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MATHEMATICAL ESSAY AS METHOD FOR ASSESSING MATHEMATICAL COMMUNICATION SKILLS

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Mathematics communications skills are essential to learning and understanding mathematics. Objective of assessing mathematical communication is to evaluate how students use mathematical language, ask questions and elaborate answers, use mathematical symbols, and read algorithms, alternate between data representations, and apply technology. One of many arising questions, due to this matter, are methods most suitable for investigation of students' achievements in using apt mathematical language. The aim of this paper is to investigate weather (mathematical) expository essay is appropriate method for assessing mathematical communication. The literature analysis showed that it is possible to investigate each integral part presumed by mathematical communication as an assessment element, hence authors advocate mathematical essay as an appropriate method for assessing mathematical communication. The authors suggested framework for assessing students' mathematical essays across five levels of expertise. In assessment frame a special emphasis was put on mathematical communication and framework for writing mathematical essay, considering structural and substantive components.

Keywords: *mathematical communication; mathematical essay; mathematical language; assessment*

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Introduction

Continuous development of cognition on teaching and learning as well as emergence of new paradigms and approaches in educational sciences field are mostly accompanied by educational policies changes i.e., educational reforms that defines new teaching and learning outcomes. Modern educational reforms require the abandonment of traditional teaching, in which the teacher as a transmitter of knowledge is at the centre, the students are passive, and all attention is focused on the content that the students are required to learn. Teaching planning, based on the contents to be taught, is abandoned, and turned towards educational standards to be achieved, competencies to be developed and outcomes to be reached. Accordingly, changes are taking place in the field of mathematics education. One of the initiators of such changes was The National Council of Teachers of Mathematics (hereinafter: NCTM) in the United States of America, which published a series of documents with standards for learning, teaching and evaluation in school mathematics (1989; 1991; 1995; 2000). In each of the documents, mathematical communication skills are highlighted as essential to learning, understanding and doing mathematics (Cai *et al.*, 1996). NCTM (2000) identify mathematical communication as one of five basic standards for the acquisition and use of mathematical knowledge, on which the curriculum should be focused. Rowan *et al.* (1990) emphasized many desirable goals of communicating mathematically that concern deepening understanding, lead to shared understanding, support students as learners, contribute to establishing pleasurable learning environment and allow teacher to obtain insight into students' thinking and learning process as well as on knowledge acquainted. According to NCTM mathematical communication skills refer to students' ability to: "organize and consolidate their mathematical thinking through communication; communicate their mathematical thinking coherently and clearly to peers, teachers, and others; analyse and evaluate the mathematical thinking and strategies of others; use the language of mathematics to express mathematical ideas precisely" (2000, p. 402). Research suggested, that encouraging students to talk, listen, read, write, and reflect on their mathematical learning and problem solving will improve both their communication skills and knowledge of their own thinking. As a result, students are able to understand concepts better, discover new

perspectives on the issue, and express their thoughts mathematically (Rahman *et al.*, 2012).

In order to successfully respond to the challenges of the development of the knowledge society and the world market, the European Union has determined eight basic competencies for lifelong education (Fuchs *et al.*, 2011). When defining mathematical competence and basic competences in science and technology, essential knowledge, skills, and attitudes related to this competence were recognized, among which is the ability to communicate in mathematical language (EC 962/2006). As a result, many good examples of educational policies can be found that have accepted the stated competences and emphasized the importance of teaching, learning, and assessing mathematical communication skills as part of their curriculum and other reforms. One of the good examples is the mathematics curriculum in the Republic of Croatia, which with the implementation of the curricular reform in 2019, and bringing new curriculum for the subject of mathematics for primary and secondary schools, set, as one of the goals of mathematics learning: to apply mathematical language in oral and written expression, structuring, analysing, understanding and evaluation of information using different ways of presenting mathematical ideas, processes and results in mathematical context and real life (NN, 7/2019). In addition, mathematical communication is set as one of the three elements for evaluating the achievement of educational outcomes and is described as follows:

- to use appropriate mathematical language (standard mathematical symbols, notation, and terminology) in oral and written expression;
- to use appropriate mathematical data representation;
- to interchange between different mathematical representation;
- to express one' thoughts in complete, relevant and condense mathematical sentences;
- to ask and answer questions beyond originally proposed question;
- to organize data into logical structure;
- proper use of technology.

Based on these objectives, the ability of mathematical communication is or rather should be in focus of teaching and learning mathematics, is conclusion that present itself. Since becoming great part of learning mathematics there has been a lot of emerging questions regarding integrating as well as teaching skills of communicating mathematically.

Some of many arising questions, due to this matter, is how to assess mathematical communication? What types of tasks and methods are most suitable for investigation of students' achievements in using apt mathematical language? To attempt to answer some of these questions the research was conducted with following aim of research:

- to investigate (mathematical) essay as appropriate method for assessing mathematical communication.

Since literature analysis revealed expository (mathematical) essay as an adequate assessment method for mathematical communication, authors wanted to further their contribution and provided framework for assessing students' mathematical essays with emphasis on mathematical communication.

2. Review of literature

2.1. Research foci

Investigating educational policies across the world we witness a great role mathematical communication is given in teaching and learning mathematics. It is evident that mathematical communication is presented as one of the central outcomes in national curricula for mathematics. However, there is limited research on integration of mathematical communication in teaching praxis and even less research on assessment of mathematical communication skills. Looking into praxis we see that many uncertainties arose regarding summative assessment and assessment in general after implementation of curricular reform. By identifying integral parts of mathematical communication in accordance with the characteristics of mathematical communication from the subject curriculum of mathematics in Croatia, we aimed to compare them to key features of an expository essay to determine whether is possible to assess mathematical communication of students through mathematical essay in qualitative manner. Based on research conducted we concluded mathematical essay to be adequate method for mathematical communication assessment and wanted to further assist in eradication of uncertainties by providing the framework for assessment of mathematical essay with emphasis on mathematical communication.

2.2. Assessment of mathematical communication

Mathematics educators and scientists agree on communication as a component essential to learning, doing, and understanding mathematics (Cai *et al.*, 1996). Hence, there is a need to determine methods, techniques and criteria for assessing mathematical communication. Objective of assessing mathematical communication is, among other things, to evaluate how students use mathematical language, ask questions and elaborate answers, use mathematical symbols, and read algorithms, alternate between data representations, and apply technology (Cai *et al.*, 1996). In literature review, in order to assess mathematical communication skills, there can be found that suggestions for using tasks that were developed to test students' proficiency according to mathematical representation, ability to interpret of mathematical symbols, and problem-solving strategies through the use of solutions (Uyen *et al.*, 2021). However, research has shown that closed tasks, accurate, fast, and precise calculation of procedural tasks is still in the centre of teaching and assessing mathematics (Trupčević & Glasnović Gracin, 2014; Dijanić & Debelec, 2015). Research shows, that although for decades educational policies have been emphasizing teaching and learning mathematical communication, "students are lacking" in development of mentioned skills (Rustam & Ramlan, 2017). Rohid *et al.* (2019), report in their study that only one of three students is able to express mathematically their ideas, Rahman *et al.* (2012), in their research identify students' difficulties in communicating mathematically and especially in conveying their knowledge to their peers in written form. Curricular reform in Croatia (2019), brought new elements for assessing mathematical competences. Mathematical communication, alongside with acquisition of knowledge and skills and problem solving is used as element to examine whether educational outcomes of teaching and learning mathematics are met, thus through new curricular reform, great importance is given to mathematical communication.

Yet, taking a step further and looking into practice there are some insecurities regarding assessment and research reported, assessment as the most demanding part of teaching and learning process as well (Petrić, 2002). Likewise, research on implementation of curricular reform in Croatia reported on lack of confidence and assuredness in teachers on ones' knowledge on assessment. Furthermore, common opinion

of teachers is that more opportunities and time is needed in order to experience sense of competence in that area (Bradfield *et al.*, 2019). There is no doubt that introduction of new elements of assessment, such as mathematical communication, played significant role in emergence of those insecurities on how to assess mathematical communication, emerging across all levels of teaching mathematics, from primary through secondary education. Alongside those insecurities, emerged by introduction of curricular reform, it is evident from teaching praxis that the teachers haven't change anything crucial in their assessment praxis. Exams were not adjusted for assessing mathematical communication, on the contrary, the very same exam that was used before, was assessed with two different grades, i.e., used to assess two different elements. One of which was mathematical communication and the other acquisition of knowledge. We could say that previously mentioned became dominant method used for assessing mathematical communication. I.e., more often than not, teacher would select several tasks on the existing test and decide that those tasks are to be assessed as mathematical communication. Unfortunately, even in literature there are limited sources to be found for educators, as there are only few recommendations, for the method one should use when assessing mathematical communication such as evaluating complicated assignments to receive a complete picture of students' learning gains or using open-ended tasks and group discussions (e.g., Cai *et al.*, 1996; Viseu & Oliveira, 2012).

2.3. Writing in mathematics classroom

Writing activities in mathematics classroom and their potential benefits for mathematical learning have received meaningful attention in mathematics education literature over the past 30 years. Many authors argue that more frequently students reported writing one or more paragraph in science and social studies, the higher their writing achievement (Wilcox & Monroe, 2011). Relying on those data we can assume that incorporating writing more frequently in mathematics classroom can support better achievements in writing mathematics. Nevertheless, research has not shown the same results for mathematics and social studies (Wilcox & Monroe, 2011). However, research did show that

writing activities are rarity in mathematics classroom and lack thereof those activities.

Analysing literature some of the possible reasons for lack of writing activities in mathematics emerged. Shield and Galbraith (1998) found reason in lack of detailed model for analysing students' writing products. There is also evidence that social science teacher finds it more natural to integrate writing in their lessons than mathematics teacher does (Jones & Thomas, 2006; Varelas *et al.*, 2008). Applebee and Langer (2006; according to Wilcox & Monroe, 2011) state that "the role of writing in mathematics instruction has not been well conceptualized", despite of mathematical communication skill being recognized and identified as aim of learning mathematics long time ago and worldwide (e.g., NCTM, 1989). The main reason of neglecting writing activities in mathematics curricula Bosse and Faulconer (2008) found in standardized tests. The authors state that standardized tests show increase in open-ended tasks, therefore integrating writing activities in mathematics classroom and curricula is no longer interesting. Consequently, there is a need to raise awareness of teachers about long term benefits of writing activities, despite of lack of that kind of tasks on the standardized tests and raise awareness about avoiding the trap of *teaching to test effect*.

Quinn and Wilson (2010) reported on teacher beliefs and practices about writing in mathematics classroom. Examined teachers agreed on writing in mathematics being extremely important for assessment, explaining that it gives students the opportunities to show their knowledge of a particular concept from different perspectives. Although acknowledging each benefit of using writing activities that could provide an alternate perspective on students' knowledge, utter representation of knowledge on particular concept or their cognitive modelling (e.g., expository writing or essays) and possessing positive attitudes towards them, still, they are used the rarest because of insufficient time (Quinn & Wilson, 2010). Moreover, numerous teachers are not appropriately skilled for integration of writing in mathematics whether it is used as tool for teaching, learning or assessment (Bosse & Faulconer, 2008). It is not rare that teachers would assign writing reports on famous mathematicians, their biographies and similar, which is fulfilling the function

of mathematics popularization (Bosse & Faulconer, 2008), and not the function of developing mathematical communicational skills.

Except for substantial evidence and proposed reasons of lack of writing activities in mathematics, vast number of authors presents benefits of writing activities for enhancing mathematical competences such as: writing mathematics may lead to students increasing in content knowledge (Borasi & Rose, 1989), as well as recognizing and generating connections (Bradley, 1990), furthermore, students' understanding of mathematical vocabulary is significantly connected to their understanding of mathematics and ability to solve problems (Stahl & Fairbanks 1986). Students demonstrate greater understanding and learning through writing to learn mathematics (e.g., Miller, 1992; Rose, 1989; Porter & Masingila, 2000). Writing about and analysing problems, even without solving them, helped students develop higher-level thinking skills (LeGere, 1991). Writing activities develop mathematical communicational skills, as well as deepening mathematical knowledge and understanding (Bosse & Faulconer, 2008), mathematical thinking (Shepard, 1993), especially conceptual and metacognitive (NCTM, 1989; LeGere, 1991).

> "When students elaborate on knowledge, they not only understand it in greater depth, but they recall it much more easily." (Marzano *et al.*, 2001, p.74)

To benefit from writing mathematics, student must value its usefulness. That is why the purpose of such assignments have to be clearly articulated and understood (Burton & Morgan, 2000). It reveals misconceptions and "helps" them to see students' thinking process, as it gives feedback about their teaching process (Quinn &Wilson, 2010). Yet, students who aren't trained to write mathematically don't have benefits from activities which include writing in mathematics classroom (Shibli, 1992; Moore, 1993) and by assessing their knowledge through use of writing activities, teachers are not to receive real depiction of their knowledge.

The literature offers many ways to integrate writing in mathematics lessons, which would assume writing about mathematics and writing mathematically. Widely known activities in that context are learning logs, think – write – share, note – taking/note – making, shared writing, alphabet books, sketch-to-stretch, graphical organizers, class

books, pictorial journals, expository writing, creative writing, proof writing, rewriting lecture notes and many others. Not all mentioned activities are suitable for each educational outcome. Different methods and methods are useful for different aspects of assessment (Allen, 1991, according to Bosse & Faulconer, 2008). For instance, learning logs or note-taking/note-making could be used as method for assessment as learning (formative assessment), shared writing and alphabet books could be used for assessment for learning (formative assessment), and expository writing as assessment of learned (summative assessment).

2.4. Expository writing in mathematics classroom – Mathematical essay

Expository writing is used when there is the purpose to describe and explain (Shield & Galbraith, 1998). Expository writing assumes analysis of a topic, the writer explains or defines a topic, using facts, statistics, and examples (Shield & Galbraith, 1998). Essay, however, is predominantly used type of task when expository writing is required from students. Essay is literary – scientific piece of writing which cultivates particular topic or a problem, hence it is appropriate for examination of level of familiarity and understanding of topic in general (Brozović & Kovačec, 1999). Essay is to be assigned to examine knowledge and understanding of definitions as well as making relations among concepts and their comparison (Vizek Vidović *et al.*, 2003). By writing an essay noticeable becomes students' ability to analyse and synthetize data and facts, knowledge relations of various areas and possibility to justify procedure or statement. Consequently, student has to be admirably familiar with the essay topic (Staščik & Glavaš, 2019). In order to write essay of quality, it is not enough to be familiar with the content on the level of terms and definitions, nonetheless wider and deeper knowledge is necessary to independently write longer and sensible text (Staščik & Glavaš, 2019). Furthermore, writing an essay provides students with opportunity to express their knowledge in creative manner as opposed to other methods of knowledge assessment (Staščik & Glavaš, 2019). Although classification of essay provides various types of essays, we are going to concentrate on expository school essay in teaching and learning mathematics.

Expository essay is predominant in knowledge assessment because it presents the acquired knowledge and analyses and interprets it. Concepts, phenomena, and ideas are presented through interpretation which is done on the basis of description and discussion. Expository essay in mathematics is going to differ from the other types by its form (Staščik & Glavaš, 2019). Contrary to other types of essays, mathematical essay would include students' sketches, drawings, calculations, constructions, proofs, and use of mathematical language etc., moreover, students are encouraged to do so (Staščik & Glavaš, 2019).

2.4.1. Assessing mathematical essay

There are several methods proposed for assessing mathematical essay in analysed literature. Van Dormolen (1985) developed two categories: aspects of mathematics and levels of language for describing all content in mathematics textbooks that can be applicable to expository writing in mathematics.

Table 1. Van Dormolen categories describing content of mathematical textbooks (Van Dormolen, 1985)

Aspects of mathematics	Levels of language
Theoretical Definitions, theorems, axioms….	**Exemplary** Specific case or example
Algorithmic Explicit methods or "how to do" a specific operation or procedure	**Generalised** Language conveying the general meaning of a concept or procedure
Logical Statements about the way one should work using the theory – theory extension	**Procedural** Step by step instructions
Methodological "how to do" rules – general; heuristic; rules need interpretation to provide an answer	**Descriptive** Stating the meaning of an idea or the appearance of an object or diagram
Conventional e.g., how to name diagram, write proof…	

Leinhardt (1987) examined the lessons of expert teachers and identified some features of explanation which could be seen also in students' writing. Those features are as following:

- Kernel – general expressions that have to be learned as knowledge;
- Goal statement – identification of goal and indicating progress towards goal;
- Demonstration – worked example which may be elaborated with symbolic representation, verbal description, diagram representation or statement of convention;
- Legitimisation – justification for the procedure or known principles, compelling logic, cross-checks;
- Link to prior knowledge – reference to everyday life;
- Practice exercises – creating set of questions to be answered by the reader.

Further, Shepard (1993) proposed progressive levels of mathematics writing and those would be: initial, intermediate and terminal level. In initial level mere record knowledge through transcription, or report knowledge through summaries were shown as well as generalized narrative of some personal examples in own words (Shepard, 1993). Intermediate level would require low-level logic explaining how to solve problems, what is wrong with incorrect working, analogic explaining how concepts are related, why procedures apply or not apply (Shepard, 1993). Terminal level of writing would assume analogic-tautologic proposing alternative ways understanding how different they are as well as tautologic of producing new method (Shepard, 1993).

In addition, Shield and Galbraith (1998) investigated the writing in the student's textbook and the examples of student writing and discovered that they share several similarities. Both centre on presenting example of the procedure, which is presented as a purely algorithmic element of mathematics. In the student's writing, the technique is occasionally stated in general terms, and on rare occasions, a connection to prior knowledge is made, which is a regular pattern in textbooks. There are no explanations of the benefits of the methods or the reasons those procedures work, in either textbook or students' writing. Even the writing style of the students is authoritative, much like the writing in textbooks. Consequently, Shield and Galbraith (1998) developed a set

of descriptions for recognizable levels of elaboration of mathematical ideas which reflect thinking and understanding as shown in Table 2.

Table 2. Shields' and Galbraiths' (1998) levels of understanding and descriptions (Shield & Galbraith, 1998)

Level	Description
Approach	Presenting unelaborated demonstration of the algorithm, but lack sufficient definition to be kernel
Recount	Reproducing complete demonstration of the concept or procedure
	The aspect of mathematics is algorithmic
	May be elaborated symbolically and verbally
	Exemplary – related only to the specific example
Generalise	Demonstration of concept or procedure which is elaborated verbally, symbolically or diagrammatically when appropriate
	Inclusion of definition or procedural statement in relative language that is kernel stated
	Algorithmic aspect of mathematics
	May include some practice exercises for the readers
	Generalize the method into an algorithm
Link	Include some links with prior knowledge or everyday applications
	Relational
Integrate	Integrating relevant ideas to arrive at an elaborated presentation
	Including theoretical and methodological aspects of mathematics in relative language
	Logical organization of ideas
	All aspects of mathematics

Staščik and Glavaš (2019) gave their frame for assessing mathematical essay considering not only levels of knowledge, but structure

design, content coverage and text coherence which are described in Table 3.

Table 3. Frame for assessing mathematical essay (Staščik & Glavaš, 2019)

Elements for assessing mathematical essay	Description
Text coherence	Shaping text as whole with visually separated parts: introduction, central part (elaboration) followed by conclusion and logic organization of text
Content coverage	Including all key elements in writing and elaboration
Structure design	Using mathematical language and appropriate writing style

Besides, Staščik and Glavaš (2019) classified knowledge across six levels: facts, understanding, applying, analysing, synthetizing, and evaluating heavily leaning on Blooms' taxonomy of knowledge (Bloom, 1956). In regards of presenting mathematical knowledge in essay knowledge on the level of facts would encompass appropriate terminology, definitions and defining key concepts. In order to present knowledge on understanding level one should explain and elaborate relations, provide examples and estimations and select appropriate method. One including statement of theories, models, and structures as well as demonstration of tasks would demonstrate mathematical knowledge of analysis level. Connecting and systematization as well as logical organizing of knowledge would indicate synthesizing level of knowledge. If essay author would provide some discussion of known and presented methods or express its conclusion on significance and relevance of procedures or concepts it would be considered as presenting mathematical knowledge, on given topic, at evaluating level. The novelty in Staščik and Glavaš (2019) framework is that they took into consideration structure design of a mathematical essay.

2.5. Influences on mathematical essay performance

Numerous research (e.g., Baroody & Ginsburg, 1990) has shown that most of writing activities in teaching and learning mathematics is predominantly reproduction of (assigned) text which is almost identical to the ones in mathematical textbooks, by style. Research have shown that mathematics textbook greatly influence teaching, by playing central role for teacher's lesson preparation, selecting of worked examples, giving homework, organizing teaching, and using teaching methods (Glasnović Gracin & Domović 2009; Glasnović Gracin & Jukić Matić, 2016). Baroody and Ginsburg (1990) also say that teachers teaching is influenced by textbooks which can be seen in using *tell-show-do* approach. Bosse and Faulconer (2008, 9) advocate that mathematics textbooks have particular form "numerous pages of discussion, examples, more discussion, and more examples all followed by a few pages of compacted homework exercises". Students find mathematics textbooks important, but mostly use them when doing homework and for exam preparation, focusing on exercises (Jukić Matić & Glasnović Gracin, 2020). Pinne (1983) states that students make attention to the explanatory text only when they get stuck during homework. Considering that, it is not surprising that students' style in expository writing closely resemble to the style of mathematics textbooks (Shield & Galbraith, 1998).

Previous experience of different writing activities, writing essays and continuous assessment of students' progress is irrelevant (in Croatian or English), since students feel as novice when it is about mathematical writing. That's why it is important for teacher to have this in mind and carefully plan initial activities and assignments (Bosse & Faulconer, 2008). Assignments have to be appropriate with students' age and prior knowledge (vocabulary, concepts, procedures, ideas, etc.), and connected to accomplishing specific learning goal (Bosse & Faulconer, 2008). The mathematical essay writing performance is objectively influenced by students' knowledge and type of knowledge, instructions provided as well as how familiar they are with assessment criteria, and subjectively language and writing skills as well as subjectivity of revisor (Staščik & Glavaš, 2019).

In order to ensure writing essays would be beneficial for students and that their work is of quality, the teacher has an important role to pre-

pare students for writing a mathematical essay. This implies that teachers are required to determine, during teaching planning phase, that they will evaluate a certain topic using a mathematical essay in order to constructively align the learning outcomes and teaching activities with the assessment method. In order for teaching and learning to be successful and beneficial, it must be constructively aligned, i.e., it is important to evaluate what and how it is taught, and students must be familiar with the criteria, methods and elements of assessment in advance (Sorić *et al.*, 2018). Moreover, students are not to be expected to know how to independently write such essay successfully without having previous experience of writing or without practical practice of writing mathematical essays exercise or math essays. It is necessary to provide students with opportunities to practice the skill of mathematical writing, where they are introduced to the form of mathematical essay, integral elements of essays, methods of assessing essays, etc. (Staščik & Glavaš, 2019).

Essay assignment consists of topic alongside few writing instructions which students will take in consideration when writing the essay. Instructions have to be content based and refer students to structure and context of the essay. Students should encompass all given instructions; however, the order of instructions is not binding to forming the essay.

If during teaching phase, teacher assign students only closed tasks or procedural tasks, leaves out defining concepts, features, and relations among them it is unrealistic to expect for students to be able to write mathematical essay, where all mentioned above is required. Therefore, teacher has to bear in mind plausible instructions (learning outcomes) and expectations for the essay content. It is needed to emphasise and pose questions to students that could as well be part of essay instructions, provide investigative assignments and direct them to utilize literature.

Staščik and Glavaš (2019) gave the proposal of structural as well as content instructions that should be addressed before starting a process of writing a mathematical essay. Proposed structural instructions are to emphasize that essay is formed as a unity with content and visually separated paragraphs: introduction, body, conclusion, as well as that student should use formal language and the sentences should be precise and clear (Staščik & Glavaš, 2019).

Content instructions are divided in three sections: introduction, body, and conclusion. In introduction students are to pay attention to present the topic and give an insight into the context of the problem and state the mathematical field to which the topic belongs (state weather student have been already familiar with the topic (when, how, context?) (Staščik & Glavaš, 2019). As for the main part of essay i.e., the body instruction will depend on topic, given guidelines and students' level of knowledge. Nevertheless, students should define key concepts, state properties of concepts and elaborate on them, make classifications, explain relations between concepts, compare and establish causal relations, give examples, demonstrate tasks, represent methods for problem solving, discuss on selecting methods, make correlations with real life, analyse and synthesize, logically organize ideas, prove, state theories, models and structures.

In the conclusion students should express an opinion on a given topic, on the importance and relevance of learning it (Staščik & Glavaš, 2019).

3. Analysis

3.1. Estimation of adequacy of mathematical essay for assessing mathematical communication

According to before mentioned aspects of mathematical communication and essay features authors attempt to verify essay as adequate technique to assess mathematical communication. In Table 4 there are presented aspects which construct mathematical communication (according to Croatian curriculum for mathematics) as well as essay features (according to the analysed literature) that cater to these aspects.

Table 4. Analysis and comparison of mathematical communication aspects and features of essay

Mathematical communication	Essay features
To use appropriate mathematical language (standard mathematical symbols, notation, and terminology) in oral and written expression.	Recognition and understanding of definitions. Using mathematical language.

To organize data into logical structure.	Making relations between concepts and their comparison. Analysis and synthesis.
To express one' thoughts in complete, relevant and condense mathematical sentences.	Requires deeper and broader knowledge to write a longer and meaningful text.
To use appropriate mathematical data representation.	Allows student showing its knowledge in creative way.
To interchange between different mathematical representation.	Assumes drawing, sketching, constructing, calculating, using mathematical language, etc.
Proper use of technology.	–

If one were to evaluate or assess use of appropriate mathematical language (standard mathematical symbols, notation, and terminology) of written expression, one could do that by evaluating essay feature of recognition and understanding of definitions and use of mathematical language in an essay. Furthermore, for assessing organization of data into logical structure, one should examine relations made between concepts and their comparison as well as their analysis and synthesis as an essay feature. For assessing expression of students' thoughts in complete, relevant and condense mathematical sentences is to search for deeper and broader knowledge written in a longer and meaningful text. Use of appropriate mathematical data representation would be assessed through creative and various ways student presented their knowledge. Finally, drawing, sketching, constructing, calculating, using mathematical language is suitable for evaluating interchange between different mathematical representation. The only aspect of mathematical communication not covered by writing an essay is proper use of technology, although we advocate that limitation could be easily taken under control in preparation phase of assigning an essay.

Taking results of this analysis i.e., comparison of mathematical communication integral elements and essay features, into consideration it is noticeable that essay is appropriate method of assessing mathematical communication. Based on results emerged during theory and research investigation authors of this paper advocate for mathematical

essay as valuable method for investigating students' mathematical communication skills.

4. Providing method/frame for assessing mathematical essay with emphasis on mathematical communication

Since theory examination and deep investigation on research concerning both mathematical communication skills and writing expository essay in schools provided authors with a prove of adequacy of an expository essay as a valuable assessment method for mathematical communication, the authors undertook the attempt to propose frame for assessing mathematical essay with emphasis on mathematical communication. Coming from Croatian educational system in which grading scale is based on five levels it came natural for the authors to create five levels of expertise for writing mathematical essay. The five levels of expertise would commence at novice level, thorough advanced beginner, competent, proficient, and come to final, expert level, as presented in Table 5. We assigned each level of expertise a presumed level of mathematical knowledge, a presumed level of structural design and text coherence, and a required presentation of mathematical communication skills, as it is shown in Table 5. Authors would advocate that this kind of assessment frame and method would serve as a tool to better assess the width and depth of each students' mathematical communication skills as well as develop sense of readiness and confidence to conduct mathematical communication assessment.

Table 5. Providing method for assessing mathematical essay with emphasis on mathematical communication

Levels of expertise for writing mathematical essay	**Knowledge**	**Structure design & text coherence**	**Mathematical communication**
Novice	listing facts, definitions, theorems, axioms but it is not kernel recording and transcription, without elaboration	absence of visual and content text formatting no clear goal in the introduction, does not reflect the title paragraphs are disorganised with no clear transitions	use appropriate mathematics terminology, partially correct use of mathematical symbols
Advanced beginner	report-summaries kernel reproducing complete demonstration of concept or "how to do" a specific operation or procedure worked specific case or example elaborating verbally describing relationships	visual text formatting stating goal but it is unfocused, placed in introduction but partly reflects the title paragraphs are vaguely organised with no transitions	use appropriate mathematics terminology, correct use of mathematical symbols
Competent	generalised narrative-personal examples in own words providing step by step instructions	goal is clearly formed and placed in introduction and linked to the title visual and	use appropriate mathematical representation for data presentation expresses ideas

	low-level logic-explaining how to solve problems, what is wrong with incorrect working justification for the procedure or known principles, compelling logic, cross-check reference to everyday life demonstrating tasks estimating, comparing, classification	content text formatting paragraphs are loosely organized with no clear transitions between ideas	in complete, sensible and condense mathematical sentences appropriate use of simple technology
Proficient	statements about the way one should work using the theory-theory extension link to prior knowledge and other educational areas explaining how concepts are related stating examples in new situations causative-consequential relations analysing generalise the method into an algorithm	goal is clearly formed there is indication of progress towards goal there is clear transition from one idea to another within paragraph	interchange between various mathematical representations organise data in logical structure appropriate use of technology.
Expert	providing alternate use for concept or procedure other	the goal is tightly linked to the conclusion	posing questions that surpass margins of

	than being taught proposing alternative ways of understanding and discussing producing new method logical organisation of ideas expressing conclusions about significance and relevance proofing including practice exercises for the readers	well organised paragraphs with clear transitions among ideas and logically structured	originally proposed question

5. Conclusion

As educational policies and research have shown mathematical communication has a great role not only in teaching and learning but understanding, appreciating and engaging with mathematics. By providing students with open-ended situations and looking into ways students arrived to their solutions helps teachers to shed a light on students' conceptions and misconceptions.

In this paper authors aimed to give theoretical foundations for the evaluation weather mathematical essay is appropriate method for assessing mathematical communication and to suggest framework for assessing mathematical essay with emphasis on mathematical communication. Literature review and theory investigation has shown that writing activities should prompt and enhance mathematical communication skills. However, looking into classrooms and practice those implications are not realized. There are many reasons mentioned that would be cause of students' poor mathematical communication skills even when writing activities are incorporated in mathematics lessons. Some of them are importance of writing skills are not conceptualized, and there is uncertainty about assessing those kinds of activities. Re-

search brought theory analysis, on essay features and mathematical communication components, that has proven mathematical essay to be appropriate for assessing each aspect of mathematical communication except for appropriate use of technology. It is evident that assigning essay writing activities in teaching and learning mathematics can prompt teachers while assessing students' mathematical communication skills and hinder any uncertainties on the appropriate method. In this research framework for assessing mathematical essay with emphasis on mathematical communication is proposed. Framework consists of five levels of expertise, from novice to expert.

Knowledge requirements as well as requirements for structure design and text coherence is given for each level. In order to eradicate any uncertainty, authors noted aspects of mathematical communication for all levels as well. Certainly, assessing mathematical communication through expository essay has its limitations as a method regarding the age of students. When certain essay structure, level of previous knowledge and use of mathematical terminology is excepted it is hardly to assume students of wide grade level would be able to master such a technique. The authors would propose use of the technique as it is for high school level students and the development or rather simplification of this method in order to cater to needs of assessing young learners' mathematical communicational skills, since it is given that allowing students to talk or write about their thinking empower them as learners (Rowan *et al.*, 1990). Furthermore, the authors believe that additional research on this topic would be extremely beneficial and required. The validity of the framework for assessing mathematical expository essays should be examined and means of modifying this technique for evaluating mathematical communication on each grade level, i.e., students of wider age range, should be proposed.

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ESEJ IZ MATEMATIKE KAO METODA VREDNOVANJA MATEMATIČKE KOMUNIKACIJE

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Za razumijevanje i učenje matematike vrlo je važna razvijenost vještine matematičke komunikacije. Cilj vrednovanja matematičke komunikacije je procjena načina korištenja matematičkog jezika, sposobnosti postavljanja pitanja, elaboriranja i argumentiranja vlastitih odgovora, upotrebljavanja i tumačenja matematičkih simbola i čitanja algoritama, sposobnost prelaska iz jednog zapisa (prikaza podataka) u drugi te sposobnost primjene tehnologije. Jedno od pitanja koje se samo nameće je: koje su metode i načini vrednovanja najprimjereniji za procjenu vještina matematičke komunikacije? Cilj ovoga rada je istražiti je li interpretativni (matematički) esej prikladna metoda za vrednovanje vještine matematičke komunikacije. Analiza je pokazala da je esejom moguće ispitati gotovo svaku sastavnicu vještine matematičke komunikacije zbog čega autori navode da je matematički esej prikladna metoda vrednovanja matematičke komunikacije. Dodatno, autori su razvili okvir za vrednovanje matematičkog eseja na pet razina uspješnosti, a pri čemu je poseban naglasak stavljen na aspekt vrednovanja matematičke komunikacije, uzimajući u obzir strukturu i sadržaj samog eseja.

Ključne riječi: *matematički esej; matematička komunikacija; matematički jezik; vrednovanje*