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A note on the number of plane partitions and r-component multipartitions of n

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Abstract. Using elementary methods, we prove new formulas for $\mathrm{pp}(n)$, the number of plane partitions of n, $\mathrm{pp}_r(n)$, the number of plane partitions of n with at most r rows, $\mathrm{pp}^s(n)$, and the number of strict plane partitions of n and $\mathrm{pp}^{so}(n)$, the number of symmetric plane partitions of n. We also give new formulas for $P_r(n)$, the number of r-component multipartitions of n.

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1. Introduction

Let n be a positive integer. We denote $[n] = \{1, 2, \ldots, n\}$. A partition of n is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of positive integers such that $|\lambda| = \lambda_1 + \cdots + \lambda_m = n$. We define $\mathrm{p}(n)$ as the number of partitions of n and for convenience, we define $p(0) = 1$. This notion has the following generalization: A *plane partition* of n is an array $(n_{ij})_{i,j\in[n]}$ of nonnegative integers such that

$$\sum_{i,j\in[n]} n_{ij} = n \text{ and } n_{ij} \geq n_{i'j'} \text{ for all } i, j, i', j' \in [n] \text{ such that } i \leq i' \text{ and } j \leq j'.$$

If $n_{ij} > n_{i(j+1)}$ whenever $n_{ij} \neq 0$, then we shall call such a partition *strict*. If $n_{ij} = n_{ji}$ for all i and j, then the partition is called *symmetric*.

For example, there are 6 plane partitions of $n = 3$, namely:

$$\begin{matrix} 3\,0\,0 \\ 0\,0\,0, \\ 0\,0\,0 \end{matrix} \quad \begin{matrix} 2\,1\,0 \\ 0\,0\,0, \\ 0\,0\,0 \end{matrix} \quad \begin{matrix} 2\,0\,0 \\ 1\,0\,0, \\ 0\,0\,0 \end{matrix} \quad \begin{matrix} 1\,1\,1 \\ 0\,0\,0, \\ 0\,0\,0 \end{matrix} \quad \begin{matrix} 1\,1\,0 \\ 1\,0\,0, \\ 0\,0\,0 \end{matrix} \quad \begin{matrix} 1\,0\,0 \\ 1\,0\,0. \\ 1\,0\,0 \end{matrix}$$

Note that four of them are strict partitions and two of them are symmetric. Moreover, three of them have nonzero entries only on the first row and five of them have nonzero entries on the first two rows.

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We denote by $\mathrm{pp}(n)$ the total number of plane partitions of n and, define $\mathrm{pp}(0) = 1$. The properties of $\mathrm{pp}(n)$ have been extensively studied in literature. As curiosity, we mention that the function $\mathrm{pp}(n)$ appears in physics in connection with the enumeration of small black holes in string theory, see [5, Appendix E].

Let $k \geq 1$ be an integer. We denote by $\mathrm{pp}_r(n)$ the number of plane partitions with at most r rows, and $\mathrm{pp}_k(0) = 1$. Note that $\mathrm{pp}_1(n) = \mathrm{p}(n)$ for all $n \geq 0$. Also, if $k \geq n$, then $\mathrm{pp}_k(n) = \mathrm{pp}(n)$. In the example above, we have $\mathrm{pp}_1(3) = \mathrm{p}(3) = 3$, $\mathrm{pp}_2(3) = 5$ and $\mathrm{pp}_3(3) = \mathrm{pp}(3) = 6$.

We denote by $\mathrm{pp}^s(n)$ the number of strict plane partitions of n, and by $\mathrm{pp}^{so}(n)$ the number of strict plane partitions of n with odd parts. We set $\mathrm{pp}^s(0) = \mathrm{pp}^{so}(n) = 1$. It is well known that $\mathrm{pp}^{so}(n)$ also counts the number of symmetric plane partitions of n.

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \geq 0$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})}, \ |z| < 1. \tag{1}$$

See [1, Chapter 5] for further details.

Our aim is to provide new formulas for $\mathrm{pp}(n)$, $\mathrm{pp}_r(n)$, $\mathrm{pp}^s(n)$ and $\mathrm{pp}^{so}(n)$ using their generating functions and the relation with the restricted partition function $\mathrm{p}_{\mathbf{a}}(n)$; see Proposition 1. In Theorem 1, we prove a new formula for $\mathrm{pp}(n)$ in terms of binomial coefficients

$$f_{s,\ell} = |\{(j_1, \ldots, j_s) \ : \ j_1 + \cdots + j_s = \ell, \ 0 \leq j_i \leq \ell \text{ for all } 1 \leq i \leq s\}| = \binom{s + \ell - 1}{\ell}.$$

Similarly, we provide formulas for $\mathrm{pp}_r(n)$, $\mathrm{pp}^s(n)$ and $\mathrm{pp}^{so}(n)$ in Theorem 2, Theorem 3 and Theorem 4, respectively. Furthermore, using a result from [3] regarding the restricted partition function, we deduce other formulas for $\mathrm{pp}(n)$, $\mathrm{pp}_r(n)$, $\mathrm{pp}^s(n)$ and $\mathrm{pp}^{so}(n)$; see Theorem 6.

An r-component multipartition of n is an r-tuple $\lambda = (\lambda^1, \ldots, \lambda^r)$ of partitions of n such that $|\lambda| = |\lambda^1| + \cdots + |\lambda^r| = n$; see [2]. We denote by $P_r(n)$ the number or r-component multipartitions of n and $P_r(0) = 1$. In Proposition 3, we show that

$$\mathrm{pp}_r(n) = \sum_{\substack{0 \leq t_1 \leq r-1 \\ \vdots \\ 0 \leq t_{r-1} \leq 1}} (-1)^{t_1 + \cdots + t_{r-1}} P_r(n - t_1 - 2t_2 - \cdots - (r-1)t_{r-1}),$$

where $P_r(j) = 0$ for $j < 0$.

In Theorem 7, we prove a new formula for $P_r(n)$ and deduce from it a new expression for $\mathrm{pp}_r(n)$; see Corollary 1. Moreover, in Theorem 8 we obtain other formula for $P_r(n)$.

2. New formulas for the number of plane partitions

- Let $n \geq r$ be two positive integers.
- Let $\mathrm{pp}(n)$ be the number of plane partitions of n. We define $\mathrm{pp}(0) = 1$.
- Let $\mathrm{pp}_r(n)$ be the number of plane partitions of n with at most r rows. We also define $\mathrm{pp}_r(0) = 1$. Note that if $n \leq r$, then $\mathrm{pp}(n) = \mathrm{pp}_r(n)$.
- Let $\mathrm{pp}^s(n)$ be the number of strict plane partitions of n. We set $\mathrm{pp}^s(0) = 1$.
- Let $\mathrm{pp}^{so}(n)$ be the number of strict plane partitions of n with odd parts. We set $\mathrm{pp}^{so}(n) = 1$. As shown in [6], $\mathrm{pp}^{so}(n)$ is equal to the number of symmetric plane partitions of n.

MacMahon [8] proved that

$$\sum_{n=0}^{\infty} \mathrm{pp}(n) z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^n} \text{ for } |z| < 1. \tag{2}$$

A refinement of this result is as follows:

$$\sum_{n=0}^{\infty} \mathrm{pp}_r(n) z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^{\min\{n,r\}}} \text{ for } |z| < 1, \tag{3}$$

see [1, Equation (10.1)].

Gordon and Houten [7] proved that

$$\sum_{n=0}^{\infty} \mathrm{pp}^s(n) z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^{\lfloor (n+1)/2 \rfloor}} \text{ for } |z| < 1.$$

Gordon [6] also proved that

$$\sum_{n=0}^{\infty} \mathrm{pp}^{so}(n) z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^{2n+1})} \prod_{n=1}^{\infty} \frac{1}{(1-z^{2n})^n} \text{ for } |z| < 1.$$

Give n, k two positive integers, we denote by $n^{[k]}$ the sequence $n, n, \ldots, n$ of length k, e.g. $2^{[3]} = 2, 2, 2$. We consider the following sequences of integers:

$$\begin{aligned}
&\mathbf{n} := (1, 2^{[2]}, 3^{[3]}, \ldots, n^{[n]}),\\
&\mathbf{n}_r := (1, 2^{[\min\{2,r\}]}, 3^{[\min\{3,r\}]}, \ldots, n^{[\min\{n,r\}]}),\\
&\mathbf{n}^s := (1, 2^{[1]}, 3^{[2]}, 4^{[2]}, \ldots, n^{[\lfloor (n+1)/2 \rfloor]}) \text{ and}\\
&\mathbf{n}^{so} := (1, 2^{[1]}, 3^{[1]}, 4^{[2]}, \ldots, n^{[\lambda(n)]}),
\end{aligned}$$

where $\lambda(n) = \begin{cases} 1, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases}.$

Proposition 1. *Let $n \geq r$ be two positive integers. We have that:*

(1) $\mathrm{pp}(n) = p_{\mathbf{n}}(n)$ *for all* $n \geq 0$.

(2) $\mathrm{pp}_r(n) = p_{\mathbf{n}_r}(n)$ *for all* $n \geq 0$.

(3) $\mathrm{pp}^s(n) = p_{\mathbf{n}^s}(n)$ *for all* $n \geq 0$.

(4) $\mathrm{pp}^{so}(n) = p_{\mathbf{n}^{so}}(n)$ *for all* $n \geq 0$.

Proof. (1): From equation (1), it follows that

$$\sum_{m=0}^{\infty} p_{\mathbf{n}}(m) z^m = \prod_{j=1}^{n} \frac{1}{(1-z^j)^j} \tag{4}$$
$$= (1+z+z^2+\cdots)(1+z^2+z^4+\cdots)\cdots(1+z^n+z^{2n}+\cdots).$$

On the other hand, from equation (2), we have

$$\sum_{m=0}^{\infty} \mathrm{pp}(m) z^m = (1+z+z^2+\cdots)(1+z^2+z^4+\cdots) \tag{5}$$
$$\cdots(1+z^n+z^{2n}+\cdots)(1+z^{n+1}+\cdots)\cdots .$$

Comparing (4) and (5), it follows that

$$\mathrm{pp}(m) = p_{\mathbf{n}}(m) \text{ for all } 0 \leq m \leq n,$$

thus, in particular, $\mathrm{pp}(n) = p_{\mathbf{n}}(n)$, as required.

(2, 3, 4): The proof is similar to the proof of (1), so we omit it. □

For all $1 \leq s \leq n$ and $\ell \geq 0$ we consider:

$$f_{s,\ell} = |\{(j_1,\ldots,j_s) \ : \ j_1+\cdots+j_s = \ell, \text{ where } 0 \leq j_i \leq \ell \text{ for all } 1 \leq i \leq s\}|. \tag{6}$$

Note that $f_{s,\ell}$ is equal to the number of monomials of degree ℓ in s variables, therefore:

$$f_{s,\ell} = \binom{s+\ell-1}{\ell}, \text{ for all } 1 \leq s \leq n \text{ and } \ell \geq 0. \tag{7}$$

These binomial coefficients $f_{s,\ell}$ play a central role in the following.

Theorem 1. *Let $n \geq 1$ be an integer. We have that*

$$\mathrm{pp}(n) = \sum_{(\ell_1,\ldots,\ell_n)\in \mathbf{A}_n} \prod_{s=2}^{n} \binom{s+\ell_s-1}{\ell_s},$$

where $\mathbf{A}_n := \{(\ell_1,\ldots,\ell_n) \in \mathbb{N}^n \ : \ \ell_1 + 2\ell_2 + \cdots + n\ell_n = n\}$.

Proof. From Proposition 1(1) it follows that

$$\begin{aligned}\mathrm{pp}(n) = \#\{(j_1,\dots,j_{\binom{n+1}{2}}) \in \mathbb{N}^{\binom{n+1}{2}} : j_1 + 2j_2 + 2j_3 + \cdots + nj_{\binom{n}{2}+1} \\ + \cdots + nj_{\binom{n+1}{2}} = n\}.\end{aligned} \tag{8}$$

We denote $\ell_1 = j_1,\ \ell_2 = j_2 + j_3, \dots, \ell_n = j_{\binom{n}{2}+1} + \cdots + j_{\binom{n+1}{2}}$. From (6) and (8) it follows that

$$\mathrm{pp}(n) = \sum_{(\ell_1,\dots,\ell_n)\in\mathbf{A}_n} \prod_{s=2}^{n} f_{s,\ell_s}. \tag{9}$$

Therefore, from (9) and (7) we get the required formula. □

Remark 1. Note that $p(n) = \#\mathbf{A}_n$ for all $n \geq 1$.

Example 1. If we take $n = 3$ in Theorem 1, we have $\mathbf{A}_3 = \{(3,0,0),(1,1,0),(0,0,1)\}$ and thus

$$\mathrm{pp}(3) = \sum_{(\ell_1,\ell_2,\ell_3)\in\mathbf{A}_3} \prod_{s=2}^{3} \binom{\ell_s + s - 1}{\ell_s} = 1 + \binom{1}{1}\binom{2}{1} + \binom{3}{1} = 6.$$

Theorem 2. *Let $n > r \geq 2$ be two positive integers. We have that*

$$\mathrm{pp}_r(n) = \sum_{(\ell_1,\dots,\ell_n)\in\mathbf{A}_n} \prod_{s=2}^{n} \binom{\ell_s + \min\{s,r\} - 1}{\ell_s}.$$

Proof. From Proposition 1(2) it follows that

$$\begin{aligned}\mathrm{pp}_r(n) = \#\{(j_1,\dots,j_{rn-\binom{r}{2}}) \in \mathbb{N}^{rn-\binom{r}{2}} : j_1+2j_2+2j_3+\cdots+rj_{\binom{r}{2}+1}+\cdots+rj_{\binom{r+1}{2}} \\ +\cdots+nj_{rn-\binom{r+1}{2}+1}+\cdots+nj_{rn-\binom{r}{2}} = n\}.\end{aligned} \tag{10}$$

We denote $\ell_1 = j_1,\ \ell_2 = j_2+j_3,\dots,\ell_n = j_{rn-\binom{r+1}{2}+1}+\cdots+j_{rn-\binom{r}{2}}$. From (10) and (6) it follows that:

$$\mathrm{pp}_r(n) = \sum_{(\ell_1,\dots,\ell_n)\in\mathbf{A}_n} \prod_{s=2}^{n} f_{\min\{r,s\},\ell_s} \tag{11}$$

Therefore, from (11) and (7) we get the required formula. □

Example 2. Let $n = 3$ and $r = 2$. Since $\mathbf{A}_3 = \{(3,0,0),(1,1,0),(0,0,1)\}$, according to Theorem 2, the number of plane partitions of 3 with at most 2 rows is:

$$\mathrm{pp}_2(3) = \sum_{(\ell_1,\ell_2,\ell_3)\in\mathbf{A}_3} \prod_{s=2}^{3} \binom{\ell_s+1}{\ell_s} = 1 + \binom{2}{1} + \binom{2}{1} = 5.$$

Theorem 3. *Let $n \geq 3$ be an integer. We have that*

$$\mathrm{pp}^s(n) = \sum_{(\ell_1,\dots,\ell_n)\in\mathbf{A}_n} \prod_{s=3}^{n} \binom{\ell_s + \lfloor\frac{s+1}{2}\rfloor - 1}{\ell_s}.$$

Proof. Assume $n = 2p$. From Proposition 1(3) it follows that

$$\mathrm{pp}^s(n) = \#\{(j_1,\ldots,j_{p^2+p}) : j_1+2j_2+3j_3+3j_4+\ldots+nj_{p^2+1}+\cdots+nj_{p^2+p}=n\} \quad (12)$$

We denote $\ell_1 = j_1,\ \ell_2 = j_2,\ \ell_3 = j_3 + j_4, \ldots, \ell_n = j_{p^2+1} + \cdots + j_{p^2+p}$. From (12) and (6) it follows that

$$\mathrm{pp}^s(n) = \sum_{(\ell_1,\ldots,\ell_n)\in \mathbf{A}_n} \prod_{s=3}^{n} f_{\lfloor \frac{s+1}{2} \rfloor, \ell_s} \quad (13)$$

Therefore, from (13) and (7) we get the required formula. The case $n = 2p + 1$ is similar. $\square$

Example 3. Let $n = 3$. Since $\mathbf{A}_3 = \{(3, 0, 0), (1, 1, 0), (0, 0, 1)\}$, according to Theorem 3, the number of strict plane partitions of 3 is:

$$\mathrm{pp}^s(3) = \sum_{(\ell_1,\ell_2,\ell_3)\in \mathbf{A}_3} \binom{\ell_3+1}{\ell_3} = 1 + 1 + \binom{2}{1} = 4.$$

Theorem 4. *Let $n \geq 3$ be a positive integer. We have that*

$$\mathrm{pp}^{so}(n) = \sum_{(\ell_1,\ldots,\ell_n)\in \mathbf{A}_n} \prod_{s=2}^{\lfloor \frac{n}{2} \rfloor} \binom{\ell_s+s-1}{\ell_s}.$$

Proof. Assume $n = 2p$. From Proposition 1(4) it follows that

$$\mathrm{pp}^s(n) = \#\{(j_1,\ldots,j_{p^2+p}) \ :\ j_1 + 2j_2 + 3j_3 + 4j_4 + 4j_5 + 5j_6 + \ldots + (n-1)j_{\binom{p}{2}} + nj_{\binom{p}{2}+1} + \cdots + nj_{\binom{p+1}{2}} = n\}.$$

We denote $\ell_1 = j_1,\ \ell_2 = j_2,\ \ell_3 = j_3, \ldots,\ \ell_{n-1} = j_{\binom{p}{2}},\ \ell_n = j_{\binom{p}{2}+1} + \cdots + j_{\binom{p+1}{2}}$. From (12) and (6) it follows that

$$\mathrm{pp}^s(n) = \sum_{(\ell_1,\ldots,\ell_n)\in \mathbf{A}_n} \prod_{s=2}^{\lfloor \frac{n}{2} \rfloor} f_{s,\ell_{2s}} \quad (14)$$

Therefore, from (14) and (7) we get the required formula. The case $n = 2p + 1$ is similar. $\square$

Example 4. Let $n = 3$. Since $\mathbf{A}_3 = \{(3, 0, 0), (1, 1, 0), (0, 0, 1)\}$ and $\lfloor \frac{3}{2} \rfloor = 1$, according to Theorem 4, the number of symmetric plane partition of 3 is:

$$\mathrm{pp}^{so}(3) = \mathrm{p}(3) = \#\mathbf{A}_3 = 3.$$

The unsigned Stirling numbers $\left[r \atop k \right]$'s are defined by

$$\binom{n+r-1}{r-1} = \frac{1}{n(r-1)!} n^{(r)} = \frac{1}{(r-1)!} \left(\left[r \atop r \right] n^{r-1} + \cdots \left[r \atop 2 \right] n + \left[r \atop 1 \right] \right).$$

We recall the following result:

Theorem 5 (see [3, Theorem 2.8(2)] and [4])**.** *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a sequence of positive integers and let* D *be the least common multiple of* $a_1, \ldots, a_r$*. We have that*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{m=0}^{r-1} \sum_{\substack{0 \le j_1 \le \frac{D}{a_1}-1, \ldots, 0 \le j_r \le \frac{D}{a_r}-1 \\ a_1 j_1 + \cdots + a_r j_r \equiv n (\bmod D)}} \sum_{k=m}^{r-1} \begin{bmatrix} r \\ k+1 \end{bmatrix} \times (-1)^{k-m} \binom{k}{m} D^{-k} (a_1 j_1 + \cdots + a_r j_r)^{k-m} n^m.$$

Theorem 6.

(1) *For* $n \ge 1$ *we have*

$$\mathrm{pp}(n) = \frac{1}{(\binom{n+1}{2}-1)!} \sum_{m=0}^{\binom{n+1}{2}-1} \sum_{\substack{0 \le \ell_1 \le D_n - 1, \ldots, 0 \le \ell_n \le D_n - n \\ \ell_1 + 2\ell_2 + \cdots + n\ell_n \equiv n (\bmod D_n)}} \prod_{s=2}^{n} \binom{\ell_s + s - 1}{\ell_s} \times \sum_{k=m}^{\binom{n+1}{2}-1} \begin{bmatrix} \binom{n+1}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D_n^{-k} (\ell_1 + 2\ell_2 + \cdots + n\ell_n)^{k-m} n^m,$$

where D_n *is the least common multiple of* $1, 2, \ldots, n$.

(2) *For* $n \ge 3$ *and* $2 \le r \le n-1$ *we have*

$$\mathrm{pp}_r(n) = \frac{1}{(nr - \binom{r}{2} - 1)!} \sum_{m=0}^{nr-\binom{r}{2}-1} \sum_{\substack{0 \le \ell_1 \le D_n - \min\{1,r\}, \\ \cdots \\ 0 \le \ell_n \le D_n - \min\{n,r\} \\ \ell_1 + 2\ell_2 + \cdots + n\ell_n \equiv n (\bmod D_n)}} \prod_{s=2}^{n} \binom{\ell_s + \min\{s,r\} - 1}{\ell_s} \times \sum_{k=m}^{nr-\binom{r}{2}-1} \begin{bmatrix} nr - \binom{r}{2} \\ k+1 \end{bmatrix} (-1)^{k-m} \binom{k}{m} D_n^{-k} (\ell_1 + 2\ell_2 + \cdots + n\ell_n)^{k-m} n^m.$$

(3) *For* $n \ge 1$ *we have*

$$\mathrm{pp}^s(n) = \frac{1}{(\lfloor \frac{n+1}{2} \rfloor \cdot \lfloor \frac{n+2}{2} \rfloor - 1)!} \sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor \cdot \lfloor \frac{n+2}{2} \rfloor - 1} \sum_{\substack{0 \le \ell_1 \le D_n - 1, \\ \ldots, \\ 0 \le \ell_n \le D_n - \lfloor \frac{n+1}{2} \rfloor \\ \ell_1 + 2\ell_2 + \cdots + n\ell_n \equiv n (\bmod D_n)}} \prod_{s=3}^{n} \binom{\ell_s + \lfloor \frac{s+1}{2} \rfloor - 1}{\ell_s} \times \sum_{k=m}^{\lfloor \frac{n+1}{2} \rfloor \cdot \lfloor \frac{n+2}{2} \rfloor - 1} \begin{bmatrix} \lfloor \frac{n+1}{2} \rfloor \cdot \lfloor \frac{n+2}{2} \rfloor \\ k+1 \end{bmatrix} (-1)^{k-m} \times \binom{k}{m} D_n^{-k} (\ell_1 + 2\ell_2 + \cdots + n\ell_n)^{k-m} n^m.$$

(4) *For $n \geq 1$ we have*

$$\begin{aligned}\mathrm{pp}^{so}(n) =& \frac{1}{\left(\binom{\lfloor \frac{n}{2} \rfloor}{2} + \varepsilon(n) - 1\right)!} \\ &\times \sum_{m=0}^{\binom{\lfloor \frac{n}{2} \rfloor}{2}+\varepsilon(n)-1} \sum_{\substack{0 \leq \ell_1 \leq D_n - \lambda(1), \ldots, 0 \leq \ell_n \leq D_n - \lambda(n) \\ \ell_1 + 2\ell_2 + \cdots + n\ell_n \equiv n (\bmod D_n)}} \prod_{s=2}^{\lfloor \frac{n}{2} \rfloor} \binom{\ell_s + s - 1}{\ell_s} \\ &\times \sum_{k=m}^{\binom{\lfloor \frac{n}{2} \rfloor}{2}+\varepsilon(n)-1} \begin{bmatrix} \binom{\lfloor \frac{n}{2} \rfloor}{2} + \varepsilon(n) \\ k+1 \end{bmatrix} (-1)^{k-m} \\ &\times \binom{k}{m} D_n^{-k} (\ell_1 + 2\ell_2 + \cdots + n\ell_n)^{k-m} n^m,\end{aligned}$$

where $\lambda(n) = \begin{cases} 1, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases}$ *and* $\varepsilon(n) = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$.

Proof. (1): Note that the length of the sequence $\mathbf{n} = (1, 2^{[2]}, \ldots, n^{[n]})$ is $\binom{n+1}{2}$. From Proposition 1(1) and Theorem 5 it follows that

$$\begin{aligned}\mathrm{pp}(n) = \mathrm{p}_{\mathbf{n}}(n) =& \frac{1}{\left(\binom{n+1}{2} - 1\right)!} \sum_{m=0}^{\binom{n+1}{2}-1} \sum_{(j_1, \ldots, j_{\binom{n+1}{2}}) \in \mathbf{C}_n} \sum_{k=m}^{\binom{n+1}{2}-1} \begin{bmatrix} \binom{n+1}{2} \\ k+1 \end{bmatrix} \\ &\times (-1)^{k-m} \binom{k}{m} D_n^{-k} (j_1 + 2j_2 + 2j_3 + \cdots + nj_{\binom{n}{2}+1} + \cdots + nj_{\binom{n+1}{2}})^{k-m} n^m, \quad (15)\end{aligned}$$

where

$$\begin{aligned}\mathbf{C}_n = \{&(j_1, \ldots, j_{\binom{n+1}{2}}) : 0 \leq j_1 \leq D_n - 1, 0 \leq j_2 \leq \frac{D_n}{2} - 1, 0 \leq j_3 \leq \frac{D_n}{2} - 1, \ldots, 0 \\ &\leq j_{\binom{n}{2}+1} \leq \frac{D_n}{n} - 1, \ldots, 0 \leq j_{\binom{n+1}{2}} \leq \frac{D_n}{n} - 1 \text{ such that} \\ &\quad j_1 + 2j_2 + 2j_3 + \cdots + nj_{\binom{n}{2}+1} + \cdots + nj_{\binom{n+1}{2}} \equiv n (\bmod D_n)\}.\end{aligned}$$

We let $\ell_1 = j_1,\ \ell_2 = j_2 + j_3,\ \ldots, \ell_n = j_{\binom{n}{2}+1} + \cdots + j_{\binom{n+1}{2}}$.

Note that if $(j_1, j_2, \ldots, j_{\binom{n+1}{2}}) \in \mathbf{C}_n$, then $0 \leq \ell_t \leq D_n - t$ for all $1 \leq t \leq n$, and moreover, $\ell_1 + 2\ell_2 + \cdots + n\ell_n \equiv n (\bmod D_n)$. From (15), using a similar argument as in the proof of Theorem 1, we get the required result.

(2): Note that the length of the sequence $\mathbf{n}_r := (1, 2^{[\min\{2,r\}]}, 3^{[\min\{3,r\}]}, \ldots, n^{[\min\{n,r\}]})$ is $\binom{r+1}{2} + r(n-r) = nr - \binom{r}{2}$. The rest of the proof is similar to the proof of (1), using Proposition 1(2), Theorem 2 and Theorem 5.

(3): Note that the length of the sequence $\mathbf{n}^s := (1, 2^{[1]}, 3^{[2]}, 4^{[2]}, \ldots, n^{[\lfloor (n+1)/2 \rfloor]})$ is p^2+p if $n = 2p$, and $(p+1)^2$ if $n = 2p+1$. Hence, in both cases, the length is $\lfloor \frac{n+1}{2} \rfloor \cdot \lfloor \frac{n+2}{2} \rfloor$. The rest of the proof is similar to the proof of (1), using Proposition 1(3), Theorem 3 and Theorem 5.

(4): Note that the length of the sequence $\mathbf{n}^{so} := (1, 2^{[1]}, 3^{[1]}, 4^{[2]}, \ldots, n^{[\lambda(n)]})$ is $\binom{p}{2}$ if $n = 2p$, and $\binom{p}{2} + 1$ if $n = 2p+1$. Hence, in both cases, the length is $\binom{\lfloor \frac{n}{2} \rfloor}{2} + \varepsilon(n)$. The rest of the proof is similar to the proof of (1), using Proposition 1(4), Theorem 4 and Theorem 5. □

3. New formulas for the number of multipartitions with r components

If we denote by $P_r(n)$ the number of r-component multipartitions of n, we have that

$$\sum_{n=0}^{\infty} P_r(n) z^n = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^r}, \tag{16}$$

where $P_r(0) = 0$ by convention; see [2]. We consider the sequence

$$\mathbf{n}^r = (1^{[r]}, 2^{[r]}, \ldots, n^{[r]}).$$

Proposition 2. *With the above notations, we have $P_r(n) = \mathrm{p}_{\mathbf{n}^r}(n)$ for all $n \geq 0$.*

Proof. It follows from (1) and (16), using a similar argument as in the proof of Proposition 1(1). □

Theorem 7. *Let $n \geq 4$ and $r \geq 2$ be two positive integers such that $n > r$. We have that*

$$P_r(n) = \sum_{(\ell_1,\ldots,\ell_n) \in \mathbf{A}_n} \prod_{s=1}^{n} \binom{\ell_s + r - 1}{\ell_s}.$$

Proof. From Proposition 2 it follows that

$$\begin{aligned} P_r(n) = \#\{(j_1, \ldots, j_{nr}) : j_1 + \cdots + j_r + 2j_{r+1} + \cdots + 2j_{2r} \\ + \cdots + nj_{nr-r+1} + \cdots + nj_{nr} = n\} \end{aligned} \tag{17}$$

We denote $\ell_1 = j_1 + \cdots + j_r$, $\ell_2 = j_{r+1} + \cdots + j_{2r}, \ldots, \ell_n = j_{nr-r+1} + \cdots + j_{nr}$. From (17) and (6) it follows that

$$P_r(n) = \sum_{(\ell_1,\ldots,\ell_n) \in \mathbf{A}_n} \prod_{s=1}^{n} f_{r,\ell_s} \tag{18}$$

Therefore, from (18) and (7) we get the required formula. □

Example 5. Let $n = 4$ and $r = 2$. Since $\mathbf{A}_4 = \{(4,0,0,0), (2,1,0,0), (1,0,1,0), (0,2,0,0), (0,0,0,1)\}$ and $f_{2,\ell} = \ell + 1$ for $0 \leq \ell \leq 4$, from Theorem 7 we have that

$$P_2(4) = 5 + 3 \cdot 2 + 2 \cdot 2 + 3 + 2 = 20.$$

Let $\mathrm{pp}_r(n)$ be the number of plane partitions of n with at most r rows. From (3) and (16) it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{pp}_r(n)z^n &= \prod_{j=1}^{r-1}(1-z^j)^{r-j}\sum_{n=0}^{\infty}P_r(n)z^n \\ &= \sum_{\substack{0\leq t_1\leq r-1\\ 0\leq t_2\leq r-2\\ \vdots\\ 0\leq t_{r-1}\leq 1}}(-1)^{t_1+\cdots+t_{r-1}}\sum_{n=0}^{\infty}P_r(n-t_1-2t_2-\cdots-(r-1)t_{r-1})z^n, \quad (19)\end{aligned}$$

where $P_r(j)=0$ for $j<0$. As a direct consequence of (19), we get the following result:

Proposition 3. *For all $n\geq 0$ we have that*

$$\mathrm{pp}_r(n) = \sum_{\substack{0\leq t_1\leq r-1\\ \vdots\\ 0\leq t_{r-1}\leq 1}}(-1)^{t_1+\cdots+t_{r-1}}P_r(n-t_1-2t_2-\cdots-(r-1)t_{r-1}).$$

As a consequence of Proposition 3 and Theorem 7, we get:

Corollary 1. *For all $n\geq 4$ and $n>r\geq 2$, we have that*

$$\begin{aligned}\mathrm{pp}_r(n) = &\sum_{\substack{0\leq t_1\leq r-1\\ \vdots\\ 0\leq t_{r-1}\leq 1\\ n<4+t_1+2t_2+\cdots+(r-1)t_{r-1}}}(-1)^{t_1+\cdots+t_{r-1}}P_r(n-t_1-2t_2-\cdots-(r-1)t_{r-1}) \\ &+ \sum_{\substack{0\leq t_1\leq r-1\\ \vdots\\ 0\leq t_{r-1}\leq 1\\ n\geq 4+t_1+2t_2+\cdots+(r-1)t_{r-1}}}(-1)^{t_1+\cdots+t_{r-1}}\sum_{\substack{(\ell_1,\ldots,\ell_n)\in\\ \mathbf{A}_{n-t_1-2t_2-\cdots-(r-1)t_{r-1}}}}\prod_{s=1}^{n}\binom{\ell_s+r-1}{\ell_s}.\end{aligned}$$

Theorem 8. *Let $n\geq 4$ and $r\geq 2$ be two positive integers such that $n\geq r$. We have that*

$$\begin{aligned}P_r(n) = &\frac{1}{(nr-1)!}\sum_{m=0}^{nr-1}\sum_{\substack{0\leq\ell_1\leq D_n-1,\ldots,0\leq\ell_n\leq D_n-n\\ \ell_1+2\ell_2+\cdots+n\ell_n\equiv n(\bmod D_n)}}\prod_{s=1}^{n}\binom{\ell_s+r-1}{\ell_s} \\ &\times\sum_{k=m}^{nr-1}\begin{bmatrix}nr\\ k+1\end{bmatrix}(-1)^{k-m}\binom{k}{m}D_n^{-k}(\ell_1+2\ell_2+\cdots+n\ell_n)^{k-m}n^m.\end{aligned}$$

Proof. From Proposition 2 and Theorem 5 it follows that

$$P_r(n) = \frac{1}{(nr-1)!}\sum_{m=0}^{nr-1} \sum_{\substack{0\leq j_1\leq D_n-1,\ldots,0\leq j_r\leq D-1\\ \vdots \\ 0\leq j_{nr-r+1}\leq \frac{D_n}{n}-1,\ldots,0\leq j_{nr}\leq \frac{D_n}{n}-1\\ j_1+\cdots+j_r+\cdots+nj_{nr-r+1}+\cdots+nj_{nr}\equiv n(\bmod\ D_n)}} \sum_{k=m}^{nr-1} \begin{bmatrix} nr \\ k+1 \end{bmatrix} (-1)^{k-m}$$
$$\times\binom{k}{m} D_n^{-k}(j_1+\cdots+j_r+\cdots+nj_{rn-r+1}+\cdots+nj_{rn})^{k-m} n^m. \tag{20}$$

We let $\ell_1 = j_1+\cdots+j_r, \ldots, \ell_n = j_{nr-r+1}+\cdots+j_{rn}$. It is easy to see that $0\leq \ell_t \leq D_n - t$ for $1\leq t\leq n$. Therefore, from (20), using a similar argument as in the proof of Theorem 7, there follows the required formula. □

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