathcal{N}_M^\lambda(z)\end{bmatrix}$ can be obtained. In fact, we use (8) to get

$$\boldsymbol{N}_M^\lambda(z) := \boldsymbol{N}_M(z^\lambda) = \boldsymbol{S}_M^\lambda(z)\,\boldsymbol{D}_M. \tag{10}$$

Here, we have

$$\boldsymbol{S}_M^\lambda(z) = \begin{bmatrix}1 & z^\lambda & z^{2\lambda} & \dots & z^{M\lambda}\end{bmatrix},$$

and $\boldsymbol{D}_M$ is as defined in (8).

3.3. Convergence of GNPs in L_2 norm

Note that in this paper we will consider the sequence of GNPs on $[0,1]$. Clearly, all of these polynomials are positive on this interval. We are going to investigate the convergence properties of the GNPs in a rigorous manner. It should be emphasized that here we only consider the CFFD. For similar arguments related to the LCFD we refer to [17, 36]. Let us assume that a function $w(z) \in L_2([0,1])$ is given. We are then capable of representing $w(z)$ as a linear of combination of GNPs as follows:

$$w(z) = \sum_{m=0}^{\infty} \pi_m \mathcal{N}_m^\lambda(z), \quad z \in [0,1].$$

Here, we seek for the unknowns π_m, $m \geq 0$. For practical computing, we restrict our discussion to a finite-dimensional subspace $Y_M^\lambda \subseteq L_2([0,1])$ defined as

$$Y_M^\lambda := \mathrm{Span}\langle \mathcal{N}_0^\lambda(z), \mathcal{N}_1^\lambda(z), \dots, \mathcal{N}_M^\lambda(z)\rangle.$$

It follows that we take only the first $(M+1)$ GNPs and approximate $w(z)$ as

$$w(z) \approx w_M^\lambda(z) := \sum_{m=0}^{M} \pi_m \mathcal{N}_m^\lambda(z), \quad z \in [0,1]. \tag{11}$$

In a concise format, one may write the approximate solution $w_M^\lambda(z)$ in the form

$$w_M^\lambda(z) = \boldsymbol{N}_M^\lambda(z)\,\boldsymbol{\Pi}_M^\lambda, \tag{12}$$

where the vector of unknowns π_m is introduced as

$$\boldsymbol{\Pi}_M^\lambda := \begin{bmatrix}\pi_0 & \pi_1 & \dots & \pi_M\end{bmatrix}^T.$$

The next result asserts that by increasing M the difference between $w(z)$ and its approximation $w_M^\lambda(z)$ in the series form (11) will tend to zero. To continue, we define $E_M^\lambda(z) := w(z) - w_M^\lambda(z)$ and by $\|\cdot\|_2$ we denote the 2-norm on $[0,1]$.

Theorem 2. *Let us assume that for $\ell = 1, 2, \dots, M+1$ we have ${}^{CF}D_z^{\ell\,\lambda}(z) \in C[0,1]$. Further, suppose that $w_M^\lambda(z) = \boldsymbol{N}_M^\lambda(z)\,\boldsymbol{\Pi}_M^\lambda$ represents the (finest) best approximation to $w(z)$ in the space Y_M^λ. Then, the following upper bound for the error $E_M^\lambda(z)$ is valid:*

$$\|E_M^\lambda(z)\|_2 \leq \frac{C_{\max}}{\sqrt{2\lambda(M+1)+1}} \left(\frac{2}{K_{\min}}\right)^{M+1} \sum_{j=0}^{M+1} \frac{1}{\Gamma(1+j\lambda)},$$

where $\left|{}^{CF}D_z^{(M+1)\lambda} w(z)\right| \leq C_{\max}$, $\forall z \in [0,1]$ *and* $K_{\min} \geq |K(\lambda)|$ *for any* $\lambda \in [0,1]$.

Proof. Since $0 < \lambda \leq 1$, we can employ Theorem 1 for the function $w(z)$ by setting $c = 0, d = 1$. Let us consider the M terms representation to find that

$$\mathcal{W}_{M+1}(z) = \sum_{\ell=0}^{M} \left({}^{CF}D_z^{\ell\lambda} w(0) \right) \left[\sum_{j=0}^{\ell} C_\ell^j \, \phi^{\ell-j}(\lambda) \, \psi^j(\lambda) \frac{z^{j\lambda}}{\Gamma(1+j\lambda)} \right], \quad z \in [0,1].$$

Owing to Corollary 1, we get the upper bound

$$\begin{aligned} |w(z) - \mathcal{W}_M(z)| &\leq C_{\max} \left(\frac{2}{K_{\min}} \right)^{M+1} \sum_{j=0}^{M+1} \frac{z^{j\lambda}}{\Gamma(1+j\lambda)} \\ &\leq C_{\max} \left(\frac{2}{K_{\min}} \right)^{M+1} \sum_{j=0}^{M+1} \frac{z^{(M+1)\lambda}}{\Gamma(1+j\lambda)}, \quad \forall z \in [0,1]. \end{aligned} \tag{13}$$

We now utilize the fact that $w_M^\lambda(z)$ is the finest approximation to $w(z)$ out of Y_M^λ. Consequently, one finds that

$$\|w(z) - w_M^\lambda(z)\|_2 \leq \|w(z) - v(z)\|_2, \quad \forall v \in Y_M^\lambda.$$

A special choice for $v(z)$ is $\mathcal{W}_M(z)$ in the last inequality. Therefore, we conclude

$$\|w(z) - w_M^\lambda(z)\|_2^2 \leq \|w(z) - \mathcal{W}_M(z)\|_2^2 = \int_0^1 |w(z) - \mathcal{W}_M(z)|^2 \, dz.$$

Now, by virtue of (13) we arrive at

$$\|w(z) - w_M^\lambda(z)\|_2^2 \leq \left[C_{\max} \left(\frac{2}{K_{\min}} \right)^{M+1} \sum_{j=0}^{M+1} \frac{1}{\Gamma(1+j\lambda)} \right]^2 \int_0^1 z^{2(M+1)\lambda} dz.$$

The desired conclusion is obtained after evaluating the definite integral followed by taking the square root. □

4. GNPs matrix collocation strategy

As previously mentioned, we aimed to write the solution of delay model (1) in the form (11). To proceed, we can write the approximate solution $w_M^\lambda(z)$ as a combination of two relations (10) and (12). Therefore, we have

$$w_M^\lambda(z) = \boldsymbol{N}_M^\lambda(z) \, \boldsymbol{\Pi}_M^\lambda = \boldsymbol{S}_M^\lambda(z) \, \boldsymbol{D}_M \, \boldsymbol{\Pi}_M^\lambda. \tag{14}$$

To find the $(M+1)$ unknown coefficients π_m, one needs $(M+1)$ collocation points on $[0,1]$ to be utilized in the GNPs matrix collocation methodology. Let $\mathcal{C}_M$ denote the set of collocation nodes with equally spaced points defined by

$$\mathcal{C}_M := \{ z_m = m/M \mid m = 0, 1, \ldots, M \}. \tag{15}$$

By evaluating the approximate solution (14) at the elements of $\mathcal{C}_M$, the following result is obtained, the proof of which is straightforward.

Lemma 2. *The matrix representation of* (14) *can be written as*

$$\boldsymbol{W}_M = \boldsymbol{S}_M\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}, \tag{16}$$

where

$$\boldsymbol{W}_M = \begin{pmatrix} w_M^{\lambda}(z_0) \\ w_M^{\lambda}(z_1) \\ \vdots \\ w_M^{\lambda}(z_M) \end{pmatrix}, \quad \boldsymbol{S}_M = \begin{pmatrix} \boldsymbol{S}_M^{\lambda}(z_0) \\ \boldsymbol{S}_M^{\lambda}(z_1) \\ \vdots \\ \boldsymbol{S}_M^{\lambda}(z_M) \end{pmatrix}.$$

In the next stage, our goal is to constitute the matrix form of $w(qz)$ in (1). First of all, note that owing to (14) we arrive at

$$w_M^{\lambda}(qz) = \boldsymbol{S}_M^{\lambda}(qz)\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}. \tag{17}$$

It is an easy task to see that

$$\boldsymbol{S}_M^{\lambda}(qz) = \boldsymbol{S}_M^{\lambda}(z)\, \boldsymbol{E}_{q,\lambda}, \quad \boldsymbol{E}_{q,\lambda} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & q^{\lambda} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q^{M\lambda} \end{pmatrix}_{(M+1)\times(M+1)}. \tag{18}$$

By combining two preceding relations (14) and (17), we get the following formula:

$$w_M^{\lambda}(qz) = \boldsymbol{S}_M^{\lambda}(z)\, \boldsymbol{E}_{q,\lambda} \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}. \tag{19}$$

Lemma 3. *At the collocation points* (15)*, the matrix representation of* (19) *can be written as*

$$\boldsymbol{W}_{M,q} = \boldsymbol{S}_M\, \boldsymbol{E}_{q,\lambda}\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}, \tag{20}$$

where the vector $\boldsymbol{S}_M$ *and the matrix* $\boldsymbol{E}_{q,\lambda}$ *are defined in* (16) *and* (18)*, respectively, and we also have*

$$\boldsymbol{W}_{M,q} = \begin{pmatrix} w_M^{\lambda}(qz_0) \\ w_M^{\lambda}(qz_1) \\ \vdots \\ w_M^{\lambda}(qz_M) \end{pmatrix}.$$

In the ultimate step, we derive matrix forms for the fractional operator $D_\star^{\rho}\, w(z)$ as the CFFD and the LCFD. By computing the ρ-LC/FF derivative of the approximate solution $w_M^{\lambda}(z)$ in (14) we get

$$D_\star^{\rho}\, w_M^{\lambda}(z) = \left(D_\star^{\rho}\, \boldsymbol{S}_M^{\lambda}(z)\right)\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}. \tag{21}$$

Depending on the fractional derivative $D_\star^{\rho}$, the following two cases may take place:

a) First, we consider the LCFD ${}^{LC}D_z^{\rho}\, w(z)$. Therefore, it would be sufficient to calculate the ρ-LC derivative of $\boldsymbol{S}_M^{\lambda}(z)$. To this end, properties (2) and (3) need to be taken into account. Upon referring to Algorithm 1, we obtain the derivative vector $\boldsymbol{S}_M^{\rho,\lambda}(z)$ defined mathematically as

$$\boldsymbol{S}_M^{\rho,\lambda}(z) := {}^{LC}D_z^{\rho}\, \boldsymbol{S}_M^{\lambda}(z).$$

As an illustration, by choosing $M = 3$, $\lambda = 1/2$, and $\rho = 3/4$ we get

$$\boldsymbol{S}_3^{\frac{3}{4},\frac{1}{2}}(z) = \begin{bmatrix} 0 & 0 & \dfrac{1}{\Gamma(5/4)} z^{\frac{1}{4}} & \dfrac{\Gamma(5/2)}{\Gamma(7/4)} z^{\frac{3}{4}} \end{bmatrix}.$$

One can easily verify that the cost of Algorithm 1 is linear $\mathcal{O}(M+1)$.

procedure $[\boldsymbol{S}_M^{\rho,\lambda}]$= calculate_DS$(\rho, \lambda, M)$
$\boldsymbol{S}_M^{\rho,\lambda}[1] := 0;$
for $m := 1, \ldots, M$ **do**
    if $(m\lambda - \rho \ < \ 0)$ **then**
        $\boldsymbol{S}_M^{\rho,\lambda}[m+1] := 0;$
    else
        if $((m\lambda < \lceil \rho \rceil)$ && $(m\alpha - \lfloor m\lambda \rfloor == 0))$ **then**
            $\boldsymbol{S}_M^{\rho,\lambda}[m+1] := 0;$
        else
            $\boldsymbol{S}_M^{\rho,\lambda}[m+1] := \dfrac{\Gamma(m\lambda+1)}{\Gamma(m\lambda+1-\rho)} z^{m\lambda-\rho};$
        end if
    end if
end for
end;

Algorithm 1: *Calculating the ρ-LC derivative of $\boldsymbol{S}_M^{\lambda}(z)$*

b) In the second case, our fractional derivative is a CF operator. As mentioned above, one only needs to calculate the ρ-CF derivative of the vector $\boldsymbol{S}_M^{\lambda}(z)$. Since computing the ρ-derivative is not an easy task in this case, we consider only $\lambda = 1$ here. Thus, the ρ-CF derivative of $\boldsymbol{S}_M^{\lambda}(z)$ is obtained via using two properties (4) and (5). It follows that

$$\boldsymbol{S}_M^{\rho,1}(z) := {}^{CF}D_z^{\rho}\, \boldsymbol{S}_M^{1}(z) = \begin{bmatrix} 0 & {}^{CF}D_z^{\rho}\, z & {}^{CF}D_z^{\rho}\, z^2 & \ldots & {}^{CF}D_z^{\rho}\, z^M \end{bmatrix}. \tag{22}$$

In either case, a) or b), we can draw the following relation from (21):

$$D_\star^{\rho}\, w_M^{\lambda}(z) = \boldsymbol{S}_M^{\rho,\lambda}(z)\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}. \tag{23}$$

In summary, we have the next matrix representation for the fractional derivatives:

Lemma 4. *At the collocation points* (15)*, the matrix representation of* (23) *can be written as*

$$\boldsymbol{W}_M^{\rho,\lambda} = \mathbb{S}_M^{\rho,\lambda}\, \boldsymbol{D}_M\, \boldsymbol{\Pi}_M^{\lambda}, \tag{24}$$

where

$$\boldsymbol{W}_M^{\rho,\lambda} = \begin{pmatrix} D_\star^\rho\, w_M^\lambda(z_0) \\ D_\star^\rho\, w_M^\lambda(z_1) \\ \vdots \\ D_\star^\rho\, w_M^\lambda(z_M) \end{pmatrix}, \quad \mathbb{S}_M^{\rho,\lambda} = \begin{pmatrix} \boldsymbol{S}_M^{\rho,\lambda}(z_0) \\ \boldsymbol{S}_M^{\rho,\lambda}(z_1) \\ \vdots \\ \boldsymbol{S}_M^{\rho,\lambda}(z_M) \end{pmatrix}.$$

We are now ready to construct the fundamental matrix equation by inserting the the set of collocation points $\mathcal{C}_M$ defined in (15) into the model problem (1). We have

$$D_\star^\rho\, w(z_m) - w(z_m) - q\, w(qz_m) = g(z_m), \quad z_m \in \mathcal{C}_M.$$

Utilizing the matrix format, we are able to write the last set of equations compactly as

$$\boldsymbol{W}_M^{\rho,\lambda} - \boldsymbol{W}_M - \boldsymbol{Q}_M\, \boldsymbol{W}_{M,q} = \boldsymbol{G}_M. \tag{25}$$

Here, the constant matrix $\boldsymbol{Q}_M$ on the left-hand side of the equality and the vector $\boldsymbol{G}_M$ on the right-hand side are defined by

$$\boldsymbol{Q}_M = \begin{pmatrix} q & 0 & \dots & 0 \\ 0 & q & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q \end{pmatrix}, \quad \boldsymbol{G}_M = \begin{pmatrix} g(z_0) \\ g(z_1) \\ \vdots \\ g(z_M) \end{pmatrix}.$$

Next, we substitute the obtained relations (16), (20), and (24) into (25). The resultant fundamental matrix equations has the following structure:

$$\boldsymbol{V}_M\, \boldsymbol{\Pi}_M^\lambda = \boldsymbol{G}_M, \quad \text{or} \quad [\boldsymbol{V}_M; \boldsymbol{G}_M], \tag{26}$$

where $\boldsymbol{V}_M := \left(\mathbb{S}_M^{\rho,\lambda} - Q_M\, S_M\, E_{q,\lambda} - \boldsymbol{S}_M\right) \boldsymbol{D}_M$.

The final matrix equation (26) is obviously a linear system with $(M+1)$ unknowns π_m for $m = 0, 1, \dots, M$ to be found as the GNPs coefficients. Still, the initial condition $w(0) = w_0$ is not implemented and entered into (26). Let us pay attention to the matrix form of the approximate solution in (14). We let $z \to 0$ to reach at

$$\boldsymbol{V}_{M,0}\, \boldsymbol{\Pi}_M^\lambda = \boldsymbol{G}_{M,0}, \qquad \boldsymbol{V}_{M,0} := \boldsymbol{S}_M^\lambda(0)\, \boldsymbol{D}_M.$$

For convenience, we replace the first row of matrix $[\boldsymbol{V}_M; \boldsymbol{G}_M]$ by the row matrix $[\boldsymbol{V}_{M,0}; \boldsymbol{G}_{M,0}]$. Let us denote the resultant modified fundamental matrix equation by

$$\bar{\boldsymbol{V}}_M\, \boldsymbol{\Pi}_M^\lambda = \bar{\boldsymbol{G}}_M, \quad \text{or} \quad [\bar{\boldsymbol{V}}_M; \bar{\boldsymbol{G}}_M]. \tag{27}$$

Now we get the GNPs coefficient after solving the resultant algebraic system of linear equation (27). So, we determine the vector $\boldsymbol{\Pi}_M^\lambda$. This implies that the unknowns π_m, for $m = 0, 1, \dots, M$ in the series solution (11) will be known. Consequently, the approximate solution $w_M^\lambda(z)$ of model (1) is obtained.

From an algorithmic point of view, all steps of the proposed GNPs matrix collocation approach are summarized in Algorithm 2. While the inputs of this algorthim are M, ρ, λ, $\boldsymbol{D}_M, \mathcal{C}_M, g(z)$, w_0, the output is the approximate solution $w_M^\lambda(z)$ of model problem (1). Note that we adopt the standard notation ":" for the entire columns or rows of a matrix in MATLAB, i.e., $\boldsymbol{A}(i,:)$ shows the row i of the matrix $\boldsymbol{A}$.

procedure $[w_M^\lambda(z)]$=GNPs_col$(M, \rho, \lambda, \boldsymbol{D}_M, \mathcal{C}_M, g(z), w_0)$
$n := M + 1;$
$\boldsymbol{S}_M^\lambda(z) := \begin{bmatrix} 1 & z^\lambda & z^{2\lambda} & \dots & z^{M\lambda} \end{bmatrix};$
$\boldsymbol{S}_M^{\rho,\lambda}(z) := D_\star^\rho \, \boldsymbol{S}_M^\lambda(z);$ {Referring to Algorithm 1 or relation (22)}
$\boldsymbol{E}_{q,\lambda} :=$ **Diag** $(1, q^\lambda, \dots, q^{M\lambda})$; $\boldsymbol{Q}_M :=$ **Diag** $(q, q, \dots, q)$; $\{\boldsymbol{E}_{q,\lambda}, \boldsymbol{Q}_M \in \mathbb{R}^{n\times n}\}$
{Using the collocation points $\mathcal{C}_M$ in (15)}
$\boldsymbol{S}_M := \boldsymbol{0}$; $\mathbb{S}_M^{\rho,\lambda} = \boldsymbol{0}$; $\{\boldsymbol{S}_M, \mathbb{S}_M^{\rho,\lambda} \in \mathbb{R}^{n\times n}\}$
$\boldsymbol{G}_M := \boldsymbol{0}$; $\{\boldsymbol{G}_M \in \mathbb{R}^{n\times 1}\}$
for $i := 0, \dots, M$ **do**
  $\boldsymbol{S}_M[i,:] := \boldsymbol{S}_M^\lambda(z_i);$
  $\mathbb{S}_M^{\rho,\lambda}[i,:] := \boldsymbol{S}_M^{\rho,\lambda}(z_i);$
  $\boldsymbol{G}_M[i] := g(z_i);$
end for
Fa_Sys:=$\left(\mathbb{S}_M^{\rho,\lambda} - Q_M \, S_M \, E_{q,\lambda} - \boldsymbol{S}_M\right) \boldsymbol{D}_M$; rhs_Sys:=$\boldsymbol{G}_M$;
{Implementing the I.C.}
Fa_Sys[1,:]:=$\boldsymbol{S}_M^\lambda(0) \, \boldsymbol{D}_M$; rhs_Sys[1]:=$w_0$;
{Solve the fundamental matrix equation (27)}
$\boldsymbol{\Pi}_M^\lambda$:=**LinSolve** (Fa_Sys, rhs_Sys);
$w_M^\lambda(z) := \boldsymbol{S}_M^\lambda(z) \, \boldsymbol{D}_M \, \boldsymbol{\Pi}_M^\lambda;$
end;

Algorithm 2: *An algorithmic description of the GNPs collocation procedure*

5. Computation results

The applications of the presented matrix collocation algorithm based on GNPs are investigated here for the delay model (1) of fractional order in the sense of LC and CF derivatives. To do so, three examples with numerical values are investigated to show the accuracy of this matrix algorithm. We use MATLAB software version 2021a and its graphical capabilities to show the applicability of our proposed approach. We also compare our outcomes with the results of other existing recent algorithms, and conclude that the proposed matrix technique in this research paper is better than others. Furthermore, the corresponding achieved absolute errors defined by

$$\mathcal{E}_{M,\lambda}(z) := |w(z) - w_M^\lambda(z)|, \quad z \in [0,1]. \tag{28}$$

Next, we define the maximum absolute values of errors (MAE) by

$$\mathcal{E}_{M,\infty} := \max_{z\in[0,1]} |w(z) - w_M^\lambda(z)|.$$

The estimated numerical order of convergence (EOC) is calculated by the following formula:

$$\mathrm{Eoc}_{M,\infty} := \log_2\left(\frac{\mathrm{Eoc}_{M,\infty}}{\mathrm{Eoc}_{2M,\infty}}\right).$$

Example 1. *We consider model* (1) *with* $q = \frac{1}{10}$ *given as [37]*

$$D_\star^\rho \, w(z) = w(z) + \frac{1}{10}\, w(\frac{1}{10} z) + g(z),$$

with initial condition $w(0) = 1$. The source function $g(z)$ in the case of the LC is

$$g_{LC}(z) = \frac{2\rho}{\Gamma(3-\rho)} z^{2-\rho} - \frac{11}{10} - \rho\, z^2 - \frac{\rho}{1000} z^2,$$

while in the case of the CF derivative it is given by

$$g_{CF}(z) = 2z - \frac{1-\rho}{\rho}\left(1 - e^{-\frac{\rho}{1-\rho} z}\right) - \frac{11}{10} - \rho\, z^2 - \frac{\rho}{1000} z^2.$$

The true analytical solution of this test case is $w(z) = \rho\, z^2 + 1$.

For this test example, let us take $M = 2$ and $\lambda = 1$. First, we consider the LC fractional derivative. By employing the GNPs collocation procedure with various fractional orders $\rho = 0.25, 0.5, 0.75, 0.9$, and $\rho = 1$, we arrive at the following approximate solutions:

$$^{LC}w_2^1(z) = \begin{cases} 0.25\, z^2 + 1.003542965 \times 10^{-15}\, z + 1.0, \\ 0.5\, z^2 - 1.346523244 \times 10^{-16}\, z + 1.0, \\ 0.75\, z^2 + 5.24820565 \times 10^{-18}\, z + 1.0, \\ 0.9\, z^2 + 4.323779765 \times 10^{-17}\, z + 1.0, \\ 1.0\, z^2 + 1.703183936 \times 10^{-108}\, z + 1.0. \end{cases}$$

The former approximations abstained with the LCFD are visualized in Fig. 1. The exact solution associated with each ρ is depicted by a solid line. Furthermore, the associated achieved absolute errors defined by (28) are visualized in this figure on the right part. It can be clearly observed that our results are accurate up to the machine epsilon.

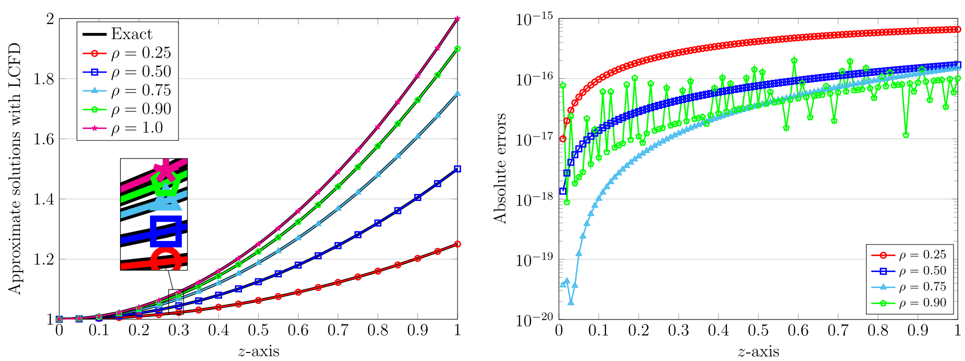

Figure 1: *Plot of approximate solutions using the GNPs approach (left) and related absolute errors (right) in Example 1 with $M = 2, \lambda = 1$, and various ρ. The fractional derivative is the LC.*

In order to validate our results, we compare the absolute errors achieved by the GNPs collocation technique and the outcomes of the existing procedure in the literature. We consider the modified operational matrix method (MOMM) developed in [37] with $n = 20$ basis functions. The results are tabulated in Table 1. Looking at Table 1 we conclude that our proposed technique using fewer bases produces more accurate

	GNPs collocation method ($M=2$)					MOMM ($n=20$) [37]				
z	$\rho=0.25$	$\rho=0.5$	$\rho=0.75$	$\rho=0.9$	$\rho=1$	$\rho=0.25$	$\rho=0.5$	$\rho=0.75$	$\rho=0.9$	$\rho=1$
0.2	1.9_{-16}	2.83_{-17}	5.1_{-18}	1.0_{-17}	0	2.1_{-12}	3.2_{-12}	2.2_{-12}	1.0_{-14}	1.0_{-15}
0.4	3.5_{-16}	5.95_{-17}	2.3_{-17}	2.6_{-17}	0	2.3_{-12}	3.5_{-12}	2.4_{-12}	2.2_{-14}	1.4_{-15}
0.6	4.8_{-16}	9.35_{-17}	5.2_{-17}	4.6_{-17}	0	2.5_{-12}	3.7_{-12}	2.7_{-12}	2.3_{-14}	1.9_{-15}
0.8	5.8_{-16}	1.30_{-16}	9.4_{-17}	7.2_{-17}	0	2.6_{-12}	3.9_{-12}	2.9_{-12}	2.5_{-14}	2.1_{-15}
1.0	6.6_{-16}	1.70_{-16}	1.5_{-16}	1.0_{-16}	0	2.9_{-12}	4.0_{-12}	3.0_{-12}	1.6_{-14}	2.3_{-15}

Table 1: *A comparison of absolute errors in GNPs matrix collocation procedure in Example 1 for $\lambda=1$, $M=2$, $\rho=0.25,0.5,0.75,0.9,1$, and various $z\in[0,1]$. The fractional derivative is the LC.*

results. On the other hand, our procedure is simple in terms of implementation than the MOMM.

In the second part, we consider CF fractional derivative in Example 1. With the same number of bases $M=2$ and $\lambda=1$, here we set $\rho=0.01,0.25,0.5,0.75,1$. The results using the GNPs collocation method are given by

$$ {}^{CF}w_2^1(z)=\begin{cases} 0.01\,z^2+1.66605856\times10^{-13}\,z+1.0,\\ 0.25\,z^2-6.059884083\times10^{-16}\,z+1.0,\\ 0.5\,z^2-7.970331941\times10^{-17}\,z+1.0,\\ 0.75\,z^2+6.527864988\times10^{-17}\,z+1.0,\\ 0.99\,z^2-2.648055286\times10^{-18}\,z+1.0.\end{cases} $$

Obviously, the results in the case of the CF are accurate enough compared to the results obtained with the LC operator.

Example 2. *The second test model with $q=\frac{1}{4}$ is considered as [37]*

$$ D_\star^\rho\,w(z)=w(z)+\frac{1}{4}\,w(\frac{1}{4}z)+g(z). $$

The initial condition is $w(0)=0$. Here, we only consider the LC operator, for which the function $g(z)$ is as follows:

$$ g_{LC}(z)=-z-\rho\,z^{\frac{3}{2}}-\frac{z}{16}-\frac{\rho}{32}z^{\frac{3}{2}}+\frac{z^{1-\rho}}{\Gamma(2-\rho)}+\frac{3\rho\sqrt{\pi}}{4\Gamma(\frac{3}{2}-\rho)}z^{\frac{3}{2}-\rho}. $$

The true solution of this test case is given by $w(z)=\rho\,z^{\frac{3}{2}}+z$.

The main motivation for considering this test case is to use generalized version of NPs by employing a value of $\lambda\neq1$ in the computations in contrast to the last example. However, we first consider $\lambda=1$. We set $M=3$ and $\rho=0.5$, by running our matrix collocation algorithm to obtain

$$ w_3^1(z)=-0.07975593903\,z^3+0.4101913739\,z^2+1.18875505\,z, $$

which is clearly far from the given exact solution consisting of the fractional power 3/2. Also, the maximum value of the absolute error is about 10^{-2}. Note that increasing M still does not considerably improve the accuracy of the solution. To show this fact, we take $M = 3j$, for $j = 1, 2, \ldots, 8$ and plot the associated absolute errors in Fig. 2. It can be seen that for $M = 24$, the achieved maximum value of error is about 10^{-4}.

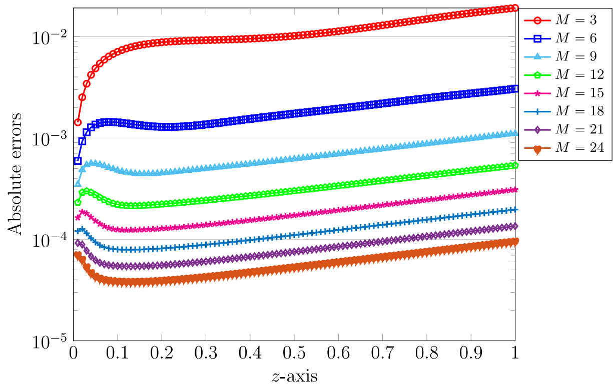

Figure 2: *Plot of achieved absolute errors using the GNPs collocation method in Example 2 with* $\rho = 0.5, \lambda = 1$ *and various* $M = 3, 6, \ldots, 24$.

The remedy is here to use a suitable value of λ as the local power of basis functions. As an example, let us take $\lambda = \rho = 1/2$ and $M = 3$. The resulting vector $\boldsymbol{S}_M^{\lambda}(z)$ becomes

$$\boldsymbol{S}_3^{\frac{1}{2}}(z) = \begin{bmatrix} 1 & z^{\frac{1}{2}} & z & z^{\frac{3}{2}} \end{bmatrix}.$$

This indicates that we can generate the components of the exact solution from $\boldsymbol{S}_3^{\frac{1}{2}}(z)$. After employing the proposed matrix algorithm, we get

$$w_3^{\frac{1}{2}}(z) = 1.0\, z - 8.496769826 \times 10^{-16}\, z^{\frac{1}{2}} + 0.5\, z^{\frac{3}{2}},$$

which obviously matches the true solution up to machine epsilon. The graphical representations of the preceding approximate solution together with the related exact solution are depicted in Fig. 3. Clearly, we obtain an optimal accuracy by choosing an appropriate value of λ and M compared to the case $\lambda = 1$, as shown in Fig. 3.

To validate and justify our results, we do some comparisons in Table 2 for $\rho = 0.5$ and $M = 3$. We utilize two different values of $\lambda = 1, 0.5$ in this table. Analogously to Table 1, we compare our outcomes with those obtained via MMOM [37] using $n = 20$ and the same $\rho = 0.5$. The last column of this table is devoted to the outcomes of errors obtained by the operational matrix of fractional order based on Taylor basis (OMTB) [5] reported using $m = 3$ and $\sigma = 1/2$ as the power of local basis functions. The higher order accuracy of the present GNPs matrix collocation algorithm is clearly visible by the tabulated results shown in Table 2.

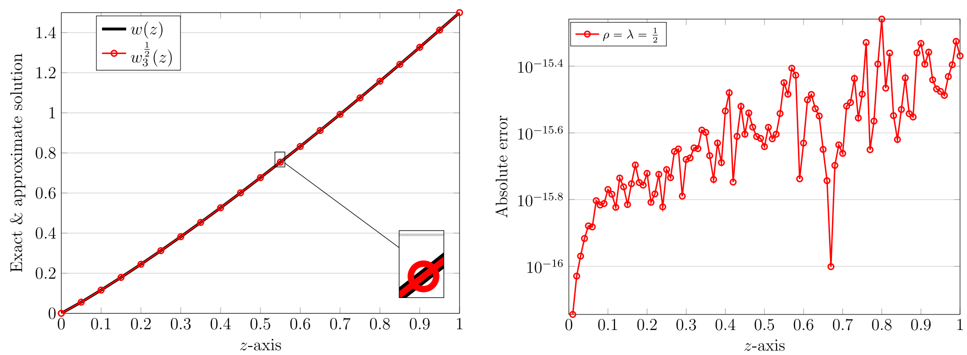

Figure 3: *Plot of the approximate solution using the GNPs approach (left) and the associated absolute errors (right) in Example 2 with $M = 3, \rho, \lambda = 1/2$. The fractional derivative is the LC.*

	GNPs collocation method ($M = 3$)		MOMM ($n = 20$) [37]	OMTB ($m = 4$) [5]
z	$\rho = 0.5, \lambda = 1$	$\rho = 0.5, \lambda = 0.5$	$\rho = 0.5$	$\rho = 0.5, \sigma = 0.5$
0.2	8.7993×10^{-3}	1.8574×10^{-16}	1.5×10^{-12}	1.4×10^{-16}
0.4	9.5372×10^{-3}	2.2250×10^{-16}	1.7×10^{-12}	0
0.6	1.1316×10^{-2}	2.7055×10^{-16}	1.8×10^{-12}	2.2×10^{-16}
0.8	1.4921×10^{-2}	3.3841×10^{-16}	2.0×10^{-12}	4.4×10^{-16}
1.0	1.9191×10^{-2}	4.2760×10^{-16}	2.1×10^{-12}	2.2×10^{-16}

Table 2: *A comparison of absolute errors in the GNPs matrix collocation procedure in Example 2 for $M = 3$, $\rho = 0.5$, $\lambda = 1, 0.5$ and various $z \in [0, 1]$. The fractional derivative is the LC.*

Example 3. *The last test case is considered with $q = \frac{1}{\alpha}$, $\alpha > 0$ as follows:*

$$D_{\star}^{\rho} w(z) = w(z) + \frac{1}{\alpha} w(\frac{1}{\alpha} z) + g(z).$$

The initial condition is $w(0) = 1$. Here, we only consider the CF operator, for which the function $g(z)$ is as follows:

$$g_{CF}(z) = -\frac{1-\rho}{2-\rho} e^{\rho z} - \frac{1}{2-\rho} e^{-\frac{\rho}{1-\rho}} - \frac{1}{\alpha} e^{\frac{\rho z}{\alpha}}.$$

The true analytical solution of the above test example is given by $w(z) = e^{\rho z}$.

We first set $\alpha = 5$. The value of λ is taken as one for the CF operator. With $\rho = 0.5$ and using $M = 3, 6$, we get the following approximations:

$$\begin{aligned} {}^{CF}w_3^{1}(z) &= 0.02710011463\, z^3 + 0.1207966828\, z^2 + 0.5009133865\, z + 1.0, \\ {}^{CF}w_6^{1}(z) &= 0.00002797819277\, z^6 + 0.0002512212292\, z^5 + 0.002610821785\, z^4 \\ &\quad + 0.02083081124\, z^3 + 0.1250004729\, z^2 + 0.4999999647\, z + 1.0. \end{aligned}$$

We plot the above approximate solutions in Fig. 4. In addition to $M = 3, 6$, we also depicted a numerical solution for $M = 9$ in this figure. The true exact solution is shown by a thick line. On the right, we visualize the associated absolute errors achieved for these values of $M = 3, 6, 9$. Evidently, by increasing M, the achieved errors decrease exponentially, which shows the convergence of the proposed GNPs collocation approach.

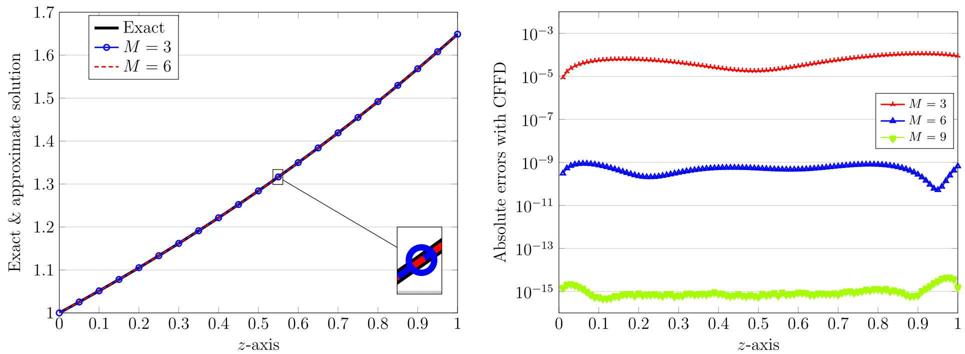

Figure 4: *Plot of a numerical solution using the GNPs approach (left) and related absolute errors (right) in Example 3 with $M = 3, 6, 9, \rho = 1/2, \lambda = 1$. The fractional derivative is the CF.*

The effects of utilizing diverse values of delay parameter q are examined in Table 3 for the last test example. Here, we utilize $M = 8$, $\rho = 0.25$, and $\lambda = 1$. The results of absolute errors for $q = 2, 4, 6, 8, 10, 20, 50$, and $q = 100$ are tabulated in Table 3. The behavior of MAE and its related estimated order of convergence are given in Table 4. In these experiments, we used $\rho = 0.25, 0.5, 0.75$, $\lambda = 1$ and different $M = 1, 2, 4, 8$. It can be evidently seen that the attained rate of convergence grows exponentially as the number of bases increases.

z	$q = 2$	$q = 4$	$q = 6$	$q = 8$	$q = 10$	$q = 20$	$q = 50$	$q = 100$
0.2	9.607_{-15}	8.157_{-18}	6.507_{-16}	1.216_{-16}	3.559_{-16}	1.967_{-16}	1.215_{-16}	1.439_{-16}
0.4	1.961_{-14}	3.692_{-16}	1.487_{-15}	2.295_{-16}	1.520_{-15}	7.484_{-16}	1.640_{-16}	1.763_{-16}
0.6	3.262_{-14}	1.151_{-16}	1.326_{-15}	8.907_{-17}	1.850_{-15}	1.048_{-15}	1.878_{-16}	6.815_{-17}
0.8	5.108_{-14}	1.626_{-17}	1.415_{-15}	3.599_{-16}	2.139_{-15}	8.621_{-16}	2.703_{-16}	1.251_{-16}
1.0	7.397_{-14}	4.755_{-16}	1.659_{-15}	1.260_{-16}	2.172_{-15}	1.543_{-15}	2.958_{-16}	6.868_{-18}

Table 3: *A comparison of absolute errors in the GNPs matrix collocation procedure in Example 3 using $M = 8$, $\rho = 0.25$, $\lambda = 1$, and various $z \in [0, 1], q > 0$. The fractional derivative is the CF.*

We now go beyond the unit interval $[0, 1]$ and consider the computational domain as $[0, 3]$. We fix $M = 10$, $\lambda = 1$, and $q = 10$ in the next experiment. Fig. 5 shows the approximate solutions obtained via the GNPs collocation strategy for different values of $\rho = 0.01, 0.25, 0.5, 0.75$, and $\rho = 0.99$. For each ρ, the associated exact true solution is depicted by a solid line. Clearly, the presented approximate solutions are well aligned with the true solutions. We can see the related absolute errors in the same figure on the right. As one can observe the largest magnitude of errors is obtained for the two most extreme values of $\rho = 0.01, 0.99$.

	$\rho = 0.25$		$\rho = 0.50$		$\rho = 0.75$	
M	$\mathcal{E}_{M,\infty}$	$\mathrm{Eoc}_{M,\infty}$	$\mathcal{E}_{M,\infty}$	$\mathrm{Eoc}_{M,\infty}$	$\mathcal{E}_{M,\infty}$	$\mathrm{Eoc}_{M,\infty}$
1	3.5590×10^{-02}	–	1.7246×10^{-01}	–	$1.10553 \times 10^{+00}$	–
2	3.7809×10^{-04}	6.3180	2.2581×10^{-03}	6.2550	1.38571×10^{-02}	6.3180
4	8.7502×10^{-08}	9.4613	1.7989×10^{-06}	10.294	1.96575×10^{-05}	9.4613
8	2.8387×10^{-15}	20.338	3.1472×10^{-13}	22.447	1.48309×10^{-11}	20.338

Table 4: *The outcomes of MAE norms and the associated estimated numerical order of convergence* $Eoc_{M,\infty}$ *in Example 3 with* $\rho = 0.25, 0.5, 0.75$, $\lambda = 1$, $q = 5$, *and various* M.

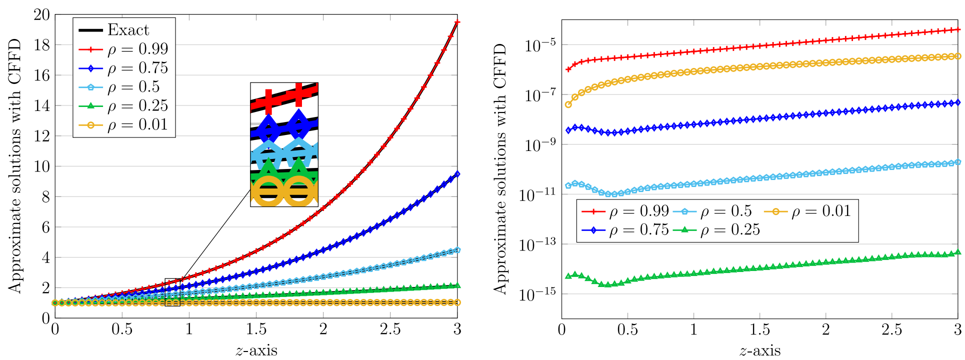

Figure 5: *Plot of the approximate solution using the GNPs approach (left) and the associated absolute errors (right) in Example 3 with* $M = 10, \lambda = 1, q = 10$, *and various* $\rho = 0.01, 0.25, 0.5, 0.75, 0.99$. *The fractional derivative is the CF.*

6. Conclusions

A class of delay differential equations with fractional order is solved by the aid of the matrix collocation method based on a novel set of polynomials called Narayana polynomials (NPs). The involved fractional derivative is interpreted in the sense of Caputo-Fabrizio and Liouville-Caputo fractional operators. To get more accurate results, a generalized version of NPs (GNPs) is introduced and the convergence properties of GNPs is established through the rigorous error analysis. Three computational test examples are provided to show the utility and accuracy of the proposed GNPs matrix collocation approach. Comparisons with the outcomes of two existing computational techniques, i.e., the modified operational matrix method (MOMM) and operational matrix of integration relying on Taylor basis (OMTB) are performed and the results are shown in figures and tables. The results indicate the present approach with a lower computational cost is more accurate than MOMM and OMTB. The proposed strategy can be straightforwardly applied to diverse important engineering model problems with various fractional derivatives.

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