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An analog of Wolstenholme's theorem: an addendum

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Abstract. Let $p \geqslant 2$ be a prime number and let a, b, m be positive integers such that $p \nmid m$. In a recent paper [1], we discussed the maximal prime power p^e, which divides the numerator of the fraction

$$\frac{1}{m} + \frac{1}{m+p^b} + \frac{1}{m+2p^b} + \cdots + \frac{1}{m+(p^a-1)p^b},$$

when written in reduced form. This short note may be regarded as an addendum to paper [1] for the case where $p = 2$, $b = 1$, $m > 1$ and $2^a \| m-1$, which was left open.

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1. Introduction

For a given prime number $p \geqslant 2$ and positive integers a, b, m such that $p \nmid m$, let us consider the fraction

$$\frac{1}{m} + \frac{1}{m+p^b} + \frac{1}{m+2p^b} + \cdots + \frac{1}{m+(p^a-1)p^b}, \tag{1}$$

and let $e = E(p, a, b, m)$ denote the maximal exponent of the prime power p^e which divides the numerator of fraction (1), when written in reduced form. In [1], we proved that

$$E(p,a,b,m) = \begin{cases} a & p > 2 \\ a & p = 2,\ b \geqslant 2 \\ 2a & p = 2,\ b = 1,\ m = 1 \\ a + c & p = 2,\ b = 1,\ m > 1,\ 2^c \| m-1,\ c < a \\ 2a & p = 2,\ b = 1,\ m > 1,\ 2^c \| m-1,\ c > a. \end{cases}$$

As one can see, all the values of $E(p, a, b, m)$ have been determined, except for the case where $p = 2$, $b = 1$, $m > 1$ and $2^a \| m - 1$. Nevertheless, we were able to

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determine in [1] that in this exceptional case $E(p,a,b,m) \geqslant 2a+1$. In this paper, we prove that if $p=2$, $b=1$, $m>1$, $2^a \| m-1$, then

$$E(p,a,b,m) = 2a+d,$$

where d is a positive integer such that $2^d \| \frac{m-1}{2^a}+1$. As an illustrative example, for $p=2$, $b=1$, $a=2$ and $m=1021$, we obtain that

$$\frac{1}{1021}+\frac{1}{1023}+\frac{1}{1025}+\frac{1}{1027}=\frac{4294946816}{1099501142025}.$$

Here $2^2 \| m-1$ and $2^8 \| \frac{m-1}{2^a}+1$, so the numerator 4294946816 is divisible by $2^{2\cdot 2+8} = 2^{12}$, but not by 2^{13}.

2. The proof of the exceptional case

For the proof of our main result we shall use the following part of Bauer's theorem from [2, p. 127]:

Proposition 1. *Let $a>1$ be a positive integer. Then*

$$\prod_{\substack{1\leqslant s<2^a\\ s\text{ is odd}}}(x-s) \equiv (x^2-1)^{2^{a-2}} \pmod{2^a},$$

where the congruence indicates that the corresponding coefficients of the polynomials are congruent modulo 2^a.

Now we are ready to prove:

Theorem 1. *Suppose that a is a positive integer and $m>1$ is an odd integer such that $2^a \| m-1$. In addition, let d be a positive integer such that $2^d \| \frac{m-1}{2^a}+1$. Then the maximal prime power 2^e which divides the numerator of the fraction*

$$\frac{1}{m}+\frac{1}{m+2}+\frac{1}{m+4}+\frac{1}{m+6}+\cdots+\frac{1}{m+2(2^a-1)},$$

when written in reduced form, is 2^{2a+d}.

Proof. Set $u=(m-1)/2^a$. Note that by our assumption u is odd and $2^d \| u+1$. In addition, set $v=(u+1)/2^d$. Note that

$$\begin{aligned}
\frac{1}{m}+\frac{1}{m+2}+\cdots+\frac{1}{m+2(2^a-1)} &= \sum_{k=0}^{2^a-1}\frac{1}{m+2k} = \sum_{k=0}^{2^a-1}\frac{1}{2^a u+(2k+1)} \\
&= \sum_{\substack{1\leqslant s<2^{a+1}\\ s\text{ is odd}}}\frac{1}{2^a u+s} \\
&= \sum_{\substack{1\leqslant s<2^a\\ s\text{ is odd}}}\left(\frac{1}{2^a u+s}+\frac{1}{2^a u+2^{a+1}-s}\right) \\
&= \sum_{\substack{1\leqslant s<2^a\\ s\text{ is odd}}}\frac{2^{a+1}u+2^{a+1}}{(2^a u+s)(2^a u+2^{a+1}-s)}
\end{aligned}$$

$$= \sum_{\substack{1\leqslant s<2^a \\ s \text{ is odd}}} \frac{2^{a+d+1}v}{\underbrace{(2^a u+s)(2^a u+2^{a+1}-s)}_{A_s}}.$$

Hence, in order to prove the claim it suffices to prove that the numerator of the fraction

$$\sum_{\substack{1\leqslant s<2^a \\ s \text{ is odd}}} \frac{1}{A_s}, \tag{2}$$

when written in reduced form, is divisible by 2^{a-1} but not by 2^a. For simplicity, for every $0 \leqslant k \leqslant 2^{a-1}$, let us denote the following kth elementary symmetric expression $e_k(A_1, A_3, \ldots, A_{2^a-1})$ by e_k. Note that the numerator of (2) is $e_{2^{a-1}-1}$. Since the A_s's are odd, it suffices to prove that $2^{a-1} \| e_{2^{a-1}-1}$. To do that, consider the polynomial

$$f(x) = \prod_{\substack{1\leqslant s<2^{a+1} \\ s \text{ is odd}}} (x - 2^a u - s).$$

It holds:

$$\begin{aligned}
f(x) &= \prod_{\substack{1\leqslant s<2^{a+1} \\ s \text{ is odd}}} (x-(2^a u+s)) = \prod_{\substack{1\leqslant s<2^a \\ s \text{ is odd}}} (x-(2^a u+s))(x-(2^a u+2^{a+1}-s)) \\
&= \prod_{\substack{1\leqslant s<2^a \\ s \text{ is odd}}} (x^2-(2^{a+1}u+2^{a+1})x+A_s) \equiv \prod_{\substack{1\leqslant s<2^a \\ s \text{ is odd}}} (x^2+A_s) \\
&= \sum_{k=0}^{2^{a-1}} e_k x^{2^a-2k} \pmod{2^{a+1}}.
\end{aligned}$$

On the other hand, by Proposition 1

$$\prod_{\substack{1\leqslant s<2^{a+1} \\ s \text{ is odd}}} (y-s) \equiv (y^2-1)^{2^{a-1}} \pmod{2^{a+1}}.$$

Thus, by taking $y = x - 2^a u$ we get

$$f(x) \equiv ((x-2^a u)^2-1)^{2^{a-1}} = \sum_{k=0}^{2^{a-1}} \binom{2^{a-1}}{k} (-1)^k \big((x-2^a u)^2\big)^{2^{a-1}-k} \pmod{2^{a+1}}.$$

Now, since $(x-2^a u)^2 \equiv x^2 \pmod{2^{a+1}}$, it follows that

$$f(x) \equiv \sum_{k=0}^{2^{a-1}} \binom{2^{a-1}}{k} (-1)^k x^{2^a-2k} \pmod{2^{a+1}}.$$

By comparing the coefficient of x^{2^a-2k} in (1) and (2) we obtain that

$$e_k \equiv \binom{2^{a-1}}{k} (-1)^k \pmod{2^{a+1}}$$

for every $0 \leqslant k \leqslant 2^{a-1}$. In particular $2^{a-1} \| e_{2^{a-1}-1}$, as required. $\square$

By combining the above with the mentioned results from [1], we have the result as follows:

Theorem 2. *Let $p \geqslant 2$ be a prime number and let a, b, m be positive integers such that $p \nmid m$. Then*

$$E(p,a,b,m) = \begin{cases} a & p > 2 \\ a & p = 2,\ b \geqslant 2 \\ 2a & p = 2,\ b = 1,\ m = 1 \\ a + c & p = 2,\ b = 1,\ m > 1,\ 2^c \| m - 1,\ c < a \\ 2a & p = 2,\ b = 1,\ m > 1,\ 2^c \| m - 1,\ c > a \\ 2a + d & p = 2,\ b = 1,\ m > 1,\ 2^a \| m - 1,\ 2^d \| \frac{m-1}{2^a} + 1. \end{cases}$$

References

[1] B. Cohen, *An analog of Wolstenholme's theorem*, Math. Commun. **28**(2023), 69–83.
[2] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, 6th edition, Clarendon Press, Oxford, , 2008.