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Dynamical Behaviours of Some Horseshoe-like Maps on the Cantor Dust $C \times C$

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Abstract. Dynamical systems are one of the important research topics that have applications in many disciplines. In particular, investigating whether a dynamical system is chaotic gives us important information about the dynamical system. In this paper, some dynamical systems are defined on the Cantor dust $C \times C$ inspired by the construction of Smale's horseshoe map. We also express these dynamical systems via ternary numbers and compute their periodic points. We thus obtain some dynamical systems which are chaotic in the sense of Devaney. Finally, we compare these dynamical systems according to whether they are topologically equivalent or not.

AMS subject classifications: 37C25, 37D45, 28A80, 37B10

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1. Introduction

Dynamical systems have an important place not only in mathematics but also in many different sciences and many studies have been conducted on this subject ([7, 8, 9, 10, 11, 12, 13]). One of the fundamental problems of dynamical systems theory is to predict how the trajectory of a point will behave. Minor differences in initial conditions can lead to significant differences in the behavior of orbits. This is known as "sensitive dependence on initial conditions". Dynamical systems that provide this feature together with both "density of periodic points" and "topological transitivity" are called Devaney chaotic and are commonly encountered in nature and in our daily life. One of the best known examples where chaos arises is the shift map defined on the code space. This dynamical system guides the identification of many different chaotic systems. Other well-known examples are the tent and doubling maps. Moreover, the Cantor set (C) and the Cantor dust ($C \times C$) have played an important role in the analysis of dynamical systems due to their generalizable structure.

The constrained tent map and Smale's horseshoe map are basic and early models of chaotic dynamical systems defined on fractals such as the Cantor set and the Cantor dust, respectively (see figures 1 and 2). The dynamics of Smale's horseshoe map is very rich and significant on the Cantor dust $C \times C$ (see Figure 2). There have

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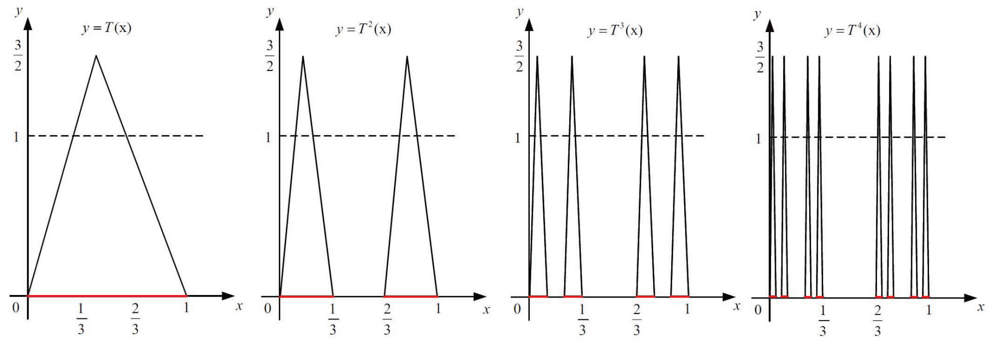

Figure 1: *The construction of tent map defined on the classical Cantor set.*

also been some recent studies on dynamical systems constructed using symmetries, expanding and folding maps on some fractals such as the Cantor dust $C \times C$, the box (Vicsek) fractal and the Sierpinski triangle, tetrahedron and propeller [1, 2, 3, 4, 14]. Moreover, in [5], the authors construct a family of chaotic dynamical systems on $C \times C$ by using the shift map and elements of the 4th Dihedral group. Furthermore, in [15], the authors define different chaotic dynamical systems on $C \times C$ with the help of the shift map, a $0-$preadded map and a $2-$preadded map. Such studies keep these construction methods up-to-date and increase the popularity of methods whose different dynamical systems are constructed on fractals. The main purpose of

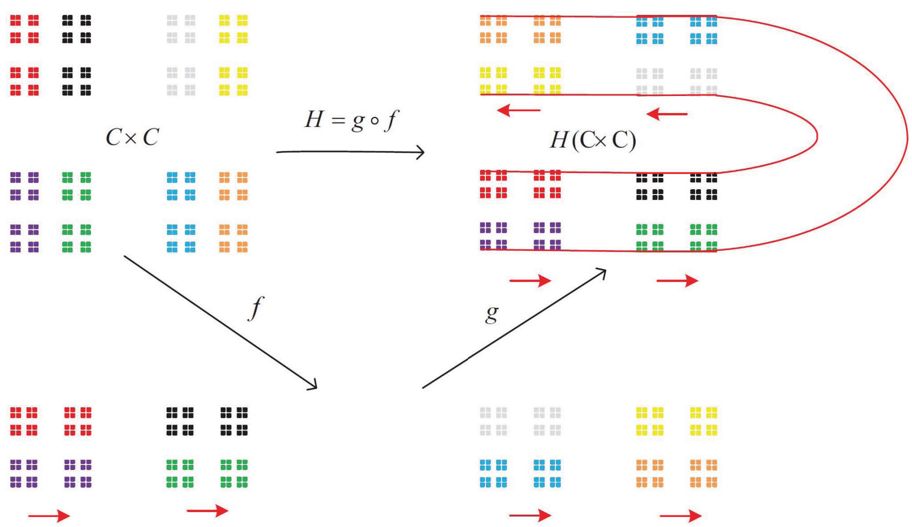

Figure 2: *The construction of Smale's horseshoe map.*

this paper is to examine different versions of Smale's horseshoe map. For this aim, we first define a new mapping, which we call in this paper the first modification of Smale's horseshoe. We then express this mapping in ternary system, and we also compute its periodic points. In propositions 1, 2 and 3, we show in detail that the first modification of Smale's horseshoe map is Devaney chaotic on $C \times C$. In addition, we obtain a map which we call the second modification of Smale's horseshoe map, which is topologically conjugate to the first modification of Smale's horseshoe map (see Proposition 4). Moreover, we obtain some periodic points of these maps. Finally, we show that the third modification of a horseshoe map is not topological conjugate to first and second modifications of Smale's horseshoe map (see Remark 3).

2. Preliminaries

In this section, we give some fundamental notions, definitions and theorems which we use throughout this paper.

Code representations of points on the Cantor dust: It is well-known that the points of a classical Cantor set can be expressed as

$$0.x_1x_2x_3\ldots = \frac{x_1}{3} + \frac{x_2}{3^2} + \cdots + \frac{x_n}{3^n} + \cdots ,$$

where $x_i \in \{0,2\}$. It follows that the Cantor dust is the set

$$C \times C = \{(0.x_1x_2x_3\ldots,\ 0.y_1y_2y_3\ldots)|\ x_i, y_i \in \{0,2\},\ i = 1,2,3,\ldots\}.$$

Definition 1 (see [8])**.** *$\{X; f\}$ is a discrete dynamical system if (X, d) is a metric space and $f : X \to X$ is a function.*

Definition 2 (see [6])**.** *A discrete dynamical system $\{X; f\}$ is chaotic in the sense of Devaney if it is topologically transitive, the set of periodic points is dense in X and it is sensitive dependent on initial conditions.*

Remark 1. *Suppose that X is a topological space with the property that no non-empty open subset U has a finite subset dense in U. If $f : X \to X$ has a dense orbit, then it is topologically transitive (for the proof see [9]).*

Definition 3 (see [10, 12])**.** *Suppose that $f : X \to X$ and $g : Y \to Y$ are two functions where (X, d_1) and (Y, d_2) are two metric spaces. The functions f and g are said to be topologically conjugate if there is a homeomorphism h such that $h \circ f = g \circ h$. In this case, the homeomorphism h is called the conjugacy map between f and g and it is denoted by $f \approx_h g$. Moreover, the dynamical systems $\{X; f\}$ and $\{Y; g\}$ are said to be topologically equivalent.*

Theorem 1 (see [10, 12])**.** *Let $\{X; f\}$ and $\{Y; g\}$ be dynamical systems and let $f \approx_h g$.*

1. *If c is m-periodic point of f, then $h(c)$ is the m-period point of g.*
2. *f is topologically transitive $\Leftrightarrow$ g is topologically transitive.*
3. *The set of periodic points of f are dense in X $\Leftrightarrow$ the set of periodic points of g are dense in Y.*

2.1. The construction of the Smale's horseshoe map

Smale's horseshoe map, H, is one of the standard examples of discrete dynamical systems. In [6], the authors give this map by using ternary representation of numbers as follows:

$$H(0.x_1x_2x_3\ldots,0.y_1y_2y_3\ldots)=\begin{cases}(0.x_2x_3x_4\ldots,0.x_1y_1y_2y_3\ldots) & \text{if } x_1=0,\\ (0.\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\ldots,0.x_1\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) & \text{if } x_1=2,\end{cases} \quad (1)$$

where $(0.x_1x_2x_3\ldots,0.y_1y_2y_3\ldots)\in C\times C$ (see Figure 2), where $\widetilde{x}_i=2-x_i$ and $\widetilde{y}_i=2-y_i$. The following lemma gives a general formula for the iterations of points under H (for the proof see [6]):

Lemma 1. *$H^n(0.x_1x_2x_3\ldots,0.y_1y_2y_3\ldots)$ equals*

$$\begin{cases}(0.x_{n+1}x_{n+2}\ldots,0.x_{n-1}x_{n-2}\ldots x_1x_ny_1y_2\ldots) & \text{if } x_n=0,\\ (0.\widetilde{x}_{n+1}\widetilde{x}_{n+2}\ldots,0.\widetilde{x}_{n-1}\widetilde{x}_{n-2}\ldots\widetilde{x}_1x_n\widetilde{y}_1\widetilde{y}_2\ldots) & \text{if } x_n=2,\end{cases}$$

where $(0.x_1x_2x_3\ldots,0.y_1y_2y_3\ldots)\in C\times C$ (for $n=2$ see Figure 3).

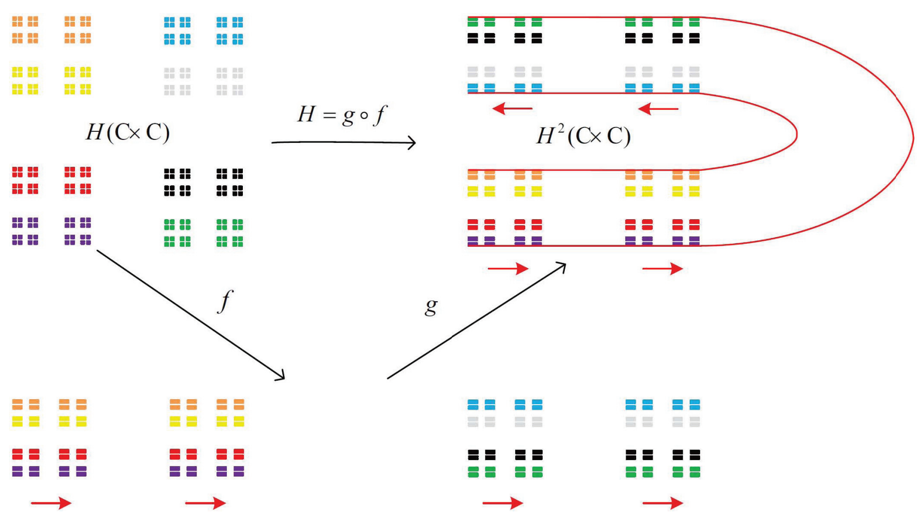

Figure 3: *The second iteration of points under Smale's horseshoe map*

Some periodic points of H:
For $x_n=0$, n-periodic points of H are formulated as follows:

$$(0.\overline{x_1x_2x_3\ldots0},0.\overline{x_{n-1}x_{n-2}x_{n-3}\ldots x_10}).$$

For $x_n=2$, n-periodic points of H are formulated as follows:

$$(0.\overline{x_1x_2x_3\ldots2\widetilde{x}_1\widetilde{x}_2\ldots\widetilde{x}_{n-1}0},0.\overline{\widetilde{x}_{n-1}\widetilde{x}_{n-2}\widetilde{x}_{n-3}\ldots\widetilde{x}_12x_{n-1}x_{n-2}\ldots x_10}).$$

By using these formula, we obtain that

fixed points of H are

$$\begin{array}{ll} (\overline{0}, \overline{0}) & , \text{ for } x_1 = 0 \\ (\overline{20}, \overline{20}) & , \text{ for } x_1 = 2 \end{array},$$

2-periodic points of H are

$$\begin{array}{ll} (\overline{0220}, \overline{2200}) & , \text{ for } x_1 = 0, x_2 = 2 \\ (\overline{2200}, \overline{0220}) & , \text{ for } x_1 = 2, x_2 = 2 \end{array},$$

3-periodic points of H are

$$\begin{array}{ll} (\overline{002220}, \overline{222000}) & , \text{ for } x_1x_2x_3 = 002 \\ (\overline{020}, \overline{200}) & , \text{ for } x_1x_2x_3 = 020 \\ (\overline{022200}, \overline{022200}) & , \text{ for } x_1x_2x_3 = 022 \\ (\overline{200}, \overline{020}) & , \text{ for } x_1x_2x_3 = 200 \\ (\overline{220}, \overline{220}) & , \text{ for } x_1x_2x_3 = 220 \\ (\overline{222000}, \overline{002220}) & , \text{ for } x_1x_2x_3 = 222 \end{array}.$$

Remark 2. *It is well-known that Smale's horseshoe map is Devaney chaotic (for the proof, see [6]).*

3. Some modified horseshoe maps

In this part, by considering the construction of Smale's horseshoe map given in [6], different styles of horseshoe maps will be constructed on $C \times C$. Geometrically, we consider the first examples that come to mind and thus we call these maps 'modified horseshoes'. We then compute periodic points of these maps and investigate whether they are Devaney chaotic or not. Therefore, we will have significant knowledge of their dynamics of them and be able to compare these maps in terms of topological conjugacy.

3.1. The first modification of Smale's horseshoe map

Let us construct a map on $C \times C$ with a horseshoe-like movement. Suppose that the function f contracts $C \times C$ three times on the vertical axis, while expanding it three times on the horizontal axis. Note that we use the same function f given in Smale's horseshoe map. The main difference here is in the definition of function g. To be more clear, the function g acts on the second term of the first component of the code representations of the points of $C \times C$, as shown in Figure 4. Thus, the map $F_1 = g_1 \circ f : C \times C \to C \times C$ is formulated as

$$F_1(x_1x_2x_3\ldots, y_1y_2y_3\ldots) = \begin{cases} (x_2x_3x_4\ldots, x_2y_1y_2y_3\ldots) & \text{if } x_1 = 0, \\ (\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\ldots, x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) & \text{if } x_1 = 2. \end{cases} \quad (2)$$

It is more complicated to examine the dynamics of this map than Smale's horseshoe map (see Figure 5). Now, we want to get information about the status of $n-$ periodic points under the function F_1. To this end, we must obtain a few iterations of the

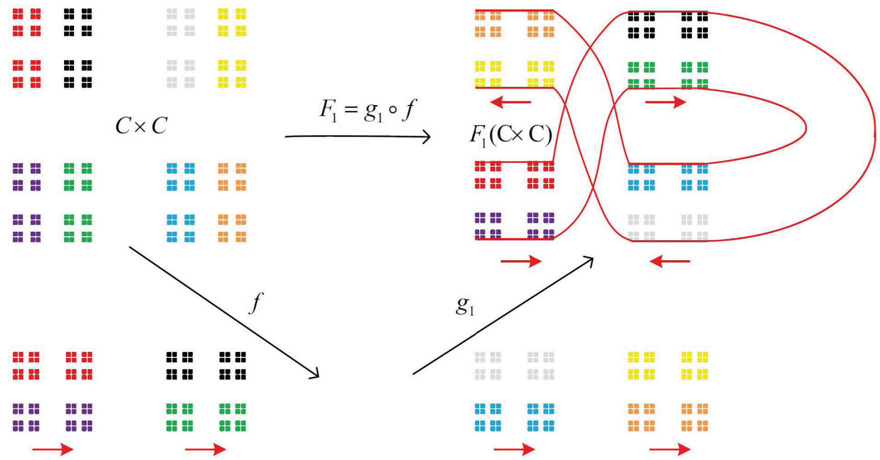

Figure 4: *The first modification of horseshoe map.*

code representations of the points under the function F_1. As seen in (2), there are two rules for F_1:

i) If the first term of the first component is zero, then this term is removed in the first component and the second term of the first component is added to the top of the terms of the second component and the tails are written the same in both components.

ii) If the first term of the first component is two, then this term is removed from the first component and the second term of the first component is added to the top of the terms of the second component and the conjugates of tails are written in both components.

In order to obtain the second iterations of the code representations of the points under the F_1:

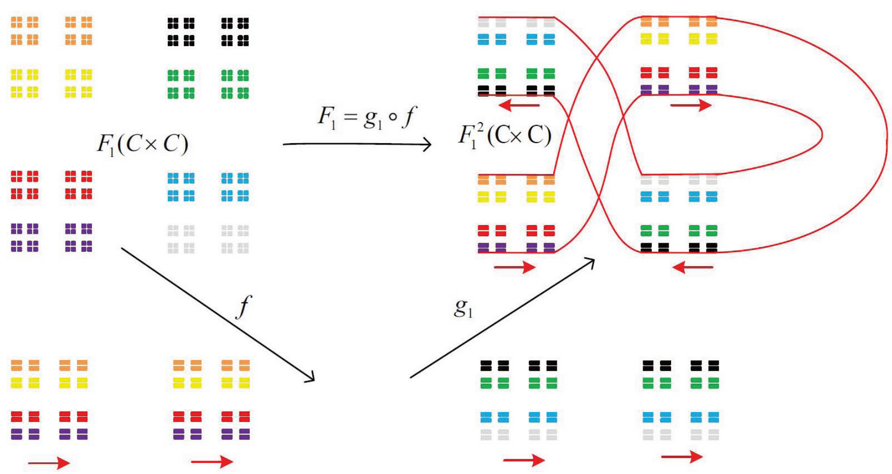

Figure 5: *The second iteration of the points under* F_1.

Let $x_1 = 0$ and $x_2 = 0$. If rule (i) is applied to $F_1(x_1x_2x_3\dots, y_1y_2y_3\dots)$, then we have $(x_3x_4x_5\dots, x_3x_2y_1y_2\dots)$.

Let $x_1 = 0$ and $x_2 = 2$. If rule (ii) is applied to $F_1(x_1x_2x_3\dots, y_1y_2y_3\dots)$, then we have $(\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots, x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)$.

Let $x_1 = 2$ and $x_2 = 0$. If rule (ii) is applied to $F_1(x_1x_2x_3\dots, y_1y_2y_3\dots)$, then we have $(x_3x_4x_5\dots, \widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots)$.

Let $x_1 = 2$ and $x_2 = 2$. If rule (i) is applied to $F_1(x_1x_2x_3\dots, y_1y_2y_3\dots)$, then we have $(\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots, \widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)$.

Consequently, ${F_1}^2 : C \times C \to C \times C$, ${F_1}^2(x_1x_2x_3\dots, y_1y_2y_3\dots)$ is expressed as

$$\begin{cases} (x_3x_4x_5\dots, x_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1 = 0, x_2 = 0 \\ (\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots, x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1 = 0, x_2 = 2 \\ (x_3x_4x_5\dots, \widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1 = 2, x_2 = 0 \\ (\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots, \widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1 = 2, x_2 = 2 \end{cases}.$$

With similar calculations, the third iterations of the code representations of the points under the F_1 are obtained as follows:

$$\begin{cases} (x_4x_5x_6\dots, x_4x_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3 = 000 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots, x_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3 = 002 \\ (x_4x_5x_6\dots, \widetilde{x}_4\widetilde{x}_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3 = 020 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots, \widetilde{x}_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3 = 022 \\ (x_4x_5x_6\dots, x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3 = 200 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots, x_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3 = 202 \\ (x_4x_5x_6\dots, \widetilde{x}_4x_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3 = 220 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots, \widetilde{x}_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3 = 222 \end{cases}.$$

If we continue the process like this, then we obtain the fourth iterations of the code representations of the points under the F_1 as follows:

$$\begin{cases} (x_5x_6x_7\dots, x_5x_4x_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0000 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, x_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0002 \\ (x_5x_6x_7\dots, \widetilde{x}_5\widetilde{x}_4x_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0020 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, \widetilde{x}_5x_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0022 \\ (x_5x_6x_7\dots, x_5\widetilde{x}_4\widetilde{x}_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0200 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, x_5x_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0202 \\ (x_5x_6x_7\dots, \widetilde{x}_5x_4\widetilde{x}_3x_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0220 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, \widetilde{x}_5\widetilde{x}_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 0222 \\ (x_5x_6x_7\dots, x_5x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2000 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, x_5\widetilde{x}_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2002 \\ (x_5x_6x_7\dots, \widetilde{x}_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2020 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, \widetilde{x}_5x_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2022 \\ (x_5x_6x_7\dots, x_5\widetilde{x}_4x_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2200 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, x_5x_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2202 \\ (x_5x_6x_7\dots, \widetilde{x}_5x_4x_3\widetilde{x}_2y_1y_2y_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2220 \\ (\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots, \widetilde{x}_5\widetilde{x}_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots)\ , & \text{for } x_1x_2x_3x_4 = 2222 \end{cases}.$$

In this way, ${F_1}^5 : C \times C \to C \times C$, ${F_1}^5(x_1x_2x_3\dots, y_1y_2y_3\dots)$ equals to

$$\begin{cases}
(x_6x_7x_8\ldots, x_6x_5x_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00000 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6\widetilde{x}_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00002 \\
(x_6x_7x_8\ldots, \widetilde{x}_6\widetilde{x}_5x_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00020 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6x_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00022 \\
(x_6x_7x_8\ldots, x_6\widetilde{x}_5\widetilde{x}_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00200 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6x_5x_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00202 \\
(x_6x_7x_8\ldots, \widetilde{x}_6x_5\widetilde{x}_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00220 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6\widetilde{x}_5x_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 00222 \\
(x_6x_7x_8\ldots, x_6x_5\widetilde{x}_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02000 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6\widetilde{x}_5x_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02002 \\
(x_6x_7x_8\ldots, \widetilde{x}_6\widetilde{x}_5\widetilde{x}_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02020 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6x_5x_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02022 \\
(x_6x_7x_8\ldots, x_6\widetilde{x}_5x_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02200 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6x_5\widetilde{x}_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02202 \\
(x_6x_7x_8\ldots, \widetilde{x}_6x_5x_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02220 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6\widetilde{x}_5\widetilde{x}_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 02222 \\
(x_6x_7x_8\ldots, x_6x_5x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20000 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6\widetilde{x}_5\widetilde{x}_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20002 \\
(x_6x_7x_8\ldots, \widetilde{x}_6\widetilde{x}_5x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20020 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6x_5\widetilde{x}_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20022 \\
(x_6x_7x_8\ldots, x_6\widetilde{x}_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20200 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6x_5x_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20202 \\
(x_6x_7x_8\ldots, \widetilde{x}_6x_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20220 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6\widetilde{x}_5x_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 20222 \\
(x_6x_7x_8\ldots, x_6x_5\widetilde{x}_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22000 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6\widetilde{x}_5x_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22002 \\
(x_6x_7x_8\ldots, \widetilde{x}_6\widetilde{x}_5\widetilde{x}_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22020 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6x_5x_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22022 \\
(x_6x_7x_8\ldots, x_6\widetilde{x}_5x_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22200 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, x_6x_5\widetilde{x}_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22202 \\
(x_6x_7x_8\ldots, \widetilde{x}_6x_5x_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22220 \\
(\widetilde{x}_6\widetilde{x}_7\widetilde{x}_8\ldots, \widetilde{x}_6\widetilde{x}_5\widetilde{x}_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4x_5 = 22222
\end{cases}.$$

As can be seen from the above calculations, it is not easy to formulate general situation. However, the first five iterations give us some observations about periodic points.

Example 1. *Let us compute the code representation of a point whose period is* 5. *Let* $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 0$. *From the definition of* F_1^5, *it must be*

$$(x_6x_7x_8\ldots, \widetilde{x}_6\widetilde{x}_5x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots)$$

We then obtain

$$\begin{aligned}
\cdots = x_{11} = \ x_6 = \ x_1 = 2 \\
\cdots = x_{12} = \ x_7 = \ x_2 = 0 \\
\cdots = x_{13} = \ x_8 = \ x_3 = 0 \\
\cdots = x_{14} = \ x_9 = \ x_4 = 2 \\
\cdots = x_{15} = x_{10} = x_5 = 0
\end{aligned}$$

and

$$\begin{array}{l} \cdots = \ y_6 = \ \widetilde{x}_6 = y_1 = 0 \\ \cdots = \ y_7 = \ \widetilde{x}_5 = y_2 = 2 \\ \cdots = \ y_8 = \ x_4 = y_3 = 2\,. \\ \cdots = \ y_9 = \ \widetilde{x}_3 = y_4 = 2 \\ \cdots = y_{10} = \widetilde{x}_2 = y_5 = 2 \end{array}$$

This shows that $(\overline{20020}, \overline{02222})$ *is one of the* 5*-periodic points.*

With similar calculations, we get the following periodic points:
Fixed points of F_1 are

$$\begin{array}{ll} (\overline{0}, \overline{0}) & \text{, for } x_1 = 0 \\ (\overline{20}, \overline{02}) & \text{, for } x_1 = 2 \end{array},$$

2−periodic points of F_1 are

$$\begin{array}{ll} (\overline{0220}, \overline{2002}) & \text{, for } x_1x_2 = 02 \\ (\overline{2200}, \overline{2200}) & \text{, for } x_1x_2 = 22 \end{array},$$

3−periodic points of F_1 are

$$\begin{array}{ll} (\overline{002220}, \overline{20}) & \text{, for } x_1x_2x_3 = 002 \\ (\overline{020}, \overline{2}) & \text{, for } x_1x_2x_3 = 020 \\ (\overline{022200}, \overline{02}) & \text{, for } x_1x_2x_3 = 022 \\ (\overline{200}, \overline{2}) & \text{, for } x_1x_2x_3 = 200 \\ (\overline{220}, \overline{0}) & \text{, for } x_1x_2x_3 = 220 \\ (\overline{222000}, \overline{20}) & \text{, for } x_1x_2x_3 = 222 \end{array},$$

4−periodic points of F_1 are

$$\begin{array}{ll} (\overline{00022220}, \overline{20220200}) & \text{, for } x_1x_2x_3x_4 = 0002 \\ (\overline{0020}, \overline{2220}) & \text{, for } x_1x_2x_3x_4 = 0020 \\ (\overline{00222200}, \overline{02022020}) & \text{, for } x_1x_2x_3x_4 = 0022 \\ (\overline{0200}, \overline{0222}) & \text{, for } x_1x_2x_3x_4 = 0200 \\ (\overline{02022020}, \overline{22000022}) & \text{, for } x_1x_2x_3x_4 = 0202 \\ (\overline{02222000}, \overline{00202202}) & \text{, for } x_1x_2x_3x_4 = 0222 \\ (\overline{2000}, \overline{2022}) & \text{, for } x_1x_2x_3x_4 = 2000 \\ (\overline{20020220}, \overline{00002222}) & \text{, for } x_1x_2x_3x_4 = 2002 \\ (\overline{20220200}, \overline{22200002}) & \text{, for } x_1x_2x_3x_4 = 2022 \\ (\overline{22020020}, \overline{02222000}) & \text{, for } x_1x_2x_3x_4 = 2202 \\ (\overline{2220}, \overline{0020}) & \text{, for } x_1x_2x_3x_4 = 2220 \\ (\overline{22220000}, \overline{20020220}) & \text{, for } x_1x_2x_3x_4 = 2222 \end{array},$$

5−periodic points of F_1 are

$$\begin{array}{ll}
(\overline{0000222220}, \overline{2022202000}) & \text{, for } x_1x_2x_3x_4x_5 = 00002 \\
(\overline{00020}, \overline{22200}) & \text{, for } x_1x_2x_3x_4x_5 = 00020 \\
(\overline{0002222200}, \overline{0202220200}) & \text{, for } x_1x_2x_3x_4x_5 = 00022 \\
(\overline{00200}, \overline{02220}) & \text{, for } x_1x_2x_3x_4x_5 = 00200 \\
(\overline{0020222020}, \overline{0200220220}) & \text{, for } x_1x_2x_3x_4x_5 = 00202 \\
(\overline{00220}, \overline{00020}) & \text{, for } x_1x_2x_3x_4x_5 = 00220 \\
(\overline{0022222000}, \overline{0020222020}) & \text{, for } x_1x_2x_3x_4x_5 = 00222 \\
(\overline{02000}, \overline{00222}) & \text{, for } x_1x_2x_3x_4x_5 = 02000 \\
(\overline{0200220220}, \overline{2000002222}) & \text{, for } x_1x_2x_3x_4x_5 = 02002 \\
(\overline{02020}, \overline{22022}) & \text{, for } x_1x_2x_3x_4x_5 = 02020 \\
(\overline{0202220200}, \overline{0220020022}) & \text{, for } x_1x_2x_3x_4x_5 = 02022 \\
(\overline{02200}, \overline{02002}) & \text{, for } x_1x_2x_3x_4x_5 = 02200 \\
(\overline{0220220020}, \overline{0222020002}) & \text{, for } x_1x_2x_3x_4x_5 = 02202 \\
(\overline{02220}, \overline{20202}) & \text{, for } x_1x_2x_3x_4x_5 = 02220 \\
(\overline{0222220000}, \overline{0002022202}) & \text{, for } x_1x_2x_3x_4x_5 = 02222 \\
(\overline{20000}, \overline{20022}) & \text{, for } x_1x_2x_3x_4x_5 = 20000 \\
(\overline{2000202220}, \overline{0020022022}) & \text{, for } x_1x_2x_3x_4x_5 = 20002 \\
(\overline{20020}, \overline{02222}) & \text{, for } x_1x_2x_3x_4x_5 = 20020 \\
(\overline{2002202200}, \overline{2200000222}) & \text{, for } x_1x_2x_3x_4x_5 = 20022 \\
(\overline{20200}, \overline{22202}) & \text{, for } x_1x_2x_3x_4x_5 = 20200 \\
(\overline{20220}, \overline{00002}) & \text{, for } x_1x_2x_3x_4x_5 = 20220 \\
(\overline{2022202000}, \overline{2022002002}) & \text{, for } x_1x_2x_3x_4x_5 = 20222 \\
(\overline{22000}, \overline{20200}) & \text{, for } x_1x_2x_3x_4x_5 = 22000 \\
(\overline{2200200220}, \overline{0002222200}) & \text{, for } x_1x_2x_3x_4x_5 = 22002 \\
(\overline{22020}, \overline{02000}) & \text{, for } x_1x_2x_3x_4x_5 = 22020 \\
(\overline{2202200200}, \overline{2222200000}) & \text{, for } x_1x_2x_3x_4x_5 = 22022 \\
(\overline{22200}, \overline{22020}) & \text{, for } x_1x_2x_3x_4x_5 = 22200 \\
(\overline{2220200020}, \overline{0220220020}) & \text{, for } x_1x_2x_3x_4x_5 = 22202 \\
(\overline{22220}, \overline{00220}) & \text{, for } x_1x_2x_3x_4x_5 = 22220 \\
(\overline{2222200000}, \overline{2000202220}) & \text{, for } x_1x_2x_3x_4x_5 = 22222
\end{array}.$$

The above calculations show that there are two types for n-periodic points of F_1: If $x_n = 0$, then n-periodic points are of the form

$$(\overline{x_1x_2x_3\dots x_{n-1}x_n}, \overline{x'_1x'_nx'_{n-1}\dots x'_2}),$$

and if $x_n = 2$, then n-periodic points are of the form

$$(\overline{x_1x_2x_3\dots x_{n-1}x_n\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3\dots\widetilde{x}_n}, \overline{x'_1x'_nx'_{n-1}\dots x'_2\widetilde{x'}_1\widetilde{x'}_n\widetilde{x'}_{n-1}\dots\widetilde{x'}_2}).$$

The other observation is also related to the second components of the map ${F_1}^n$. That is, the first $n-1$ terms that come after the first term of the second component have all the orderings of 0 and 2. For example, let us examine the periodic points of the map ${F_1}^4$. For $x_4 = 0$ and $x_4 = 2$, if we look at the first three terms after the first term of the second component of the 4-periodic points, then all sequences 000, 002, 020, 022, 200, 220, 222 are available.

With the help of these observations, we now prove that the set of periodic points of F_1 is dense in $C \times C$ and $\{C \times C; F_1\}$ has a dense orbit.

Proposition 1. *The set of the periodic points of F_1 is dense in $C \times C$.*

Proof. Let $X = (x_1x_2x_3 \dots x_{n-1}x_nx_{n+1} \dots, y_1y_2y_3 \dots y_{n-1}y_ny_{n+1} \dots) \in C \times C$ be an arbitrary point. According to the definition of the function ${F_1}^n$, there is a $2n$-periodic point in the following form:

$$(\overline{x_1x_2x_3 \dots x_{n-1}x_nw_1w_2w_3 \dots w_{n-1}w_n}, \overline{x'_1w'_nw'_{n-1} \dots w'_2w'_1x'_nx'_{n-1} \dots x'_3x'_2})$$

where $w_n = 0$ and $w_n \in \{0.2\}$ are arbitrary for $i = 1, 2, \dots, n-1$.
i) Let $y_1 = x'_1$. The first n terms in the first components of these periodic points are the same like the first n terms of X. In the second component, the first $n-1$ terms coming after x'_1 all have ordering of 0 and 2 owing to the fact that w'_is are arbitrary. Therefore, we can choose $w'_n = y_2, w'_{n-1} = y_3, \dots, w'_3 = y_{n-1}, w'_2 = y_n$. Moreover, we have $y_1 = x'_1$, and it follows that the first n terms are the same in both components.
ii) Let $y_1 \neq x'_1$. There is a $(2n+1)$-periodic point in the following form:

$$(\overline{x_1x_2x_3 \dots x_{n-1}x_n\alpha w_1w_2w_3 \dots w_{n-1}w_n}, \overline{x''_1w''_nw''_{n-1} \dots w''_2w''_1\alpha' x''_nx''_{n-1} \dots x''_3x''_2}),$$

where α is chosen to make $x'_1 \neq x''_1$. In such a case, we get $y_1 = x''_1$. The first n terms in the first component are $x_1x_2x_3 \dots x_{n-1}x_n$. Since w''_is are arbitrary in the second component and all n orderings are present, we can get $w''_n = y_2, w''_{n-1} = y_3, \dots, w''_3 = y_{n-1}, w''_2 = y_n$.

This shows that there exists a periodic point which is close enough to the point $(x_1x_2x_3 \dots x_{n-1}x_nx_{n+1} \dots, y_1y_2y_3 \dots y_{n-1}y_ny_{n+1} \dots)$ of $C \times C$. Consequently, the set of periodic points is dense in $C \times C$. ☐

Proposition 2. *F_1 has a dense orbit.*

Proof. We must find a point A of $C \times C$ such that the set

$$O_{F_1}(A) = \{A, F_1(A), F_2(A), \dots, F_1^n(A), \dots\}$$

is dense in $C \times C$. Let

$$(x_1x_2x_3 \dots x_{n-1}x_nx_{n+1} \dots, y_1y_2y_3 \dots y_{n-1}y_ny_{n+1} \dots) \in C \times C$$

be an arbitrary point. Suppose that the code representation of A is $(\alpha, \beta) = (\alpha_1\alpha_2\alpha_3 \dots, \beta_1\beta_2\beta_3 \dots)$, where α_ns consist of all combinations of 0 and 2 with blocks n. That is, we have

$$\alpha_n = 00\ 02\ 20\ \ 22$$

for $n = 2$. Thus, in the first component of A we can obtain blocks with $2r$ such that

$$\underbrace{w_{s-r} \dots w_{s-2}0x_1x_2x_3 \dots x_r}_{\text{a block with } 2r},$$

where w_ns all have ordering of 0 and 2. Moreover, for a natural number m we can get

$$F_1{}^m(\alpha_1\alpha_2\alpha_3\ldots,\beta_1\beta_2\beta_3\ldots) = (x_1x_2x_3\ldots x_r\ldots, x_1'w_{s-2}'\ldots w_{s-r}'\ldots),$$

where $w_{s-2}' = y_2, w_{s-3}' = y_3, \ldots, w_{s-r}' = y_r$. If $x_1' = y_1$, then the desired is obtained. If $x_1' \neq y_1$, then for $m \neq s$ and $s \in \mathbb{N}$ a block satisfying $x_1' = y_1$ can be found in a similar way. This completes the proof. □

Proposition 3. *F_1 has sensitive dependence on initial conditions.*

Proof. Given an arbitrary point A of $C \times C$ with code representation

$$(x_1x_2x_3\ldots x_nx_{n+1}\ldots, y_1y_2y_3\ldots y_ny_{n+1}\ldots).$$

Let us choose a point B whose code representation is

$$(x_1x_2x_3\ldots x_nx_{n+1}'\ldots, y_1y_2y_3\ldots y_ny_{n+1}'\ldots),$$

where $x_{n+1} \neq x_{n+1}'$ and $y_{n+1} \neq y_{n+1}'$. Thus, we get

$$\begin{aligned} d(A,B) &= \sqrt{\left(\sum_{i=1}^{n}\frac{x_i - x_i}{3^i} + \sum_{i=n+1}^{\infty}\frac{x_i - x_i'}{3^i}\right)^2 + \left(\sum_{i=1}^{n}\frac{y_i - y_i}{3^i} + \sum_{i=n+1}^{\infty}\frac{x_i - x_i'}{3^i}\right)^2} \\ &\leq \sqrt{\left(\sum_{i=n+1}^{\infty}\frac{2}{3^i}\right)^2 + \left(\sum_{i=n+1}^{\infty}\frac{2}{3^i}\right)^2} \\ &= \frac{\sqrt{2}}{3^n}, \end{aligned}$$

where d is Euclidean metric on $\mathbb{R}^2$. That is, B is close enough to A for n large enough. For $x_n = 0$, we have

$$\begin{aligned} F^n(A) =& (x_{n+1}x_{n+2}x_{n+3}\ldots, s_{n+1}s_ns_{n-1}\ldots y_1y_2y_3\ldots) \\ F^n(B) =& (x_{n+1}'x_{n+2}'x_{n+3}'\ldots, t_{n+1}t_nt_{n-1}\ldots y_1y_2y_3\ldots) \end{aligned}$$

or for $x_n = 2$, we get

$$\begin{aligned} F^n(A) =& (\widetilde{x}_{n+1}\widetilde{x}_{n+2}\widetilde{x}_{n+3}\ldots, v_{n+1}v_nv_{n-1}\ldots y_1y_2y_3\ldots) \\ F^n(B) =& (\widetilde{x'}_{n+1}\widetilde{x'}_{n+2}\widetilde{x'}_{n+3}\ldots, w_{n+1}w_nw_{n-1}\ldots y_1y_2y_3\ldots). \end{aligned}$$

Since $x_{n+1} \neq x_{n+1}'$, we obtain $\widetilde{x}_{n+1} \neq \widetilde{x'}_{n+1}$. In both cases, we compute

$$\begin{aligned} d(F^n(A), F^n(B)) &\geq \sqrt{\left(\frac{2}{3} - \sum_{i=2}^{\infty}\frac{2}{3^i}\right)^2 + \left(\sum_{i=1}^{\infty}\frac{0}{3^i}\right)^2} \\ &= \frac{1}{3}. \end{aligned}$$

This shows that F_1 has sensitive dependence on the initial conditions. □

3.2. The second modification of the horseshoe map

In this section, we will first describe a map which we call the second modification of a horseshoe map. Also, we show that the first and second modifications of a horseshoe map are topologically conjugate. And thus we compute its periodic points by using conjugacy map.

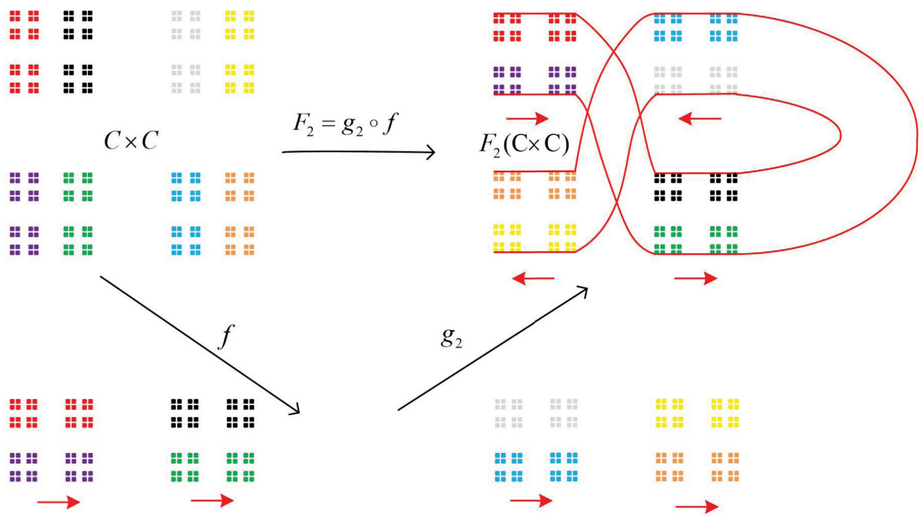

Figure 6: *The first iteration of the points under F_2.*

We define the second modification of a horseshoe map as $F_2 : C \times C \to C \times C$,

$$F_2(x_1x_2x_3\dots,y_1y_2y_3\dots) = \begin{cases} (x_2x_3x_4\dots,\widetilde{x}_2y_1y_2y_3\dots) & \text{if } x_1 = 0, \\ (\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\dots,\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) & \text{if } x_1 = 2, \end{cases}$$

(Figure 6). Therefore, ${F_2}^2 : C \times C \to C \times C$, ${F_2}^2(x_1x_2x_3\dots,y_1y_2y_3\dots)$ equals to

$$\begin{cases} (x_3x_4x_5\dots,\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots) \ , & \text{for } x_1 = 0, x_2 = 0 \\ (\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots,\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1 = 0, x_2 = 2 \\ (x_3x_4x_5\dots,x_3x_2y_1y_2y_3\dots) \ , & \text{for } x_1 = 2, x_2 = 0 \\ (\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots,x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1 = 2, x_2 = 2 \end{cases}$$

(Figure 7), and ${F_2}^3 : C \times C \to C \times C$, ${F_1}^3(x_1x_2x_3\dots,y_1y_2y_3\dots)$ equals

$$\begin{cases} (x_4x_5x_6\dots,\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\dots) \ , & \text{for } x_1x_2x_3 = 000 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots,\widetilde{x}_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1x_2x_3 = 002 \\ (x_4x_5x_6\dots,x_4x_3\widetilde{x}_2y_1y_2y_3\dots) \ , & \text{for } x_1x_2x_3 = 020 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots,x_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1x_2x_3 = 022 \\ (x_4x_5x_6\dots,\widetilde{x}_4x_3x_2y_1y_2y_3\dots) \ , & \text{for } x_1x_2x_3 = 200 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots,\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1x_2x_3 = 202 \\ (x_4x_5x_6\dots,x_4\widetilde{x}_3x_2y_1y_2y_3\dots) \ , & \text{for } x_1x_2x_3 = 220 \\ (\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots,x_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots) \ , & \text{for } x_1x_2x_3 = 222 \end{cases},$$

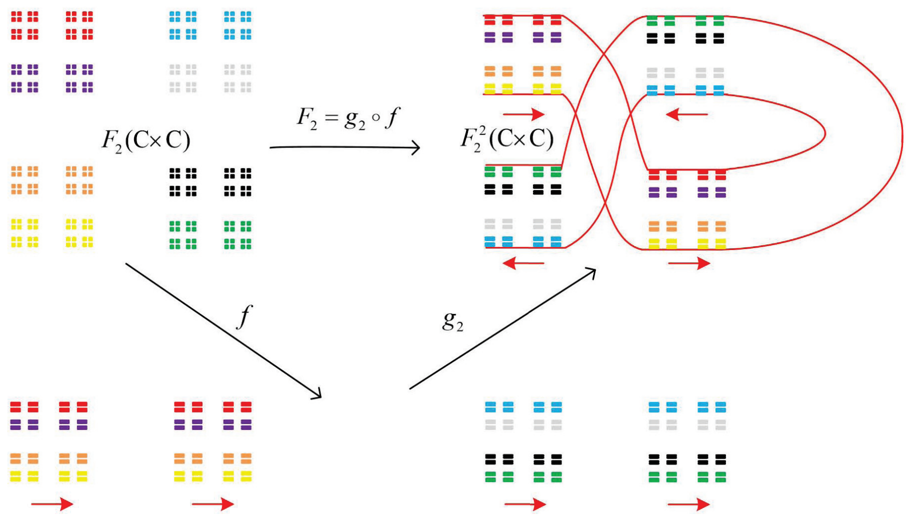

Figure 7: *The second iteration of the points under F_2.*

and ${F_2}^4 : C \times C \to C \times C$, ${F_2}^4(x_1x_2x_3\ldots, y_1y_2y_3\ldots)$ equals

$$\begin{cases}
(x_5x_6x_7\ldots, \widetilde{x}_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0000 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, \widetilde{x}_5x_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0002 \\
(x_5x_6x_7\ldots, x_5x_4\widetilde{x}_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0020 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, x_5\widetilde{x}_4x_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0022 \\
(x_5x_6x_7\ldots, \widetilde{x}_5x_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0200 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, \widetilde{x}_5\widetilde{x}_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0202 \\
(x_5x_6x_7\ldots, x_5\widetilde{x}_4x_3\widetilde{x}_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0220 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, x_5x_4\widetilde{x}_3x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 0222 \\
(x_5x_6x_7\ldots, \widetilde{x}_5\widetilde{x}_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2000 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, \widetilde{x}_5x_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2002 \\
(x_5x_6x_7\ldots, x_5x_4x_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2020 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, x_5\widetilde{x}_4\widetilde{x}_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2022 \\
(x_5x_6x_7\ldots, \widetilde{x}_5x_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2200 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, x_5\widetilde{x}_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2202 \\
(x_5x_6x_7\ldots, x_5\widetilde{x}_4\widetilde{x}_3x_2y_1y_2y_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2220 \\
(\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\ldots, x_5x_4x_3\widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)\ , & \text{for } x_1x_2x_3x_4 = 2222
\end{cases}.$$

In the following proposition, we prove that F_1 and F_2 are topologically conjugate via a homeomorphism h. Then, some periodic points of F_2 are calculated by using periodic points of F_1 and the conjugacy map h.

Proposition 4. *F_1 and F_2 are topologically conjugate via the homeomorphism $h : C \times C \to C \times C$, $h(x_1x_2x_3\ldots, y_1y_2y_3\ldots) = (x_1x_2x_3\ldots, \widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots)$.*

Proof. For all $(u_1u_2u_3\ldots, v_1v_2v_3\ldots) \in C \times C$, we get $(u_1u_2u_3\ldots, \widetilde{v}_1\widetilde{v}_2\widetilde{v}_3\ldots) \in C \times C$ such that $h(u_1u_2u_3\ldots, \widetilde{v}_1\widetilde{v}_2\widetilde{v}_3\ldots) = (u_1u_2u_3\ldots, v_1v_2v_3\ldots)$. Thus, h is surjective.

Now, we prove that h is an isometry. We must show that $d(h(A), h(B)) = d(A, B)$ for all $A = (x_1x_2x_3\ldots, y_1y_2y_3\ldots), B = (x'_1x'_2x'_3\ldots, y'_1y'_2y'_3\ldots) \in C \times C$. Since $y_i - y'_i = \widetilde{y}'_i - \widetilde{y}_i$ for $i = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(A,B) &= \left(\left(\sum_{i=1}^{\infty}\left(\frac{x_i - x'_i}{3^i}\right)\right)^2 + \left(\sum_{i=1}^{\infty}\left(\frac{y_i - y'_i}{3^i}\right)\right)^2\right)^{\frac{1}{2}} \\ &= \left(\left(\sum_{i=1}^{\infty}\left(\frac{x_i - x'_i}{3^i}\right)\right)^2 + \left(-\sum_{i=1}^{\infty}\left(\frac{\widetilde{y}_i - \widetilde{y'}_i}{3^i}\right)\right)^2\right)^{\frac{1}{2}} \\ &= d(h(A), h(B)). \end{aligned}$$

Consequently, h is a homeomorphism. Now, let us show that $h \circ F_1 = F_2 \circ h$. We compute that

$$\begin{aligned} (h \circ F_1)(x_1x_2x_3\ldots, y_1y_2y_3\ldots) &= h\left(\begin{cases} (x_2x_3x_4\ldots, x_2y_1y_2y_3\ldots) \ , \ x_1 = 0 \\ (\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\ldots, x_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) \ , \ x_1 = 2 \end{cases}\right) \\ &= \begin{cases} (x_2x_3x_4\ldots, \widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) \ , \ x_1 = 0 \\ (\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\ldots, \widetilde{x}_2y_1y_2y_3\ldots) \ , \ x_1 = 2 \end{cases} \end{aligned}$$

and

$$\begin{aligned} (F_2 \circ h)(x_1x_2x_3\ldots, y_1y_2y_3\ldots) &= F_2(x_1x_2x_3\ldots, \widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) \\ &= \begin{cases} (x_2x_3x_4\ldots, \widetilde{x}_2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) \ , \ x_1 = 0 \\ (\widetilde{x}_2\widetilde{x}_3\widetilde{x}_4\ldots, \widetilde{x}_2y_1y_2y_3\ldots) \ , \ x_1 = 2 \end{cases}. \end{aligned}$$

This completes the proof. □

Calculation of some periodic points of F_2: Now, let us calculate some periodic points of F_2 by using Theorem 1. That is, if $(p_1p_2p_3\ldots, q_1q_2q_3\ldots)$ is an n-periodic point of F_1, then $h(p_1p_2p_3\ldots, q_1q_2q_3\ldots)$ is an n-periodic point of F_2 from $F_1 \approx_h F_2$. Thus, fixed points of F_2 are

$$\begin{aligned} h(000\ldots, 000\ldots) &= (000\ldots, 222\ldots) \\ h(2020\ldots, 0202\ldots) &= (2020\ldots, 2020\ldots) \end{aligned},$$

2-periodic points of F_2 are

$$\begin{aligned} h(22002200\ldots, 22002200\ldots) &= (22002200\ldots, 00220022\ldots) \\ h(02200220\ldots, 20022002\ldots) &= (02200220\ldots, 02200220\ldots) \end{aligned},$$

3-periodic points of F_2 are

$$\begin{aligned} h(0022000220\ldots, 202020\ldots) &= (0022000220\ldots, 020202\ldots) \\ h(020020\ldots, 222\ldots) &= (020020\ldots, 000\ldots) \\ h(022200022200\ldots, 020202\ldots) &= (022200022200\ldots, 202020\ldots) \\ h(200200\ldots, 222\ldots) &= (200200\ldots, 000\ldots) \\ h(220220\ldots, 000\ldots) &= (220220\ldots, 222\ldots) \\ h(222000222000\ldots, 202020\ldots) &= (222000222000\ldots, 020202\ldots) \end{aligned},$$

Moreover, the other periodic points are computed in a similar way.

3.3. The third modification of horseshoe map

In this section, a different dynamical system is constructed by modifying the horseshoe map on $C \times C$ and some periodic points of this map are calculated. We call the map $F_3 : C \times C \to C \times C$,

$$F_3(x_1x_2x_3\ldots, y_1y_2y_3\ldots) = \begin{cases} (x_2x_3x_4\ldots, 0y_1y_2y_3\ldots) & \text{if } x_1 = 0, x_2 = 0, \\ (2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots, 0x_3x_4x_5\ldots) & \text{if } x_1 = 0, x_2 = 2, \\ (2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots, 2x_3x_4x_5\ldots) & \text{if } x_1 = 2, x_2 = 0, \\ (0\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots, 2\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) & \text{if } x_1 = 2, x_2 = 2, \end{cases}$$

the third modification of a horseshoe map (see Figure 8). Moreover, we obtain

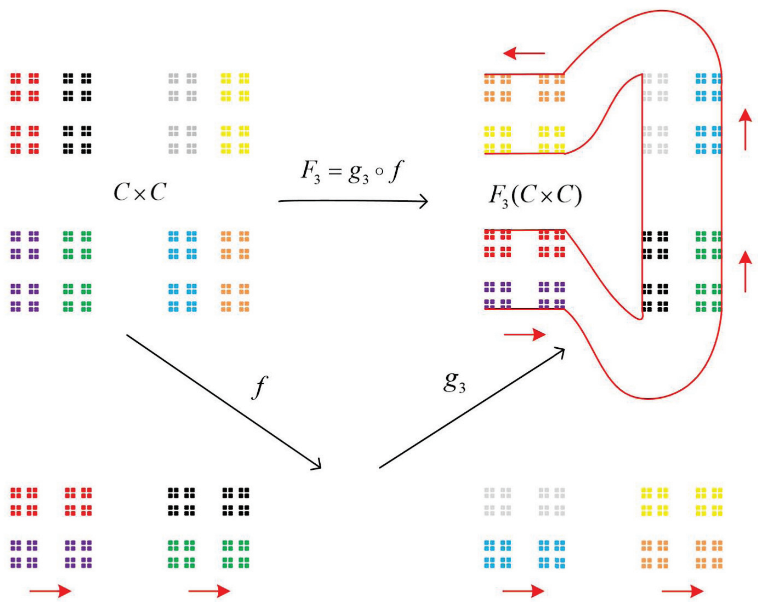

Figure 8: *The first iteration of the points under F_3.*

$$F_3{}^2(x_1x_2x_3\ldots, y_1y_2y_3\ldots) = \begin{cases} (x_3x_4x_5\ldots, 00y_1y_2\ldots) & , \text{ for } x_1x_2x_3 = 000 \\ (22\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots, 0x_4x_5x_6\ldots) & , \text{ for } x_1x_2x_3 = 002 \\ (0y_2y_3y_4\ldots, 22\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots) & , \text{ for } x_1x_2 = 02, y_1 = 0 \\ (22\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots, 2\widetilde{y}_2\widetilde{y}_3\widetilde{y}_4\ldots) & , \text{ for } x_1x_2 = 02, y_1 = 2 \\ (0y_2y_3y_4\ldots, 20\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots) & , \text{ for } x_1x_2 = 20, y_1 = 0 \\ (20\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots, 2\widetilde{y}_2\widetilde{y}_3\widetilde{y}_4\ldots) & , \text{ for } x_1x_2 = 20, y_1 = 2 \\ (20y_1y_2y_3\ldots, 0\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\ldots) & , \text{ for } x_1x_2x_3 = 220 \\ (\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots, 02\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) & , \text{ for } x_1x_2x_3 = 222 \end{cases}$$

(Figure 9), and also

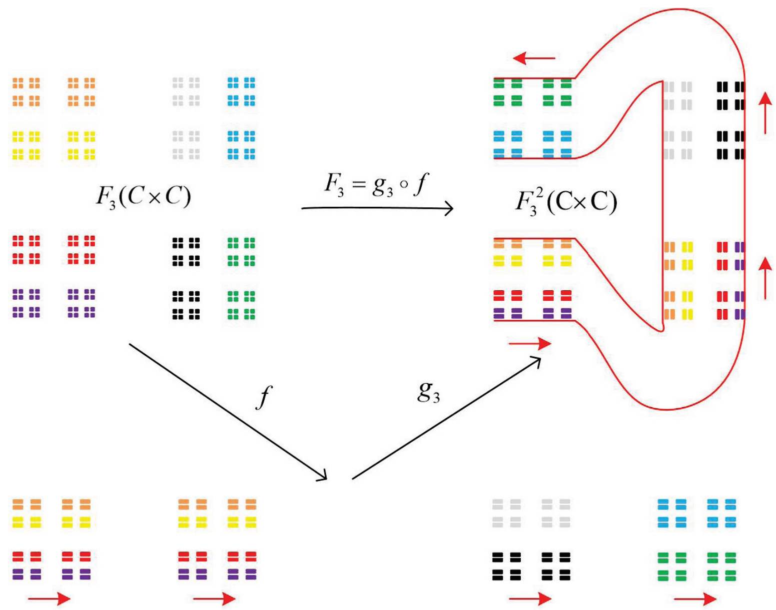

Figure 9: *The second iteration of the points under* F_3.

$$
{F_3}^3(x_1x_2x_3\dots,y_1y_2y_3\dots)=\begin{cases}
(x_4x_5x_6\dots,000y_1y_2\dots) & ,\ \text{for } x_1x_2x_3x_4=0000\\
(222\widetilde{y}_1\widetilde{y}_2\dots,0x_5x_6x_7\dots) & ,\ \text{for } x_1x_2x_3x_4=0002\\
(0\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\dots,22\widetilde{x}_4\widetilde{x}_5\dots) & ,\ \text{for } x_1x_2x_3=002\\
(y_2y_3y_4\dots,022\widetilde{x}_3\widetilde{x}_4\dots) & ,\ \text{for } x_1x_2=02, y_1y_2=00\\
(200x_3x_4\dots,0y_3y_4y_5\dots) & ,\ \text{for } x_1x_2=02, y_1y_2=02\\
(0x_3x_4x_5\dots,20y_2y_3\dots) & ,\ \text{for } x_1x_2=02, y_1=2\\
(y_2y_3y_4\dots,020\widetilde{x}_3\widetilde{x}_4\dots) & ,\ \text{for } x_1x_2=20, y_1y_2=00\\
(202x_3x_4\dots,0y_3y_4y_5\dots) & ,\ \text{for } x_1x_2=20, y_1y_2=02\\
(20y_2y_3\dots,2\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\dots) & ,\ \text{for } x_1x_2=20, y_1=2\\
(22x_4x_5x_6\dots,2y_1y_2y_3\dots) & ,\ \text{for } x_1x_2x_3=220\\
(220y_1y_2\dots,0\widetilde{x}_5\widetilde{x}_6\widetilde{x}_7\dots) & ,\ \text{for } x_1x_2x_3x_4=2220\\
(\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\dots,002\widetilde{y}_1\widetilde{y}_2\dots) & ,\ \text{for } x_1x_2x_3x_4=2222
\end{cases}
$$

(Figure 10).

Calculation of some periodic points of F_3: As seen from the definition and a few iterations of F_3, it is rather difficult to give a general formula for n-periodic points. However, we yield an important result using by the number of 2-periodic points.

Fixed points of F_3: F_3 has only one fixed point

$$(\overline{0},\overline{0}) \text{ for } x_1=0, x_2=0.$$

2-periodic points of F_3: F_3 has no 2-periodic point. Using the definition of F_3, the following contradictions are obtained:

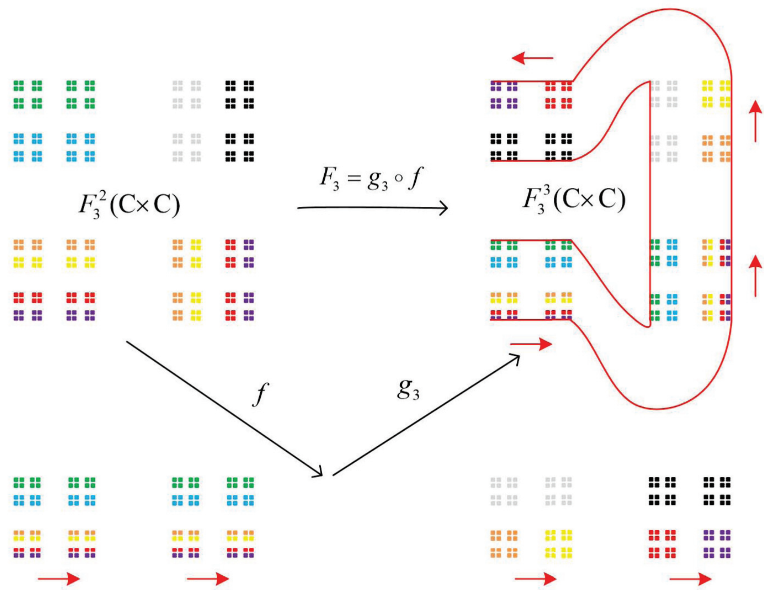

Figure 10: *The third iteration of the points under F_3.*

Let $x_1 = 0, x_2 = 0, x_3 = 2$.

$$(22\widetilde{y}_1\widetilde{y}_2\ldots, 0x_4x_5x_6\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_1 = 2.$$

Let $x_1 = 0, x_2 = 2, y_1 = 0$.

$$(0y_2y_3y_4\ldots, 22\widetilde{x}_3\widetilde{x}_4\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_3 = y_3 \text{ and } x_3 = \widetilde{y}_3.$$

Let $x_1 = 0, x_2 = 2, y_1 = 2$.

$$(22\widetilde{x}_3\widetilde{x}_4\ldots, 2\widetilde{y}_2\widetilde{y}_3\widetilde{y}_4\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_1 = 2.$$

Let $x_1 = 2, x_2 = 0, y_1 = 0$.

$$(0y_2y_3y_4\ldots, 20\widetilde{x}_3\widetilde{x}_4\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_1 = 0.$$

Let $x_1 = 2, x_2 = 0, y_1 = 2$.

$$(20\widetilde{x}_3\widetilde{x}_4\ldots, 2\widetilde{y}_2\widetilde{y}_3\widetilde{y}_4\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow y_3 = \widetilde{y}_3.$$

Let $x_1 = 2, x_2 = 2, x_3 = 0$.

$$(20y_1y_2\ldots, 0\widetilde{x}_4\widetilde{x}_5\widetilde{x}_6\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_2 = 0.$$

Let $x_1 = 2, x_2 = 2, x_3 = 2$.

$$(\widetilde{x}_3\widetilde{x}_4\widetilde{x}_5\ldots, 02\widetilde{y}_1\widetilde{y}_2\widetilde{y}_3\ldots) = (x_1x_2x_3\ldots, y_1y_2y_3\ldots) \Rightarrow x_1 = \widetilde{x}_3.$$

3-periodic points of F_3: We compute

$$\begin{array}{lll} (0\overline{2}, 2\overline{0}) & , \text{ for } & x_1 = 0, x_2 = 2, y_1 = 2 \\ (20\overline{2}, 0\overline{2}) & , \text{ for } & x_1 = 2, x_2 = 0, y_1 = 0, y_2 = 2\,. \\ (22\overline{0}, \overline{2}) & , \text{ for } & x_1 = 2, x_2 = 2, x_3 = 0 \end{array}$$

Remark 3. *Since F_3 has no 2-periodic points, the dynamical system $\{C \times C; F_3\}$ cannot be topologically equivalent to dynamical systems $\{C \times C; F\}$, $\{C \times C; F_1\}$ and $\{C \times C; F_2\}$.*

4. Conclusion

In [6], Smale's horseshoe map is expressed in the ternary system and thus its periodic points are formulated and Devaney chaos conditions are more simply investigated. In the present paper, we first modify Smale's horseshoe map in different forms and obtain different dynamical systems on $C \times C$. Compared to Smale's horseshoe map, to express and calculate the periodic points of these modifications it is more complex in ternary system. In the light of the observations obtained after a few iterations of the point under the first modification, we conclude that the dynamical system $\{C \times C; F_1\}$ is Devaney chaotic. We also show that the second modification F_2 is topologically conjugate to the first modification F_1 and thus we calculate periodic points of F_2 more easily. Finally, we construct a dynamical system that is not topologically equivalent to these dynamical systems. Similarly, many different modifications of Smale's horseshoe map on $C \times C$ can be constructed and these systems can be classified according to whether they are topologically conjugate or not.

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