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Variational and numerical approach to a quasi-steady rolling problem

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SUMMARY

In this paper, a variable stiffness parameters method (VSPM), is applied to a quasi-steady, hot strip rolling problem. A slightly compressible, rigid-viscoplastic and isotropic hardening material model is assumed for the workpiece. For the roll-workpiece interface a nonlinear friction law is supposed to hold. The problem is stated in the form of a variational inequality, containing nonlinear and nondifferentiable terms. Under certain restrictions on the material characteristics, the convergence of the VSPM is shown. Combining the VSPM with the FEM, an algorithm is proposed and applied to solve numerically a rolling problem, and the obtained results are illustrated and discussed.

Key words: *quasi-steady rolling, friction, variable stiffness parameters method (VSPM), FEM.*

1. INTRODUCTION

The flow theory of plasticity [1-2], has been successfully applied in the last few decades, for material modelling in the analyses of the rolling processes. Computer simulations of steady-state and non-steady rolling processes, [3-10], for rigid-plastic (viscoplastic), with or without hardening, incompressible or slightly compressible materials, at different friction conditions, have been performed. The computer analyses have been usually based on the discretization by finite elements, the virtual power variational principle, or Markov’s functional [1]. The strain hardening phenomena have been taken into account either by using a mixed Lagrangian-Eulerian formulation, or some flow line iterative technique. The obtained systems of nonlinear equations have been usually solved by the Newton-Raphson or a successive iterations method. Recently, a variational formulation for an isothermal, steady-state hot-rolling problem of rigid-plastic, strain-rate sensitive and slightly compressible materials, with a velocity dependent Coulomb friction law, has been derived and studied in Ref. [7]. The case of incompressible materials and constant friction law have been considered in Ref. [8]. The existence and uniqueness results have been obtained and the convergence of the successive iteration (secant-modulus, Kachanov) methods have been proved. Numerical results, for the case of incompressible materials, obtained by a finite element - secant-modulus method, and compared with other methods of aproach have been discussed in Ref. [9].

In this paper, a hot strip rolling problem for slightly compressible, rigid-viscoplastic and isotropic hardening materials, with velocity dependent friction conditions is considered. Due to the strain hardening, the problem is quasi-steady i.e. time dependent until steady-state is reached. The corresponding variational formulation is given and with the help of a proposed variable stiffness parameters method (VSPM) [10-11], under certain restrictions on the material characteristics, the existence and uniqueness results are briefly shown. Combining the VSPM with the FEM, an algorithm is proposed and applied to solve numerically a rolling problem. The influence of the friction conditions on the contact velocities, stresses, equivalent strain-rates and strains is illustrated and discussed.

2. STATEMENT OF THE PROBLEM

We consider an isothermal, quasi-steady hot-strip rolling process in a fixed domain, until steady-state is reached at a time $T\in[0,\infty)$. We suppose that the workpiece is an isotropic, rigid-plastic, strain and strain-rate sensitive, slightly compressible metallic body occupying for all $t\in[0,T]$ the domain $\Omega\subset R^k$, $k=2,3$, (Figure 1). The boundary of the domain is constituted of four open disjoint subsets $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$, as $\Gamma_1\cup\Gamma_2$ is assumed tractions free, Γ_3 is the boundary of symmetry and Γ_4 is the contact boundary. Here after any point of $\overline{\Omega}=\Omega\cup\Gamma$ we identify by its Cartesian coordinates $\boldsymbol{x}=\{x_i\}$, $(1\le i\le k)$, and use the standard indicial notations. Let $\boldsymbol{u}(\boldsymbol{x},t)=\{u_i(\boldsymbol{x},t)\}$, $\boldsymbol{\sigma}(\boldsymbol{x},t)=\{\sigma_{ij}(\boldsymbol{x},t)\}$, $\dot{\varepsilon}(\boldsymbol{x},t)=\{\dot{\varepsilon}_{ij}(\boldsymbol{x},t)\}$, $(1\le i,j\le k)$, denote the velocity vector, stress tensor and strain rate tensor respectively. We assume that workpiece material satisfies the following yield criterion and flow rule:

$$F\equiv\bar{\sigma}^2-\sigma_p^2\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)=0,\quad \dot{\varepsilon}_{ij}=\frac{3}{2}\frac{\dot{\bar{\varepsilon}}}{\bar{\sigma}}\left(s_{ij}+\frac{2}{9}d\sigma_H\delta_{ij}\right) \tag{1}$$

Here d > 0 is a constant associated with the material compressibility, such that when $d\to 0$ the classical von Mises yield criterion and flow rule for incompressible materials are approached. The equivalent stress, strain-rate and strain are given by the expressions:

$$\bar{\sigma}=\sqrt{\frac{3}{2}s_{ij}s_{ij}+d\sigma_H^2}$$

$$\dot{\bar{\varepsilon}}=\sqrt{\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}+\frac{1}{d}\dot{\varepsilon}_\upsilon^2} \tag{2}$$

$$\bar{\varepsilon}(\boldsymbol{x},t)=\int_0^t\dot{\bar{\varepsilon}}(\boldsymbol{x},\tau)\,d\tau$$

where $\sigma_H=\frac{1}{3}\sigma_{ii}$, $\dot{\varepsilon}_\upsilon=\dot{\varepsilon}_{ii}$ are the hydrostatic pressure and the volume dilatation strain rate, $s_{ij}=\sigma_{ij}-\sigma_H\delta_{ij}$, $\dot{e}_{ij}=\dot{\varepsilon}_{ij}-\frac{1}{3}\dot{e}_\upsilon\delta_{ij}$, are the components of deviatoric stress and strain-rate tensors, $1\le i,\ j\le k$. We assume that the strain and strain-rate dependent uniaxial yield limit $\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)$, is a monotonically increasing, almost everywhere differentiable function on both variables, such that:

$$c_1\le\frac{\partial\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)}{\partial\dot{\bar{\varepsilon}}}\le\frac{\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)}{\dot{\bar{\varepsilon}}}\le c_2$$

$$c_3\le\frac{\partial\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)}{\partial\bar{\varepsilon}}\frac{1}{\dot{\bar{\varepsilon}}}\le c_4\qquad\forall\bar{\varepsilon},\dot{\bar{\varepsilon}}\in[0,\infty) \tag{3}$$

where c_1, c_2, c_3 and c_4 are positive constants.

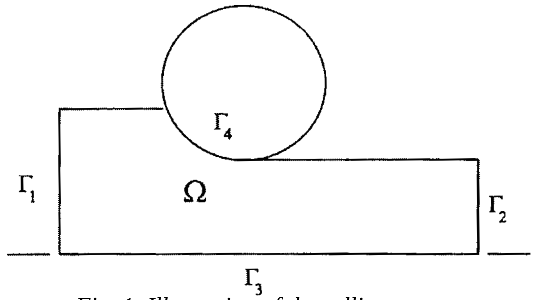

Fig. 1 Illustration of the rolling process

We state the following boundary value problem:

Problem 1: Find the velocity $\boldsymbol{u}$, stress $\boldsymbol{\sigma}$, and equivalent strain $\bar{\varepsilon}$ fields, satisfying the following equations and relations:

– equation of equilibrium

$$\sigma_{ij,j}=0\quad\text{in}\quad\Omega\times(0,T) \tag{4}$$

– equivalent strain evolution equation

$$\frac{\partial\bar{\varepsilon}(\boldsymbol{x},t)}{\partial t}=\dot{\bar{\varepsilon}}(\boldsymbol{x},t)\quad in\quad\Omega\times(0,T) \tag{5}$$

– constitutive equations

$$\sigma_{ij}=\frac{\sigma_p}{\dot{\bar{\varepsilon}}}\left(\frac{2}{3}\dot{\varepsilon}_{ij}+\left(\frac{1}{d}-\frac{2}{9}\right)\dot{\varepsilon}_\upsilon\delta_{ij}\right) \tag{6}$$

– strain-rate velocity relations

$$\dot{\varepsilon}_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{7}$$

– boundary conditions

$$\sigma_{ij}n_j=0\quad\text{on}\quad\Gamma_1\cup\Gamma_2\times(0,T) \tag{8}$$

$$\boldsymbol{\sigma}_T=\boldsymbol{0},\ u_N=0\quad\text{on}\quad\Gamma_3\times(0,T) \tag{9}$$

$$\sigma_N+\frac{u_N^+}{d_N}=0 \tag{10}$$

if $\left|\boldsymbol{\sigma}_T(\boldsymbol{u})\right|<\tau_f\left(\bar{\varepsilon}_p,\boldsymbol{u}\right)$, then $\boldsymbol{u}_T-\boldsymbol{u}_{TR}=\boldsymbol{0}$,

if $\left|\boldsymbol{\sigma}_T(\boldsymbol{u})\right|=\tau_f\left(\bar{\varepsilon}_p,\boldsymbol{u}\right)$, then $\exists$ const. $\lambda\ge 0$,

such that $\boldsymbol{u}_T-\boldsymbol{u}_{TR}=-\lambda\boldsymbol{\sigma}_T(\boldsymbol{u})$ on $\Gamma_4\times(0,T)$

– initial conditions

$$\bar{\varepsilon}(\boldsymbol{x},0)=0 \tag{11}$$

Here δ_{ij} is Kronecker symbol; $\boldsymbol{n}=\{n_i\}$ is the unit normal vector outward to Γ; $\boldsymbol{u}_N$, $\boldsymbol{u}_T$ and $\boldsymbol{\sigma}_N$, $\boldsymbol{\sigma}_T$ are the normal and tangential components of the velocity and the stress vector; $u_N^+=max\,(0,u_N)$; $d_N>0$ is a penalty constant; $\boldsymbol{u}_{TR}$ is the tangential component of the roll velocity; $\tau_f(\bar{\varepsilon}_\rho, \boldsymbol{u})$ is the shear strength limit for the roll-workpiece interface according to the accepted Coulomb-Siebel friction law [4, 9]:

$$\tau_f(\bar{\varepsilon}_\rho, \boldsymbol{u}) = min\left(-\mu_f(\boldsymbol{x})\sigma_N(\boldsymbol{u}), m_f(\boldsymbol{x})\tau_p\left(\bar{\varepsilon}_\rho, \dot{\bar{\varepsilon}}_\rho(\boldsymbol{u})\right)\right) \quad (12)$$

where $\mu_f(\boldsymbol{x})$ is the coefficient of friction and $m_f(\boldsymbol{x}) \in (0,1]$ is the friction factor. $\tau_f(\bar{\varepsilon}_\rho, \dot{\bar{\varepsilon}}_\rho)$ is the strain and strain-rate dependent shear yield limit for the material of the roll-workpiece interface, where the subindex ρ denotes that the strains and strain-rates are appropriately modified [9, 12], in order to be well defined on Γ_4. The Coulomb-Siebel friction law combines the stick-slipping and stick-shearing friction mechanisms and expresses the fact that the shear stresses can not exceed the shear yield limit. The normal compliance and interface friction models, proposed in Ref. [12], could also be used; however, a good approximation of the normal compliance behaviour at high loading during rolling, is the penalty normal stress - normal velocity relation, used here.

3. VARIATIONAL FORMULATION

We denote by:

$$\boldsymbol{V} = \{\boldsymbol{v}: \boldsymbol{v} \in (H^1(\Omega))^k,\ \upsilon_N = 0 \text{ on } \Gamma_3\} \quad (13)$$

the space of kinematically admissible velocities. Then the variational formulation of Problem 1 is the following one:

Problem 2: Find $\boldsymbol{u}(t) \in \boldsymbol{V}$ and $\bar{\varepsilon}(t) \in H^0(\Omega)$, satisfying for all $t \in [0, T]$:

$$a(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + \langle k_N(\boldsymbol{u}), \boldsymbol{v}-\boldsymbol{u}\rangle + j(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{v}) - j(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{u}) \geq 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \quad (14)$$

$$\int_\Omega \frac{\partial \bar{\varepsilon}}{\partial t} \gamma\, dx = \int_\Omega \dot{\bar{\varepsilon}}(t) \gamma\, dx, \quad \forall \gamma \in H_0(\Omega) \quad (15)$$

and the initial conditions $\bar{\varepsilon}(0) = 0$.

Here above we have used the notations:

$$a(\bar{\varepsilon}, \boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \frac{\sigma_p(\bar{\varepsilon}, \boldsymbol{w})}{\dot{\bar{\varepsilon}}(\boldsymbol{w})} \left[\frac{2}{3} \dot{\varepsilon}_{ij}(\boldsymbol{u}) \dot{\varepsilon}_{ij}(\boldsymbol{v}) + \left(\frac{1}{d} - \frac{2}{9}\right) \dot{\varepsilon}_\upsilon(\boldsymbol{u}) \dot{\varepsilon}_\upsilon(\boldsymbol{v})\right] dx \quad (16)$$

$$\langle k_N(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_{\Gamma_4} \frac{u_N^+ \upsilon_N}{d_N}\, d\Gamma \quad (17)$$

$$j(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{v}) = \int_{\Gamma_4} \tau_f(\bar{\varepsilon}_\rho, \boldsymbol{u}) |\boldsymbol{v}_T - \boldsymbol{u}_{TR}|\, d\Gamma \quad (18)$$

These relations express correspondingly the virtual power of actual stresses, contact pressure and friction forces. Since the functional $j(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{v})$ is nondifferentiable, we introduce the following regularized one:

$$j_{d_T}(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{v}) = \int_{\Gamma_4} \tau_f(\bar{\varepsilon}_\rho, \boldsymbol{u}) \sqrt{|\boldsymbol{v}_T - \boldsymbol{u}_{TR}|^2 + d_T^2} \quad (19)$$

where $d_T > 0$ is a constant. Then Problem 2 obtains the following equality form:

Problem 3: Find $\boldsymbol{u}(t) \in \boldsymbol{V}$ and $\bar{\varepsilon}(t) \in H^0(\Omega)$, satisfying for all $t \in [0,T]$:

$$a(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \langle k_N(\boldsymbol{u}), \boldsymbol{v}\rangle + \langle j'_{d_T}(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{u}), \boldsymbol{v}\rangle = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \quad (20)$$

$$\int_\Omega \frac{\partial \bar{\varepsilon}}{\partial t} \gamma\, dx = \int_\Omega \dot{\bar{\varepsilon}}(t) \gamma\, dx \quad \forall \gamma \in H^0(\Omega) \quad (21)$$

and the initial conditions $\bar{\varepsilon}(0) = 0$.

Here we have used the notation:

$$\left\langle j'_{d_T}(\bar{\varepsilon}, \boldsymbol{u}; \boldsymbol{u}), \boldsymbol{v}\right\rangle = \int_{\Gamma_4} \tau_f(\bar{\varepsilon}_\rho, \boldsymbol{u}) \frac{(\boldsymbol{u}_T - \boldsymbol{u}_{TR}) \cdot \boldsymbol{v}_T}{\sqrt{|\boldsymbol{u}_T - \boldsymbol{u}_{TR}|^2 + d_T^2}}\, d\Gamma \quad (22)$$

Actually, the solution $\boldsymbol{u}$ of Problem 3 depends on the regularization parameter, but for the sake of simplicity, we use the same notation as in Problem 2.

We further assume:

$$\mu_f(x),\ m_f(x) \in L_\infty(\Gamma_4) \quad (23)$$

Theorem: Let the friction coefficient or factor be sufficiently small and the assumptions (3) hold. Then there exists a unique solution of Problem 2, such that:

$$\boldsymbol{u} \in L_\infty(0,T;\ \boldsymbol{V}), \quad \bar{\varepsilon} \in L_\infty(0,T;\ H^0(\Omega)) \quad (24)$$

Proof: Since the proof is too length and technical and follows in general details the proofs in Refs. [7] and [11], we shall present herein only a sketch of it.

Uniqueness: Let $\{\boldsymbol{u}_1(t), \bar{\varepsilon}_1(t)\}$ and $\{\boldsymbol{u}_2(t), \bar{\varepsilon}_2(t)\}$ be two solutions of Problem 2. Then replacing with them in Eqs. (14) and (15) and taking correspondingly $\boldsymbol{v} = \boldsymbol{u}_2(t)$, $\boldsymbol{v} = \boldsymbol{u}_1(t)$ and $\gamma = \bar{\varepsilon}_2(t) - \bar{\varepsilon}_1(t)$, after taking into account Eq. (3), with the help of Gronwall lemma [11], we obtain that:

$$\|\boldsymbol{u}_2(t) - \boldsymbol{u}_1(t)\|_1 = 0, \quad \|\bar{\varepsilon}_2(t) - \bar{\varepsilon}_1(t)\|_0 = 0 \quad (25)$$

Existence: Let us consider the following auxiliary problem. Assume that for $t \in [0,T]$, $\boldsymbol{u}_n(t) \in \boldsymbol{V}$, $\bar{\varepsilon}_n(t) \in H^0(\Omega)$, $n = 0, 1, \ldots$, be known. Then:

- Find $\boldsymbol{u}_{n+1}(t) \in \boldsymbol{V}$, $\bar{\varepsilon}_{n+1}(t) \in H^0(\Omega)$, satisfying:

$$a(\bar{\varepsilon}_n, \boldsymbol{u}_n; \boldsymbol{u}_{n+1}, \boldsymbol{v}-\boldsymbol{u}_{n+1}) + \langle k_N(\boldsymbol{u}_{n+1}), \boldsymbol{v}-\boldsymbol{u}_{n+1}\rangle + j(\bar{\varepsilon}_n, \boldsymbol{u}_n; \boldsymbol{v}) - j(\bar{\varepsilon}_n, \boldsymbol{u}_n; \boldsymbol{u}_{n+1}) \geq 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \quad (26)$$

$$\int_\Omega \frac{\partial \bar{\varepsilon}_{n+1}}{\partial t} \gamma\, dx = \int_\Omega \dot{\bar{\varepsilon}}_{n+1} \gamma\, dx \quad \forall \gamma \in H^0(\Omega) \quad (27)$$

and the initial conditions $\bar{\varepsilon}_{n+1}(0) = 0$.

Following Refs. [7] and [11], it can be shown that for any n, this problem has a unique solution. It can be further shown, again with the help of Gronwall lemma, that:

$$\lim_{n\to\infty}\left\|\boldsymbol{u}_{n+1}(t)-\boldsymbol{u}_n(t)\right\|_1=0$$
$$\lim_{n\to\infty}\left\|\bar{\varepsilon}_{n+1}(t)-\bar{\varepsilon}_n(t)\right\|_0=0 \tag{28}$$

and therefore the obtained sequence $\{\boldsymbol{u}_n(t), \bar{\varepsilon}_n(t)\}$ converges, at $n\to\infty$, to a solution $\{\boldsymbol{u}(t), \bar{\varepsilon}(t)\}$ of Problem 2, which is unique.

The above auxiliary problem is a method of successive linearizations, which we call a variable stiffness parameters method, by an analogy with the method proposed in Ref. [11] for solving small strains plastic flow theory problems with hardening. This method also gives the time T, at which the process becomes steady-state, as well as the steady-state solution of Problem 2.

The existence and uniqueness of the solution of Problem 3 can be obtained analogously to Problem 2. One could also show that when the regularization parameter tends to zero, the solution of Problem 3 tends to the solution of Problem 2.

4. NUMERICAL SOLUTION AND RESULTS

Here we use the finite element method for a spatial discretization and the scheme $t_k=k\Delta t$, $k=0,1,...$, with a time step Δt, for time discretization of Problem 3. After applying the VSPM we obtain in $\boldsymbol{V}_h\subset\boldsymbol{V}$, $h\in(0,1)$ is a mesh parameter, the following finite-dimensional problem, which gives the time T, at which the process becomes steady-state, as well as the steady-state solution of Problem 3.

Problem 3$_h$: Find $\boldsymbol{u}^h_{n+1,k}\in\boldsymbol{V}_h$, $\bar{\varepsilon}^h_{n+1,k}\in H^0(\Omega)$, $n=0,1,2,...$, satisfying for arbitrary initial state $\boldsymbol{u}^h_{0,0}\in\boldsymbol{V}_h$ and every $\boldsymbol{v}^h\in\boldsymbol{V}_h$ the equation:

$$a\left(\bar{\varepsilon}^h_{n,k},\boldsymbol{u}^h_{n,k};\boldsymbol{u}^h_{n+1,k},\boldsymbol{v}^h\right)+\left\langle k_N\left(\boldsymbol{u}^h_{n+1,k}\right),\boldsymbol{v}^h\right\rangle+ \\ +\left\langle j'_{d_T}\left(\bar{\varepsilon}^h_{n,k},\boldsymbol{u}^h_{n,k};\boldsymbol{u}^h_{n+1,k}\right),\boldsymbol{v}^h\right\rangle=0 \tag{29}$$

$$\bar{\varepsilon}^h_{n+1,k}=\sum_{m=0}^{k}\dot{\bar{\varepsilon}}^h_{n+1,m}\Delta t \tag{30}$$

and the initial conditions $\bar{\varepsilon}^h_{n,0}=0$, until:

$$\left\|\boldsymbol{u}^h_{n+1,k}-\boldsymbol{u}^h_{n,k}\right\|_0\Big/\left\|\boldsymbol{u}^h_{n+1,k}\right\|_0<\delta$$

and:

$$\left\|\boldsymbol{u}^h_{n+1,k+1}-\boldsymbol{u}^h_{n+1,k}\right\|_0\Big/\left\|\boldsymbol{u}^h_{n+1,k+1}\right\|_0<\delta$$

Here $\|\cdot\|_0$ is a vector norm and δ is the tolerance. Using further isoparametric finite elements with the correponding Gauss integration rule: bilinear, four-noded quadrilaterals for discretization of Ω and linear, two-noded elements for discretization of Γ_4, we obtain from Eq. (29) the following system of non-linear equations:

$$\left(\boldsymbol{K}\left(\boldsymbol{u}^h_{n,k}\right)+\boldsymbol{K}_c\left(\boldsymbol{u}^h_{n,k}\right)\right)\left\{\boldsymbol{u}^h_{n+1,k}\right\}=\boldsymbol{F}_c\left(\boldsymbol{u}^h_{n,k}\right) \tag{31}$$

where $\boldsymbol{K}$, $\boldsymbol{K}_c$ and $\boldsymbol{F}_c$ are the velocity dependent stiffness matrix, the contact stiffness matrix and the contact load vector. The vector of nodal velocities is denoted by $\left\{u^h_{n+1,k}\right\}$. The matrix $\boldsymbol{K}$ contains the first term on the left-hand side of Eq. (29). The matrix $\boldsymbol{K}_c$, contains the second term and a part of the third term in Eq. (29). The vector $\boldsymbol{F}_c$, is formed by the remaining part of the third term in Eq. (29), containing the roll velocity.

Further we assume that the following empirical relations for the yield limit hold:

$$\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)=c(\bar{\varepsilon})\dot{\bar{\varepsilon}}^b \quad for \quad \dot{\bar{\varepsilon}}\in\left[\dot{\bar{\varepsilon}}_1,\dot{\bar{\varepsilon}}_2\right], \bar{\varepsilon}\in\left[0,\dot{\bar{\varepsilon}}_2\right] \tag{32}$$
$$\sigma_p\left(\bar{\varepsilon},\dot{\bar{\varepsilon}}\right)=c(\bar{\varepsilon})\,\dot{\bar{\varepsilon}}_q^{b-1}\dot{\bar{\varepsilon}} \quad q=1,2 \quad for$$
$$\dot{\bar{\varepsilon}}\le\dot{\bar{\varepsilon}}_1 \quad and \quad \dot{\bar{\varepsilon}}\ge\dot{\bar{\varepsilon}}_2 \quad \bar{\varepsilon}\in\left[0,\bar{\varepsilon}_2\right]$$
$$c(\bar{\varepsilon})=\alpha_0\left(1+\alpha_1\bar{\varepsilon}\right)^{\alpha_2} \quad for \quad \bar{\varepsilon}\in\left[0,\bar{\varepsilon}_2\right] \quad and$$
$$c(\bar{\varepsilon})=\alpha_0\left(1+\alpha_1\bar{\varepsilon}_2\right)^{\alpha_2} \quad for \quad \bar{\varepsilon}\ge\bar{\varepsilon}_2$$

where $\alpha_l>0$, $0\le l\le 2$, $b>0$, $\dot{\bar{\varepsilon}}_1$, $\dot{\bar{\varepsilon}}_2$ and $\bar{\varepsilon}_2$ are material constants, depending on the process conditions. It can be easily checked that the requirenments (3) are satisfied.

The initial state is a solution of the system (31) at $\dot{\bar{\varepsilon}}_1=10^{-6}$ and zero equivalent strain. We also further use $\dot{\bar{\varepsilon}}_2=10^6$ and $\bar{\varepsilon}_2=10$. The strain-rates and strains for the points of the contact boundary are defined as mean values of the strain-rates and strains for the neighbouring finite elements in the domain. Computational experiments show that the following values of the compressibility, penalty and regularization constants should be chosen: $d=10^{-3}$, $d_N=10^{-3}$, $d_T=10^{-6}$. These values ensure first the volumetric strain-rate and normal contact velocities to be as small as possible and second the solution to be obtained for a few iterations with a high accuracy. Futher $\delta=10^{-4}$ is used.

Example 1 ([6, 9]): A two-dimensional hot-rolling problem of low-carbon steel at temperature $T=1200°C$ is considered. The workpiece is $15\ mm$ long; the initial thickness is *2 mm*, which is reduced *40 %* during rolling. The roll diameter is *400 mm*; rolling velocity is $\mathbf{u}_{TR}=1256.64$ *mm/s*. The yield limit law $\sigma_p = 51.73(1+25\bar{\varepsilon})^{0.10}\dot{\varepsilon}^{0.218}$ and friction factor $m_f=1.00$ are used. The time step is chosen to be $\Delta t=0.0005$ *sec*.

For discretization of Ω and Γ_c 90 four-noded bilinear, quadrilateral elements and 20 two-noded linear elements are used correspondingly. Thus, the finite element mesh with 124 nodes and 248 total number of freedom degrees is obtained. Computational experiments on a rough and refined mesh show that this mesh is optimal since further refinement does not change the results.

The effect of the friction, on the contact velocities, stresses, effective strain-rates and strains of the analysed workpiece is illustrated in Figures 2 to 6.

The presented results correspond to the steady-state solution, obtained at the time $T=0.0025$ - 0.0035 *sec*, depending on the friction coefficients used, $\mu_f=0.1$, $\mu_f=0.2$, ..., $\mu_f=0.6$. In comparison with the results presented in Ref. [6], the results obtained here show some differences, which may be due to the different algorithms used. In comparison with the results presented in Ref. [9], where only a steady-state problem is considered, the differences are much smaller and could be explained with the inclusion of the strain hardening in the material law.

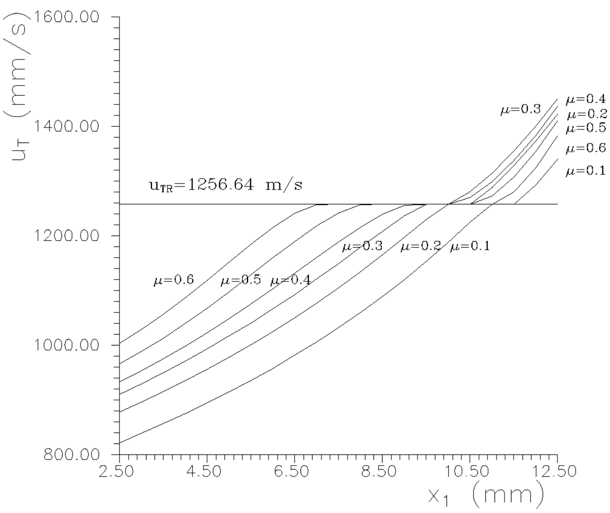

Fig. 2 Tangential velocities along the contact arc

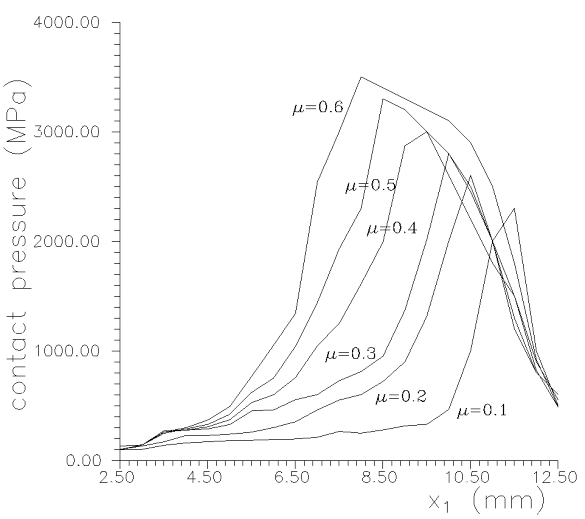

Fig. 3 Contact pressure along the contact arc

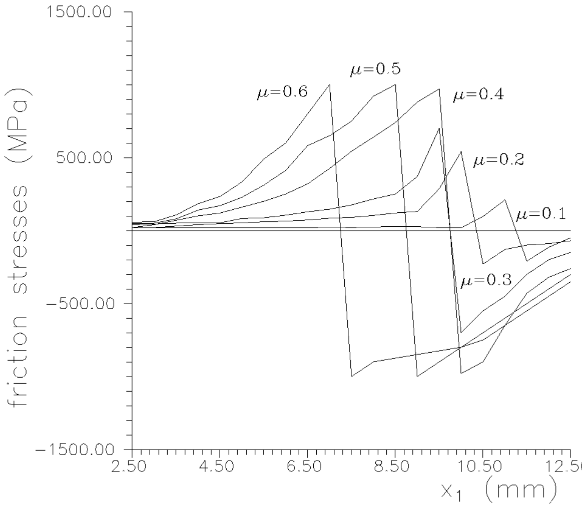

Fig. 4 Friction stresses along the contact arc

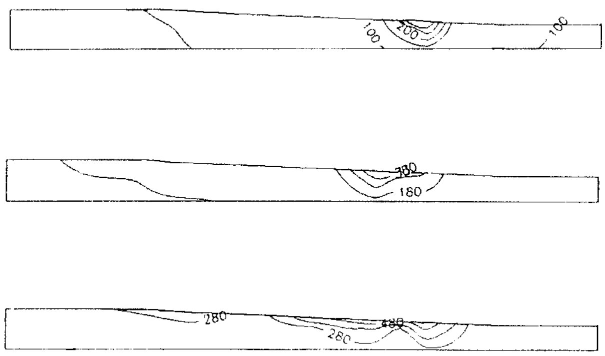

Fig. 5 Distribution of $\dot{\bar{\varepsilon}}$ at $\mu_f = 0.2$, $\mu_f = 0.4$, $\mu_f = 0.6$

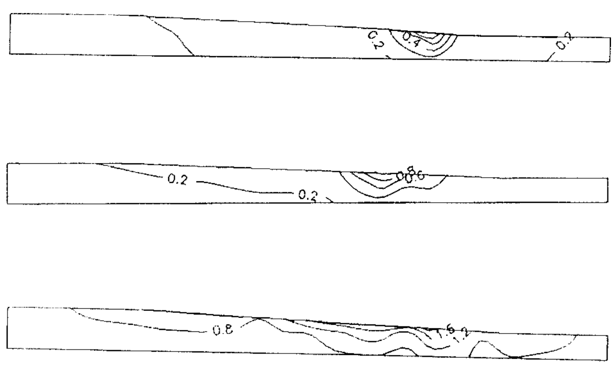

Fig. 6 Distribution of $\bar{\varepsilon}$ at $\mu_f = 0.2$, $\mu_f = 0.4$, $\mu_f = 0.6$

5. CONCLUDING REMARKS

In this paper, a VSPM is applied to a quasi-steady, hot strip rolling problem with a nonlinear friction, for slightly compressible, rigid-viscoplastic and isotropic hardening materials. The problem is formulated as a variational inequality, containing strongly nonlinear and nondifferentiable terms. Under certain restrictions on the material characteristics, the convergence of the VSPM is shown. An efficient algorithm, combining the VSPM with the FEM, is proposed and used to solve an example rolling problem. The computed results are graphically illustrated and compared with the results obtained for the steady-state problem, as well as with the results obtained by other methods of approach. The computational experiments show that the proposed method could serve for obtaining precise results in the computational analysis of rolling problems.

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VARIJACIJSKI I NUMERIČKI PRISTUP PROBLEMU KVAZI-STACIONARNOG VALJANJA

SAŽETAK

U ovom radu primjenjuje se metoda parametara varijabilne krutosti (VSPM) na kvazi-stacionarni problem valjanja vruće lamele. Predlaže se neznatno stišljiv, krut-visko plastični i izotropni model stvrdnjavanja za element. Pretpostavlja se da vrijedi nelinearni zakon trenja za interakciju valjanje-element. Problem se postavlja u obliku varijacijske nejednakosti koja sadrži nelinearne i nediferencijalne izraze. Uz određene restrikcije za svojstva materijala prikazuje se konvergencija VSPM. Kombinirajući VSPM s metodom konačnih elemenata predlaže se algoritam i primjenjuje se za numeričko rješavanje jednog slučaja problema valjanja, a postignuti rezultati su prikazani i obrađeni u radu.

Ključne riječi: *kvazi-stacionarno valjanje, trenje, metoda parametara varijabilne krutosti (VSPM), MKE.*