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# Sinc approximation for numerical solutions of two-dimensional nonlinear Fredholm integral equations

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**Abstract.** A new efficient iterative method of successive approximations based on the fixed point method and sinc quadrature is proposed for two-dimensional nonlinear Fredholm integral equations of the second kind (2DNFIEs). We have provided convergence and error analysis of the suggested method. Besides, numerical stability of the method concerning the choice of the first iteration is verified.

**AMS subject classifications**: 47H09, 47H10

**Keywords**: iterative approach, sinc quadrature, numerical stability, fixed point method

## 1. Introduction

In this research, a new iterative approach based on the fixed point method and sinc quadrature is improved for the numerical solution of the (2DNFIEs):

$$X(s,t) = f(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(X(x,y))dxdy, \quad (s,t) \in J, \tag{1}$$

where $f$ and $H$ are known functions on $J = [a,b] \times [c,d]$ and $J \times J$, respectively, $X(s,t)$ is the unknown function to be determined, and $\lambda$ is a constant. This type of equations is applied in various fields of electromagnetism, plasma physics, telegraph equations, electrical engineering, queueing theory, economics, population, image processing, reformulation of boundary value problems, and in many other areas [13, 42, 31, 10, 30, 43]. Many different methods are usually used to solve Eq. (1) such as the collocations and Galerkin methods [14, 9, 2], the Haar wavelet [17, 16], the Bernoulli operational matrix method [6], triangular functions (TFs)[23, 12], Legendre wavelets [18], the hybrid function method [15, 39], Bernoulli polynomials [5], the reduced differential transform method [44], radial basis functions [32], the B-spline collocation method [28], the hat basis function [33], block-pulse functions [38], rationalized Haar functions [3], the Bernstein polynomials [34], operational matrices [29], and the degenerate kernel method [4].

In the approaches suggested above, the integral equation is converted into a system of nonlinear algebraic equations which has to be accomplished with iterative methods. It is cumbersome to solve these systems, or the solution may be unreliable.

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To reduce this problem, we attempt to provide a numerical method to approximate a solution for Eq. (1) using the fixed point method and sinc quadrature. The sinc method is a robust numerical tool for finding fast and accurate solutions in different fields of problems [25, 40]. Numerical methods have been given for solving integral equations including successive approximations methods based on quadrature rules [7, 8, 20, 21, 22, 27]. The existence and uniqueness result quoted for Eq. (1) is varied see [1, 11, 19, 26, 35]. The outline of the paper is as follows. In Section 2, we assert basic theorems and properties of the sinc function, which are referred to in the ensuing sections. In Section 3, the convergence of the method of successive approximations is studied. Numerical stability of this method is proved in Section 4. Numerical results are reported in Section 5 which confirm that the implementation of the method is considerably fast and highly accurate.

## 2. Mathematical preliminaries

### 2.1. Modulus of continuity

**Definition 1.** *Let $g : J \to \mathbb{R}$ be a bounded function, the oscillation of $g$ on $J$ is the quantity*

$$\begin{aligned}\omega_J(g,\delta) = \sup\{&|g(x,y) - g(x',y')|; x, x' \in [a,b];\\ &y, y' \in [c,d];; \sqrt{(x-x')^2 + (y-y')^2} \le \delta\}.\end{aligned}$$

*Furthermore, $\omega_J(g,\delta)$ is called a uniform modulus of continuity of $g$ if $g \in C\big(J\big)$.*

**Theorem 1.** *The following properties hold:*

*(i)* $|g(x,y) - g(x',y')| \le \omega_J(g, \sqrt{(x-x')^2 + (y-y')^2})$ *for all* $x, x' \in [a,b]$ *and* $y, y' \in [c,d]$,

*(ii)* $\omega_J(g,\delta)$ *is a non-decreasing function in* $\delta$,

*(iii)* $\omega_J(g,0) = 0$,

*(iv)* $\omega_J(g,\delta_1+\delta_2) \le \omega_J(g,\delta_1) + \omega_J(g,\delta_2)$ *for any* $\delta_1, \delta_2 \ge 0$,

*(v)* $\omega_J(g,n\delta) \le n\omega_J(g,\delta)$ *for* $n \in \mathbb{N}$ *and any* $\delta \ge 0$,

*(vi)* $\omega_J(g,\alpha\delta) \le (\alpha+1)\omega_J(g,\delta)$ *for any* $\delta, \alpha \ge 0$,

*(vii)* $\omega_J(g,\cdot)$ *is continuous at 0 iff* $g \in C(J)$,

*(viii) If* $J \subseteq J'$, *then* $\omega_J(g,\delta) \le \omega_{J'}(g,\delta)$ *for all* $\delta \ge 0$.

**Proof**. See [24]. □

**Theorem 2.** *Suppose that $g : J \to \mathbb{R}$ is an integrable, bounded function. For each $a = x_0 < x_1 < ... < x_n = b$ , $c = y_0 < y_1 < ... < y_n = d$ and each $\kappa_i \in [x_{i-1}, x_i]$ , $\nu_j \in [y_{j-1}, y_j]$, we have*

$$\Big| \int_c^d \int_a^b g(s,t)dsdt - \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1})g(\kappa_i, \nu_j)\Big|$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1})\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(g, \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}).$$

**Proof**. From Theorem 1, we have

$$\Big| \int_c^d \int_a^b g(s,t)dsdt \; - \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1})g(\kappa_i, \nu_j)\Big|$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} |g(s,t) - g(\kappa_i, \nu_j)|\, dsdt$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1})\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(g, \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}).$$

□

## 2.2. Sinc function

**Definition 2** (see [37])**.** *Let $g$ be a function defined for all $x$ on $(\infty, \infty)$ and $h > 0$. The Whittaker cardinal is written as*

$$C(g,h)(x) = \sum_{-\infty}^{\infty} g(ih)S(i,h)(x),$$

*whenever this series in (3) converges, and $S(i,h)(x)$ are the sinc functions defined by*

$$S(i,h)(x) = \frac{sin[\pi(\frac{x-ih}{h})]}{\pi(\frac{x-ih}{h})}$$

**Definition 3** (see [25])**.** *Let $h > 0$. $A(h)$ is the family of entire functions $g$ such that on the real line $g \in L^2(\mathbb{R})$ and in the complex plane, $g$ is of exponential type $\frac{\pi}{h}$, i.e.,*

$$|g(z)| \leq K \exp(\frac{\pi|z|}{h})$$

*for $\exists\ K > 0$.*

**Theorem 3** (see [25])**.** *If $g \in A(h)$, then for all $z \in \mathfrak{C}$*

$$g(z) = \sum_{i=-\infty}^{\infty} g(ih)sinc(\frac{z - ih}{h}). \quad (2)$$

**Definition 4** (see [37]). *A function $f$ is said to decay double exponentially if $\exists\ \alpha,\ K$ so that*

$$|f(t)| \leq K \exp(-\alpha e^{|t|}),\ t \in \mathbb{R},$$

*or a function $g$ is said to decay double exponentially regarding the conformal map $\phi$, if $\exists\ \alpha,\ K$ so that*

$$|g(\phi(t))\phi'(t)| \leq K \exp(-\alpha e^{|t|}),\ t \in \mathbb{R}, \tag{3}$$

where $\phi$ is called the DE transformation. By [37, 36], we have the DE formula for the definite integration of a function $f$:

$$\int_a^b f(x)dx = h \sum_{i=-N}^{N} f(\varphi(ih))\varphi'(ih)) + O(e^{\frac{-2\pi dN}{log(2\pi dN/\alpha)}}),$$

with $h = \frac{\log(\frac{2\pi dN}{\alpha})}{N}$ and

$$x = \varphi(t) = \frac{b-a}{2} \tanh(\frac{\pi}{2} \sinh(t)) + \frac{b+a}{2},$$
$$\varphi'(t) = \frac{b-a}{2} \frac{\pi/2 \cosh(t)}{\cosh^2(\pi/2 \sinh(t))},\ t \in \mathbb{R}$$

**Corollary 1.** *If $g \in A(h)$ and $g \in L^1(R)$, then*

$$h \sum_{i=-\infty}^{\infty} g(ih) = \int_{-\infty}^{\infty} g(t)dt.$$

**Proof**. Replace z by $t \in \mathbb{R}$ in (2) and integrate the result over $\mathbb{R}$ (see [25]). $\square$

**Theorem 4.** *Let $S_N = h \sum_{i=-N}^{N} \varphi'(ih)$. Then*

$$\lim_{N \to \infty} S_N = (b-a).$$

**Proof**. By Corollary (1), we have

$$\begin{aligned}
\lim_{N \to \infty} S_N &= h \sum_{i=-\infty}^{\infty} \varphi'(ih) \\
&= \int_{-\infty}^{\infty} \varphi'(t)dt \\
&= \lim_{x \to \infty} \int_{-x}^{x} \frac{b-a}{4} \frac{\pi \cosh(t)}{\cosh^2(\pi/2 \sinh(t))} dt \\
&= \lim_{x \to \infty} \frac{b-a}{4} \frac{4\pi \sinh(\pi/2 \sinh(x))}{\pi \cosh(\pi/2 \sinh(x))} \\
&= b-a.
\end{aligned}$$

$\square$

If $g(x,y)$ decays double exponentially regarding

$$\varphi_1(s) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh(s)) + \frac{b+a}{2}$$

and

$$\varphi_2(t) = \frac{d-c}{2}\tanh(\frac{\pi}{2}\sinh(t)) + \frac{d+c}{2},$$

then the function $g$ can be expanded as follows (see [37]):

$$\begin{aligned} g(\varphi_1(s),\varphi_2(t))\varphi'(s)\varphi'(t) = &\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} g(\varphi_1(ih_1),\varphi_2(jh_2))\varphi_1'(ih_1)\varphi_2'(jh_2) \\ &\times sinc(\frac{t}{h_1}-i)sinc(\frac{s}{h_2}-j) + E(t,s,h_1,h_2), \end{aligned}$$

where $h_1 = \frac{\log(2\pi d_1 N/\alpha_1)}{N}$ and $h_2 = \frac{\log(2\pi d_2 N/\alpha_2)}{M}$.
Integrating this expression regarding x and y, we have

$$\int_a^b\int_c^d g(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\varphi_1(s),\varphi_2(t))\varphi_1'(s)\varphi_2'(t)dsdt.$$

Further,

$$\begin{aligned} \int_a^b\int_c^d g(x,y)dxdy = &h_1h_2\sum_{i=-N}^{N}\sum_{j=-M}^{M} g(\varphi_1(ih_1),\varphi_2(jh_2))\varphi_1'(ih_1)\varphi_2'(jh_2) \\ &+ O(exp(\frac{-2\pi d_1}{h_1})) + O(exp(\frac{-2\pi d_2}{h_2})). \end{aligned} \tag{4}$$

## 3. Main results

### 3.1. The sequence of successive approximations

In the sequel, we shall prove the existence and uniqueness of the solution to Eq. (1) by the Banach contraction principle. Consider Eq. (1) under the following conditions:

a°. $f \in C(J,\mathbb{R})$, $\psi \in C(\mathbb{R},\mathbb{R})$ and $H \in C(J\times\mathbb{R},\mathbb{R})$,

b°. there exists $\rho \geq 0$, such that $|\psi(P_1)-\psi(P_2)| \leq \rho|P_1-P_2|, \quad \forall P_1,P_2 \in C(J)$.

c°. $\eta = \rho\lambda N_H(b-a)(d-c) < 1$, where $N_H = \max\{\big|H(s,t,x,y)\big|; s,x\in[a,b], t,y\in [c,d],\}$, according to the continuity of $H$.

Now, we define the operator $\Gamma : X \to X$ by

$$\Gamma(X)(s,t) := f(s,t) + \lambda\int_c^d\int_a^b H(s,t,x,y)\psi(X(x,y))dxdy, \quad (s,t)\in J.$$

**Theorem 5.** *Under the above assumptions, Eq. (1) has a unique solution $X^* \in C(J)$. Moreover, for any arbitrary choice of initial point $X_0 \in C(J)$, the Picard sequence defined by*

$$X_k = \Gamma(X_{k-1}), \qquad k \geq 1$$

*with the initial value $X_0 := f_0$ converges with respect to the uniform norm to the solution $X(s,t)$ of the integral equation (1) and we have*

$$\big\|X^* - X_k\|_u \leq \frac{\eta^k}{1-\eta}\|X_0 - \Gamma(X_0)\|_u, \tag{5}$$

$$\big\|X^* - X_k\|_u \leq \frac{\eta}{1-\eta}\|\Gamma(X_{k-2}) - \Gamma(X_{k-1})\|_u, \tag{6}$$

*hold for any $k \in \mathbb{N}$. Furthermore, choosing $X_0 = f$, (5) becomes*

$$\big\|X^* - X_k\|_u \leq \frac{\eta^{k+1}}{\rho(1-\eta)}N_0, \tag{7}$$

*where*

$$N_0 = \sup\{|\psi(f(s,t))|; s \in [a,b], t \in [c,d]\}.$$

**Proof**. For any $X \in C(J)$ we have

$$\begin{aligned}
\omega_J(\Gamma(X),\delta) \leq &\sup_{(s_i,t_i)\in J,\ i=1,2}\left\{\big|f(s_1,t_1) - f(s_2,t_2)\big| \ : \ \sqrt{(s_2-s_1)^2+(t_2-t_1)^2} \leq \delta\right\}\\
&+ \sup_{(s_i,t_i)\in J,\ i=1,2}\left\{ \mid \lambda\int_c^d\int_a^b H(s_1,t_1,x,y)\psi(X(x,y))dxdy\right.\\
&\left. - \lambda\int_c^d\int_a^b H(s_2,t_2,x,y)\psi(X(x,y))dxdy \mid : \ \sqrt{(s_2-s_1)^2+(t_2-t_1)^2} \leq \delta\right\}\\
\leq &\,\omega_J(f,\delta) + \lambda\int_c^d\int_a^b \omega_{st}\big(H,\delta)|\psi(X(x,y))|dxdy,
\end{aligned}$$

where

$$\omega_{st}\big(H,\delta) = \sup_{\substack{(x_i,y_i)\in J,\\ i=1,2}}\left\{\big|H(s_1,t_1,x,y) - H(s_2,t_2,x,y)\big| \ ; \sqrt{(s_2-s_1)^2+(t_2-t_1)^2} \leq \delta\right\}.$$

Take the limit as $\delta \to 0$. Then by Theorem 1-vii, we have $\omega_J(\Gamma(X),\delta) \to 0$, so $\Gamma$ maps $C(J)$ onto itself.
Furthermore, for $X, Y \in C(J)$, we have

$$\begin{aligned}
|\Gamma(X)(s,t) - \Gamma(Y)(s,t)| &= \lambda\left|\int_c^d\int_a^b K(s,t,x,y)[\psi(X(x,y)) - \psi(Y(x,y))]dxdy\right|\\
&\leq \lambda\int_c^d\int_a^b |H(s,t,x,y)||\psi(X(x,y)) - \psi(Y(x,y))|dt\\
&\leq \rho\lambda N_H(b-a)(d-c) \parallel X - Y \parallel_u,
\end{aligned}$$

for all $(s,t) \in J$. Thus,

$$\| \Gamma(X) - \Gamma(Y) \|_u \leq \eta \| X - Y \|_u .$$

By virtue of the Banach contraction principle and crucial condition c°, we infer that Eq. (1) has a unique solution and the same Banach's fixed point principle leads to estimates (5) and (6).

□

Now, we introduce a numerical scheme to solve Eq. (1). Applying formula (4) to Eq. (1), we have

$$\begin{aligned}
&\overline{X}_0(s,t) = r(s,t), \\
&\overline{X}_k(s,t) = r(s,t) + \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,\varphi_1(ih_1),\varphi_2(jh_2))\varphi_1'(ih_1)\varphi_2'(jh_2) \\
&\qquad\qquad \times \psi(\bar{X}_{k-1}(s_i,t_j)).
\end{aligned} \tag{8}$$

To determine the unknown values $\overline{X}_k(s,t)$, choosing the sinc points $s_p = \varphi_1(ph_1)$ and $t_q = \varphi_2(qh_2)$ as collocation points, we get

$$\begin{aligned}
&\overline{X}_0(s_p,t_q) = r(s_p,t_q), \\
&\overline{X}_k(s_p,t_q) = r(s_p,t_q) + \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s_p,t_q,\varphi_1(ih_1),\varphi_2(jh_2)) \\
&\qquad\qquad \times \psi(\overline{X}_{k-1}(\varphi_1(ih_1),\varphi_2(jh_2))\varphi_1'(ih_1)\varphi_2'(jh_2)).
\end{aligned} \tag{9}$$

## 3.2. Error analysis

**Theorem 6.** *Assume the conditions of Theorem 5 hold. Then the recurrence relation (8) converges to the unique solution $X^*$ of Eq. (1) and the following error estimate holds:*

$$\begin{aligned}
\|X^* - \bar{X}_k\| \leq & \frac{\eta^{k+1}}{\rho(1-\eta)} N_0 + \frac{\eta}{1-\eta}\omega_J(f,\delta) + \frac{\theta\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) \\
& + \frac{\varsigma\eta}{\rho N_H(1-\eta)}\omega_{xy}(H,\delta),
\end{aligned}$$

*where*

$$\begin{cases}
N_k = \sup\limits_{(s,t)\in J} |\psi(\bar{X}_k(s,t))|, \\
\Gamma_k = \sup\limits_{(s,t)\in J} |\psi(X_k(s,t))|, \\
\varsigma = \max\limits_{i=\overline{0,k-1}} \{N_i\}, \\
\theta = \max\limits_{i=\overline{0,k-2}} \{\Gamma_i\}.
\end{cases} \tag{10}$$

**Proof**. By (8), $\forall(s,t) \in J$, we have

$$
\begin{aligned}
&\big|X_1(s,t) - \overline{X}_1(s,t)\big| \\
\leq& \Big|\lambda \int_a^b \int_c^d H(s,t,x,y)\psi(X_0(x,y))dxdy \\
&- \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x_i,y_j)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(\bar{X}_0(x_i,y_j))\Big| \\
\leq& \big|\lambda \int_a^b \int_c^d H(s,t,x,y)\psi(X_0(x,y))dxdy \\
&- \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x,y)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j))\big| \\
&+ \big|\lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x,y)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j)) \\
&- \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x_i,y_j)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j))\big| \\
&+ \big|\lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x_i,y_j)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j)) \\
&- \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} H(s,t,x_i,y_j)\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(\bar{X}_0(x_i,y_j))\big| \\
\leq& \lambda N_H \int_a^b \int_c^d \big|\psi(X_0(x,y))dxdy \\
&- \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} \varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j))| \\
&+ \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} |H(s,t,x,y) - H(s,t,x_i,y_j)|\varphi_1'(ih_1)\varphi_2'(jh_2)\psi(X_0(x_i,y_j))| \\
&+ \lambda h_1 h_2 \sum_{i=-N}^{N} \sum_{j=-M}^{M} |H(s,t,x_i,y_j)||\varphi_1'(ih_1)\varphi_2'(jh_2)|\psi(X_0(x_i,y_j)) - \psi(\bar{X}_0(x_i,y_j))|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\big|X_1(s,t) - \overline{X}_1(s,t)\big| \leq& \lambda N_H \Big| \int_a^b \int_c^d \psi(X_0(x,y))dxdy \\
&- \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (x_{i+1} - x_i)(y_{i+1} - y_i)\psi(X_0(x_i,y_j))|
\end{aligned}
$$

$$+ N_0\lambda\omega_{xy}\big(H,\delta)h_1h_2\sum_{i=-N}^{N}\sum_{j=-M}^{M}\varphi_1'(ih_1)\varphi_2'(jh_2)$$
$$+ N_H\lambda h_1h_2\sum_{i=-N}^{N}\sum_{j=-M}^{M}\varphi_1'(ih_1)\varphi_2'(jh_2)|\psi(X_0(x_i,y_j)) - \psi(\bar{X}_0(x_i,y_j))|,$$

where

$$\omega_{xy}\big(H,\delta) = \sup_{\substack{(x_i,y_i)\in J,\\ i=1,2}}\left\{\big|H(s,t,x_1,y_1) - H(s,t,x_2,y_2)\big|\ ;\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\le\delta\right\}.$$

Based on Theorem 4, we have

$$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\varphi_1'(ih_1)\varphi_2'(jh_2) = (b-a)(d-c).$$

Furthermore, if we use the differential mean value theorem and Theorem 2, we have $s_{i+1}-s_i = \varphi(\kappa_i)h_1 \le b-a$ and $t_{i+1}-t_i = \varphi(\nu_j)h_2 \le d-c$. So, we see that

$$\begin{aligned}
&\big|X_1(s,t) - \overline{X}_1(s,t)\big|\\
&\le\lambda N_H\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(x_{i+1}-x_i)(y_{i+1}-y_i)\\
&\quad\times\omega_{[x_i,x_{i+1}]\times[y_i,y_{i+1}]}(\psi(X_0),\sqrt{((x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2})\\
&\quad+ N_0\lambda(b-a)(d-c)\omega_{xy}\big(H,\delta) + N_H\lambda(b-a)(d-c)|\psi(f(x_i,y_j)) - \psi(f(x_i,y_j))\\
&\le\lambda N_H(b-a)(d-c)\omega_J\left(\psi(f),\sqrt{\frac{(b-a)^2h_1+(d-c)^2h_2}{16}}\right)\\
&\quad+ N_0\lambda(b-a)(d-c)\omega_{xy}\big(H,\delta).
\end{aligned}$$

Now, for $k=2$ and by condition (b°), it follows that

$$\begin{aligned}
\big|X_2(s,t) - \overline{X}_2(s,t)\big| \le&\lambda N_H(b-a)(d-c)\omega_J\left(\psi(X_1),\sqrt{\frac{(b-a)^2h_1+(d-c)^2h_2}{16}}\right)\\
&+ N_1\lambda(b-a)(d-c)\omega_{xy}\big(H,\delta)\\
&+ \rho N_H\lambda(b-a)(d-c)\|X_1(x_i,y_j) - \bar{X}_1(x_i,y_j)\|_u.
\end{aligned}$$

By induction for $k\ge 3$, using condition (b°), (6) and (9), we see that

$$\begin{aligned}
\big|X_K(s,t) - \overline{X}_K(s,t)\big| \le&\lambda N_H(b-a)(d-c)\omega_J\big(\psi(X_{k-1}),\delta\big)\\
&+ N_{k-1}\lambda(b-a)(d-c)\omega_{xy}\big(H,\delta\big)\\
&+ \rho N_H\lambda(b-a)(d-c)\|X_{K-1} - \bar{X}_{K-1}\|\\
\le&\frac{\eta}{\rho}\omega_J\big(\psi(X_{k-1}),\delta\big) + \frac{\eta}{\rho N_H}N_{k-1}\omega_{xy}\big(H,\delta\big) + \eta\|X_{K-1} - \bar{X}_{K-1}\|_u,
\end{aligned}$$

where $\delta = \sqrt{\frac{(b-a)^2h_1+(d-c)^2h_2}{16}} \leq \frac{b-a}{4}\sqrt{h_1} + \frac{d-c}{4}\sqrt{h_2}$. Taking the supremum for $(s,t) \in J$, we have

$$\begin{aligned}
\|X_k - \overline{X}_k\|_u &\leq \frac{\eta}{\rho}\omega_J\big(\psi(X_{k-1}),\delta\big) + \frac{\eta}{\rho N_H}N_{k-1}\omega_{xy}\big(H,\delta\big) + \eta\|X_{K-1} - \bar{X}_{K-1}\|_u \\
\big\|X_{k-1} - \overline{X}_{k-1}\big\|_u &\leq \frac{\eta}{\rho}\omega_J\big(\psi(X_{k-1}),\delta\big) + \frac{\eta}{\rho N_H}N_{k-2}\omega_{xy}\big(H,\delta\big) + \eta\|X_{K-2} - \bar{X}_{K-2}\|_u \\
&\vdots \\
\big\|X_2 - \overline{X}_2\big\|_u &\leq \frac{\eta}{\rho}\omega_J\big(\psi(X_1),\delta\big) + \frac{\eta}{\rho N_H}N_1\omega_{xy}\big(H,\delta\big) + \eta\|X_1 - \bar{X}_1\|_u \\
\big\|X_1 - \overline{X}_1\big\|_u &\leq \frac{\eta}{\rho}\omega_J\big(\psi(f),\delta\big) + \frac{\eta}{\rho N_H}N_0\omega_{xy}\big(H,\delta\big) + \eta\|X_0 - \bar{X}_0\|_u,
\end{aligned}$$

and therefore,

$$\begin{aligned}
\|X_k - \overline{X}_k\| \leq &\frac{\eta}{\rho}\big(\omega_J(\psi(X_{k-1}),\delta) + \eta\omega_J(\psi(X_{k-2}),\delta) \\
&+ ... + \eta^{k-1}\omega_J(\psi(f),\delta)\Big) \\
&+ \frac{\omega_{xy}\big(H,\delta\big)\eta}{\rho N_H}\big(N_{k-1} + \eta M_{k-2} + \eta^2 M_{k-3} + ... + \eta^{k-1}N_0\Big). \qquad (11)
\end{aligned}$$

Since for $(s,t),(s',t') \in J$ with $\sqrt{(s-s')^2 + (t-t')^2} \leq \delta$, we have

$$\begin{aligned}
\big|\psi(X_k(s,t)) - \psi(X_k(s',t'))\big| \leq &\rho\big|X_k(s,t) - X_k(s',t')\big| \\
= &\Big|f(s,t) + \lambda\int_c^d\int_a^b H(s,t,x,y)\psi(X_{k-1}(x,y))dxdy \\
&- f(s',t') + \lambda\int_c^d\int_a^b H(s',t',x,y)\psi(X_{k-1}(x,y))dxdy\Big| \\
\leq &\rho\big|f(s,t) - f(s',t')\big| \\
&+ \rho\lambda\int_c^d\int_a^b\big|H(s,t,x,y) - H(s',t',x,y)\big|\big|\psi(X_{k-1}(x,y))\big|dxdy \\
\leq &\rho\big|f(s,t) - f(s',t')\big| + \rho\lambda(b-a)(d-c)\omega_{st}(H,\delta)\Gamma_{k-1}.
\end{aligned}$$

Consequently, we have

$$\omega_J(\psi(X_m),\delta) \leq \rho\omega_J(f,\delta) + \frac{\eta}{N_H}\omega_{st}(H,\delta)\Gamma_{k-1}. \qquad (12)$$

From (12) and (11), we get

$$\begin{aligned}
\|X_k - \overline{X}_k\|_u \leq &\eta\big(1 + \eta + \eta^2 + ... + \eta^{k-1}\big)\omega_J(f,\delta) \\
&+ \frac{\eta}{\rho N_H}\omega_{st}(H,\delta)\Big(\eta\Gamma_{k-2} + \eta^2\Gamma_{k-3} + ... + B^k\Gamma_0\Big) \\
&+ \frac{\omega_{xy}\big(H,\delta\big)\eta}{\rho N_H}\big(N_{k-1} + \eta M_{k-2} + \eta^2 M_{k-3} + ... + \eta^{k-1}N_0\Big).
\end{aligned}$$

Since $B < 1$, by (10) we obtain

$$\begin{aligned}\|X_k - \overline{X}_k\|_u \leq & \eta\left(\frac{1-\eta^k}{1-\eta}\right)\omega_J(f,\delta) \\ & + \frac{\theta\eta}{\rho N_H}\frac{\eta(1-\eta^{k-1})}{1-\eta}\omega_{st}(H,\delta) + \frac{\varsigma\eta}{\rho N_H}\frac{(1-\eta^k)}{1-\eta}\omega_{xy}\big(H,\delta\big).\end{aligned}$$

Therefore

$$\begin{aligned}\|X_k - \overline{X}_k\|_u \leq & \frac{\eta}{(1-\eta)}\omega_J(f,\delta) \\ & + \frac{\theta\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{\varsigma\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big). \qquad (13)\end{aligned}$$

Notice that $\eta < 1$. Thus from inequalities (7) and $\|X^* - \overline{X}_k\|_u \leq \|X^* - X_k\|_u + \|X_k - \overline{X}_k\|_u$, it is concluded that $\overline{X}_k$ converges uniformly to $X^*$ and the desired error estimate is obtained. □

## 4. Stability analysis

In order to study numerical stability of the computed values regarding small changes in the initial iteration, we consider another initial iteration term $Y_0(s,t) = g(s,t) \in C(J,\mathbb{R})$ such that $\exists\varepsilon > 0$, for which $\big|Y_0(s,t) - X_0(s,t)\big| < \varepsilon, \forall t, s \in J$. With $Y_0(s,t) = g(s,t)$, the new sequence $\{Y_k(s,t)\}_{k=1}^{\infty}$ is:

$$Y_k(s,t) = g(s,t) + \lambda\int_c^d\int_a^b H(s,t,x,y)\psi(Y_{k-1}(x,y))dxdy, \quad (s,t) \in J.$$

Similarly to (9), we get

$$\begin{aligned}\bar{Y}_0(s,t) =& g(s,t), \\ \overline{Y}_k(s,t) =& g(s,t) + \lambda h_1 h_2 \sum_{i=-N}^{N}\sum_{j=-M}^{M} H(s,t,\varphi_1(ih_1),\varphi_2(jh_2))\varphi_1'(ih_1)\varphi_2'(jh_2) \\ & \times \psi(\bar{Y}_{k-1}(s_i,t_j)).\end{aligned}$$

**Definition 5.** *The proposed numerical method is numerically stable regarding the selection of the initial iteration if* $\exists\ \beta_1,\ \beta_2 > 0$, *which are independent of the stepsize* $\delta$, *and a continuous function* $\eta : (0,d] \to [0,\infty)$ *with* $\lim_{\delta\to 0}\eta(\delta) = 0$ *such that:*

$$\|\ \bar{X}_k - \bar{Y}_k\ \| < \beta_1\varepsilon + \beta_2\eta(\delta),\ \ k \in \mathbb{N}\cup\{0\},$$

*where* $d = \frac{b-a}{4}\sqrt{\log(2\pi d_1/\alpha_1)} + \frac{d-c}{4}\sqrt{\log(2\pi d_2/\alpha_2)}$.

**Theorem 7.** *Under the assumptions of Theorem 6, scheme (9) is numerically stable regarding the selection of the initial iteration.*

**Proof**. In order to get numerical stability, we reintroduce the proof of Theorem 6 and we have

$$\begin{cases} H'_k = \sup\limits_{(s,t)\in J} |\psi(\bar{Y}_k(s,t))|, \\ \Gamma'_k = \sup\limits_{(s,t)\in J} |\psi(Y_k(s,t))|, \\ \varsigma' = \max\limits_{i=\overline{0,k-1}} \{H'_i\}, \\ \theta' = \max\limits_{i=\overline{0,k-2}} \{\Gamma'_i\}. \end{cases}$$

Similarly to (13), we get

$$\begin{aligned} \|Y_k - \overline{Y}_k\|_u \leq & \frac{\eta}{(1-\eta)}\omega_J(g,\delta) \\ & + \frac{\theta'\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{\varsigma'\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big). \end{aligned}$$

Using the triangle inequality, we obtain

$$\begin{aligned} \|\overline{X}_k - \overline{Y}_k\|_u \leq & \|\overline{X}_k - X_k\|_u + \|X_k - Y_k\|_u + \|Y_k - \overline{Y}_k\|_u \\ \leq & \|X_k - Y_k\|_u + \frac{\eta}{(1-\eta)}\omega_J(f,\delta) \\ & + \frac{\theta\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{\varsigma\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big), \\ & + \frac{\eta}{(1-\eta)}\omega_J(g,\delta) \\ & + \frac{\theta'\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{\varsigma'\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big). \end{aligned}$$

Since

$$\mid X_0(s,t) - Y_0(s,t) \mid < \varepsilon, \qquad \forall (s,t) \in J,$$

we have

$$\begin{aligned} \Big| X_1(s,t) - Y_1(s,t) \Big| \leq & \Big| X_0(s,t) + \lambda \int_c^d \int_a^b H(s,t,x,y)\psi(X_0(x,y))dxdy \\ & - Y_0(s,t) - \lambda \int_c^d \int_a^b H(s,t,x,y)\Psi\big(Y_0(x,y)\big)dxdy \Big| \\ < & \varepsilon + \rho\lambda N_H \int_c^d \int_a^b \mid X_0(s,t) - Y_0(s,t) \mid dxdy \\ < & \varepsilon + \rho\lambda N_H (b-a)(d-c)\varepsilon. \end{aligned}$$

By induction, we get

$$\|X_k(s,t) - Y_k(s,t)\|_u < \frac{1}{1-\eta}\varepsilon,$$

for all $(s,t) \in J$ and $k \geq 0$. Then

$$\begin{aligned}\|\overline{X}_k - \overline{Y}_k\|_u \leq & \frac{1}{1-\eta}\varepsilon + \frac{\eta}{1-\eta}\Big(\big(\omega_J(f,\delta) + \omega_J(g,\delta)\big) \\ & + \frac{(\theta+\theta')\eta}{\rho N_H}\omega_{st}(H,\delta) + \frac{\varsigma+\varsigma'}{\rho N_H}\omega_{xy}\big(H,\delta\big)\Big),\end{aligned}$$

where

$$\beta_1 = \frac{1}{1-\eta}, \qquad \beta_2 = \frac{\eta}{1-\eta},$$

and

$$\eta(\delta) = \omega_J(f,\delta) + \omega_J(g,\delta) + \frac{(\theta+\theta')\eta}{\rho N_H}\omega_{st}(H,\delta) + \frac{\varsigma+\varsigma'}{\rho N_H}\omega_{xy}\big(H,\delta\big).$$

□

**Remark 1.** *Because $\eta < 1$, we see that*

$$\lim_{\delta,\varepsilon\to 0}\big\|\overline{X}_k - \overline{Y}_k\big\|_u = 0.$$

**Remark 2** (Stopping criterion)**.** *For given $\varepsilon^{'} > 0$, the first $k \in \mathbb{N}$ is determined, for which*

$$|\ \bar{X}_k(s_p,t_q) - \bar{X}_{k-1}(s_p,t_q)\ | < \varepsilon^{'},$$

*and we end with this $k$ by employing the approximations $\bar{X}_k(s,t)$ of solution. Using the triangle inequality, we see that*

$$\begin{aligned}\|X^* - \bar{X}_k\|_u \leq & \|X^* - X_k\|_u + \|X_k - \bar{X}_k\|_u \\ \leq & \frac{\eta}{1-\eta}\|X_k - X_{k-1}\|_u + \frac{\eta}{1-\eta}\omega_J(f,\delta) \\ & + \frac{\theta\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{\varsigma\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big),\end{aligned}$$

*and*

$$\begin{aligned}\|X_k - X_{k-1}\|_u \leq & \|X_k - \bar{X}_k\|_u + \|\bar{X}_k - \bar{X}_{k-1}\|_u + \|\bar{X}_{k-1} - X_{k-1}\|_u \\ \leq & \frac{2\eta}{1-\eta}\omega_J(f,\delta) + \frac{2\theta\eta^2}{\rho N_H(1-\eta)}\omega_{st}(H,\delta) + \frac{2\varsigma\eta}{\rho N_H(1-\eta)}\omega_{xy}\big(H,\delta\big) \\ & + \|\bar{X}_k - \bar{X}_{k-1}\|_u.\end{aligned}$$

*Consequently,*

$$\begin{aligned}\|X^* - \bar{X}_k\|_u \leq & \frac{\eta}{1-\eta}\|\bar{X}_k - \bar{X}_{k-1}\|_u + \frac{2\eta^2}{(1-\eta)^2}\omega_J(f,\delta) \\ & + \frac{2\theta\eta^3}{\rho N_H(1-\eta)^2}\omega_{st}(H,\delta) + \frac{2\varsigma\eta^2}{\rho N_H(1-\eta)^2},\end{aligned}$$

*and therefore, in order to get* $\mid X^*(s_p, t_q) - \bar{X}_k(s_p, t_q) \mid < \varepsilon$ *we require*

$$\frac{2\eta^2}{(1-\eta)^2}\omega_J(f,\delta) + \frac{2\theta\eta^3}{\rho N_H(1-\eta)^2}\omega_{st}(H,\delta) + \frac{2\varsigma\eta^2}{\rho N_H(1-\eta)^2} = \chi(\delta) < \frac{3}{4}\varepsilon, \tag{14}$$

*and*

$$\frac{\eta}{1-\eta}\|\bar{X}_k - \bar{X}_{k-1}\|_u < \frac{\varepsilon}{4}.$$

*We can select the least* $N, M \in \mathbb{N}$*, for which inequality (14) is kept. Lastly, we find the lowest* $k \in \mathbb{N}$ *for which*

$$\|\bar{X}_k - \bar{X}_{k-1}\| < \frac{\varepsilon}{4}.\frac{1-\eta}{\eta} = \varepsilon^{'}.$$

*With these,* $\mid \bar{X}_k(s_p, t_q) - \bar{X}_{k-1}(s_p, t_q) \mid < \varepsilon^{'}$ *leads to* $\mid X^*(s_p, t_q) - \bar{X}_k(s_p, t_q) \mid < \varepsilon$*, and the desired accuracy* $\varepsilon$ *is achieved.*

Here, we propose an algorithm of numerical successive approximations to carry out the proposed method.

**The iterative algorithm**

**Input :** $a, b, c, d, \lambda, \varepsilon^{'}, N, M$ and the functions $H, f$.
**Step 1:** Put $h_1 = \frac{log(2\pi N)}{N}, h_2 = \frac{log(2\pi M)}{M}$ and $\bar{X}_0(s,t) := f_0(s,t)$.
**Step 2:** Compute $\bar{X}_k(s_p, t_q)$, $p = -N, ..., N, q = -M, ..., M$ by (9).
**Step 3:** Compute $T := max\{\mid \bar{x}_k(s_p, t_q) - \bar{x}_{k-1}(s_p, t_q) \mid\}$,
$p = -N, ..., N, q = -M, ..., M$ by (9).
**Step 4:** If $T < \varepsilon^{'}$, print $k$ and print $\bar{x}_k(s_p, t_q)$,
$p = -N, ..., N, q = -M, ..., M$, STOP.
Otherwise, put k:= k + 1 and go to Step 2.

## 5. Numerical experiments

In the section, $e_n(s,t) = \mid X^*(s_p, t_q) - \bar{X}_k(s_p, t_q) \mid$ is a pointwise error function in $(s,t) \in I$ and $\|e_n\|_\infty$ is a maximum absolute error, i.e., $\|e_n\|_\infty := \max\{\mid X^*(s_p, t_q) - \bar{X}_k(s_p, t_q) \mid p = -N, ..., N, q = -M, ..., M\}$. Denote

$$\rho_n = log_2\left(\frac{\|e_n\|_\infty^{(i)}}{\|e_n\|_\infty^{(i+1)}}\right),$$

where $X^*$ and $\bar{X}_k$ are the exact and the approximate solution of Eq. (1), respectively. In the above formula, $\|e_n\|_\infty^{(i)}$ denotes $\|e_n\|_\infty$ in the ith column of the tables.

We consider sinc grid points as:

$$S = \{s_{-N}, ..., s_0, ..., s_N\}$$
$$T = \{t_{-M}, ..., t_0, ..., t_M\}.$$

Let $[a,b]\times[c,d]=[0,1]\times[0,1], \lambda=1$, and let $\varphi_1=\varphi_2=\varphi$ be a conformal map defined as:

$$s=\varphi(x)=\frac{1}{2}\tanh(\frac{\pi}{2}\sinh(x))+\frac{1}{2},\quad s_p=\varphi(ph),\quad p=-N,...,N$$
$$t=\varphi(y)=\frac{1}{2}\tanh(\frac{\pi}{2}\sinh(y))+\frac{1}{2},\quad t_q=\varphi(qh),\quad q=-M,...,M.$$

Let us take $\alpha_1=\alpha_2=\frac{\pi}{2}$ and $d_1=d_2=\frac{\pi}{2},\ N=M$ for all examples. Then the step size of sinc is given as:

$$h_1=h_2=h=\frac{log(2\pi N)}{N}.$$

All computations were carried out with MAPLE 17 on a PC with 2.13 GHz frequency, Intel Pentium processor with 2 GB RAM.

**Example 1.** *Consider*

$$X(s,t)=s^2+t-\frac{0.114}{(s+1)(t^2+1)}+\int_0^1\int_0^1\frac{stxy^3}{(s+1)(t^2+1)}\sin(X(x,y))dxdy,$$
$$(s,t)\in[0,1]\times[0,1].$$

*The exact solution is given by*

$$X(s,t)=s^2+t.$$

*Applying the proposed iterative method for $N=M=25$, $\varepsilon'=10^{-20}$, we get $k=11$ iterations and in order to confirm the numerical stability, in the seventh column, we include the differences between the effective computed values $d_{p,q}=|\bar{X}_{11}(s_p,t_q)-\bar{Y}_{11}(s_p,t_q)|$, $p,q=-25,...,25$, where the change in the first term of the Picard sequence is 0.1 ($f(s,t):=f(s,t)+0.1$). For $N\in\{5,10,15,20,25\}$ and $\varepsilon'=10^{-15}$, get $k=9$. The results in Table 1 confirm the convergence of the algorithm, that is, $\|e_n\|_\infty\to 0$ when $h\to 0$.*

| $(s,t)$ | N=5 | N=10 | N=15 | N=20 | N=25 | $d_{p,q}$ |
|---|---|---|---|---|---|---|
| (0.1,0.1) | 1.668E-6 | 5.177E-8 | 1.328E-9 | 3.533E-11 | 3.944E-12 | 0.1000 |
| (0.2,0.2) | 3.059E-6 | 9.491E-8 | 2.434E-9 | 6.477E-11 | 7.230E-12 | 0.1000 |
| (0.3,0.3) | 4.235E-6 | 1.314E-7 | 3.370E-9 | 8.968E-11 | 1.001E-11 | 0.1001 |
| (0.4,0.4) | 5.244E-6 | 1.627E-7 | 4.173E-9 | 1.111E-10 | 1.239E-11 | 0.1001 |
| (0.5,0.5) | 6.118E-6 | 1.898E-7 | 4.868E-9 | 1.295E-10 | 1.446E-11 | 0.1001 |
| (0.6,0.6) | 6.882E-6 | 2.135E-7 | 5.477E-9 | 1.457E-10 | 1.627E-11 | 0.1001 |
| (0.7,0.7) | 7.557E-6 | 2.345E-7 | 6.014E-9 | 1.600E-10 | 1.786E-11 | 0.1011 |
| (0.8,0.8) | 8.157E-6 | 2.531E-7 | 6.491E-9 | 1.727E-10 | 1.928E-11 | 0.1011 |
| (0.9,0.9) | 8.693E-6 | 2.697E-7 | 6.918E-9 | 1.841E-10 | 2.055E-11 | 0.1021 |
| $\lVert e_n \rVert_\infty$ | 9.176E-6 | 2.847E-7 | 7.302E-9 | 1.943E-10 | 2.169E-11 | - |
| $\rho_n$ | - | 5.010 | 5.285 | 5.232 | 3.163 | - |
| k | 8 | 9 | 9 | 9 | 9 | - |
| Time (s) | 0.42 | 2.2 | 9.29 | 52.93 | 497.59 | - |

**Table 1:** *Numerical results in Example 1.*

**Example 2.** *Consider [3]*

$$X(s,t) = f(s,t) + \int_0^1 \int_0^1 H(s,t,x,y)\psi(X(x,y))dxdy, \ \ (s,t) \in [0,1] \times [0,1],$$

*where*

$$\begin{aligned} f(s,t) &= \frac{1}{(1+s+t)^2} - \frac{s}{6(1+t)}, \\ H(s,t,x,y) &= \frac{s}{1+t}(1+y+x), \\ \psi(\beta) &= \beta^2, \end{aligned}$$

*the exact solution is given by*

$$X^*(s,t) = \frac{1}{(1+s+t)^2}.$$

*Tables 2-3 show numerical results for different numbers of meshes, and the proposed method solutions are compared with results in [3]. Since they are from Table 3, the errors of the proposed method are much smaller than the errors described in [3] at the chosen points. Hence, our method has a moderate convergence rate.*

| $(s,t)=(\frac{1}{2^l},\frac{1}{2^l})$ | N=4 | N=8 | N=16 | N=32 | $d_{p,q}$ |
|---|---|---|---|---|---|
| 1 | 2.319E-4 | 6.882E-6 | 2.135E-7 | 4.667E-10 | 0.1000 |
| 2 | 2.061E-4 | 6.117E-6 | 1.898E-7 | 4.149E-10 | 0.1000 |
| 3 | 1.767E-4 | 5.244E-6 | 1.627E-7 | 3.556E-10 | 0.1000 |
| 4 | 1.427E-4 | 4.235E-6 | 1.314E-7 | 2.872E-10 | 0.1001 |
| 5 | 1.031E-4 | 3.059E-6 | 9.491E-8 | 2.074E-10 | 0.1001 |
| 6 | 5.622E-5 | 1.668E-6 | 5.177E-8 | 1.131E-10 | 0.1011 |
| $\|e_n\|_\infty$ | 3.092E-4 | 9.176E-6 | 2.847E-7 | 6.223E-10 | - |
| $\rho_n$ | - | 5.07 | 5.01 | 8.83 | - |
| k | 7 | 7 | 7 | 7 | - |
| Time (s) | 0 | 0.37 | 2.03 | 24.59 | - |

**Table 2:** *Numerical results in Example 2.*

| | Method in [3] | | | Present method | | |
|---|---|---|---|---|---|---|
| $N=M$ | $\|e_n\|_\infty$ | $\rho_n$ | *Iteration* | $\|e_n\|_\infty$ | $\rho_n$ | *Iteration* |
| 4 | $1.30\times10^{-2}$ | - | 3 | $3.092\times10^{-4}$ | - | 7 |
| 8 | $4.90\times10^{-3}$ | 1.4 | 3 | $9.176\times10^{-6}$ | 5.07 | 7 |
| 16 | $1.53\times10^{-3}$ | 1.7 | 3 | $2.847\times10^{-7}$ | 5.01 | 7 |
| 32 | $4.32\times10^{-4}$ | 1.8 | 3 | $6.223\times10^{-10}$ | 8.83 | 7 |

**Table 3:** *Comparison of maximum absolute errors with the method in [3] for Example 2.*

**Example 3.** *Consider [3, 16]*

$$X(s,t)=f(s,t)+\int_0^1\int_0^1(s\sin(y)+1)(X(x,y))^3dxdy,\ \ (s,t)\in[0,1]\times[0,1],$$

*where*

$$f(s,t)=s\cos(t)+\frac{1}{20}(\cos^4(1)-1)-\frac{1}{12}\sin(1)(\cos^2(1)+2),$$

*the exact solution is given by*

$$X^*(s,t)=s\cos(t).$$

*For $N=4$, $N=8$, $N=16$, $N=32$ and $\varepsilon'=10^{-15}$, the following results are obtained (see tables 4-5 ). As we can see from numerical results, when the size of N increases sinc method is much better than methods in [3] and [16].*

| $(s,t)=(\frac{1}{2^l},\frac{1}{2^l})$ | N=4 | N=8 | N=16 | N=32 | $d_{p,q}$ |
|---|---|---|---|---|---|
| 1 | 1.217E-5 | 4.417E-7 | 1.289E-8 | 2.147E-11 | 0.1012 |
| 2 | 2.231E-5 | 8.098E-7 | 2.363E-8 | 3.937E-11 | 0.1011 |
| 3 | 3.090E-5 | 1.121E-6 | 3.272E-8 | 5.451E-11 | 0.1011 |
| 4 | 3.825E-5 | 1.388E-6 | 4.051E-8 | 6.749E-11 | 0.1001 |
| 5 | 4.463E-5 | 1.619E-6 | 4.726E-8 | 7.874E-11 | 0.1001 |
| 6 | 5.021E-5 | 1.822E-6 | 5.317E-8 | 8.859E-11 | 0.1000 |
| $\|e_n\|_\infty$ | 6.695E-5 | 2.429E-6 | 7.089E-8 | 1.181E-10 | - |
| $\rho_n$ | - | 5.07 | 5.01 | 9.22 | - |
| k | 6 | 6 | 6 | 7 | - |
| Time (s) | 0 | 0.37 | 2.03 | 24.59 | - |

**Table 4:** *Numerical results in Example 3.*

| | Method in [3] | | | Method in [16] | | |
|---|---|---|---|---|---|---|
| N=M | $\|e_n\|_\infty$ | $\rho_n$ | Iteration | $\|e_n\|_\infty$ | $\rho_n$ | Iteration |
| 4 | $5.20\times10^{-2}$ | - | 4 | $4.3\times10^{-2}$ | - | - |
| 8 | $2.06\times10^{-2}$ | 1.3 | 5 | $1.7\times10^{-2}$ | 1.33 | - |
| 16 | $6.80\times10^{-3}$ | 1.6 | 6 | $5.4\times10^{-3}$ | 1.65 | - |
| 32 | $1.92\times10^{-3}$ | 1.8 | 6 | $1.5\times10^{-3}$ | 1.84 | - |

| | Present method | | |
|---|---|---|---|
| N=M | $\|e_n\|_\infty$ | $\rho_n$ | Iteration |
| 4 | $6.695\times10^{-5}$ | - | 6 |
| 8 | $2.429\times10^{-6}$ | 4.79 | 6 |
| 16 | $7.089\times10^{-8}$ | 5.10 | 6 |
| 32 | $1.181\times10^{-10}$ | 9.22 | 7 |

**Table 5:** *Comparison of maximum absolute errors in Example 3.*

## 6. Conclusion

This research provided an iterative numerical method based on sinc quadrature for solving nonlinear Eq. (1). This iterative method has two positive features. First, we do not have to deal with any system of nonlinear equations. Second, it is quite simple to apply and create an algorithm. The main results are Theorem 5, Theorem 6 and Theorem 7.

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